Ivan Hlaváček

A finite element analysis for elastoplastic bodies obeying Hencky's law

Aplikace matematiky, Vol. 26 (1981), No. 6, 449–461

SVAZEK 26 (1981) APLIKACE MATEMATIKY ČÍSLO 6

A FINITE ELEMENT ANALYSIS FOR ELASTO-PLASTIC BODIES OBEYING HENCKY'S LAW

IVAN HLAVÁČEK

(Received January 30, 1980)

INTRODUCTION

One of the simplest mathematical models describing the elasto-plastic behaviour of solid bodies is the constituent law of Hencky (see e.g. [1]). The classical boundary value problems allow a variational formulation in terms of stresses, known by the name of Haar-Kármán principle. In the papers by Mercier [2] and Falk [4], [5], approximate solutions of the boundary value problems have been studied, which consists of piecewise constant stress fields. It is the aim of the present paper to employ piecewise linear approximations of stress fields and to give some convergence results for them.

Using some results of C. Johnson and Mercier [7], we define both external and internal approximations of the set of statically admissible stress fields. The set of plastically admissible stress fields is approximated by the requirement that only the mean values of stresses over any finite element have to be plastically admissible.

The torsion problem of a twisted cylindrical bar (under Saint-Venant hypotheses) is solved in terms of stresses by a quite analogous manner. Here we apply piecewise "quasi-linear" approximations introduced by Raviart and Thomas [11].

1. PRELIMINARY DEFINITIONS

Let Ω be a polyhedral bounded domain in $\mathbb{R}^n$, $n = 2, 3$; $\mathbf{x} = (x_1, \ldots, x_n)$ a Cartesian coordinate system. Let $\mathbb{R}_\sigma$ be the space of symmetric $n \times n$ matrices (stress or strain tensors). A repeated index implies summation over the range $1, \ldots, n$.

Assume that a yield function $f : \mathbb{R}_\sigma \to \mathbb{R}$ is given, which is convex and continuous in $\mathbb{R}_\sigma$.

We introduce the following notations:

$$S = \{\tau : \Omega \to \mathbb{R}_\sigma \mid \tau_{ij} \in L_2(\Omega)\ \forall i, j\} .$$

$$\langle \sigma, \mathbf{e} \rangle = \int_\Omega \sigma_{ij} e_{ij} \, \mathrm{d}\mathbf{x} , \quad \|\sigma\|_0 = \langle \sigma, \sigma \rangle^{1/2} .$$

Let the boundary $\partial\Omega$ be decomposed as follows

$$\partial\Omega = \bar{\Gamma}_u \cup \bar{\Gamma}_\sigma , \quad \Gamma_u \cap \Gamma_\sigma = \emptyset ,$$

where Γ_u and Γ_σ are either empty or open in $\partial\Omega$. Assume that a body force vector $\mathbf{F} \in [L_2(\Omega)]^n$, a surface traction vector $\mathbf{g} \in [L_2(\Gamma_\sigma)]^n$ and a displacement vector $\mathbf{u}_0 \in [H^1(\Omega)]^n$ be given.

Henceforth $H^j(\Omega) = W^{j,2}(\Omega)$, $j = 0, 1, 2$, denotes the Sobolev space with the norm $\|\cdot\|_{j,\Omega}$, $H^0(\Omega) = L_2(\Omega)$. $P_k(M)$ is the space of polynomials of the k-th degree on the set M.

In case that $\Gamma_u = \emptyset$, the total equilibrium conditions for $\mathbf{F}$ and $\mathbf{g}$ are assumed to be satisfied.

In the space S we introduce also the energy scalar product

$$(\sigma, \tau) = \langle c^{-1}\sigma, \tau \rangle , \quad \|\sigma\| = (\sigma, \sigma)^{1/2} ,$$

where $c : S \to S$ is the isomorphism defined by the generalized Hooke's law:

$$\sigma = c\mathbf{e} \Leftrightarrow \sigma_{ij} = c_{ijkl}e_{kl} .$$

Here $c_{ijkl} \in L_\infty(\Omega)$, σ and $\mathbf{e}$ are the stress and strain tensors respectively,

$$\exists \alpha > 0 , \quad \langle c\mathbf{e}, \mathbf{e} \rangle \geqq \alpha \|\mathbf{e}\|_0^2 \quad \forall \mathbf{e} \in S .$$

The space of virtual displacements is defined as follows

$$V = \{\mathbf{v} \in [H^1(\Omega)]^n \mid \mathbf{v} = 0 \text{ on } \Gamma_u\} .$$

The set of statically admissible stress fields is

$$E(\mathbf{F}, \mathbf{g}) = \{\tau \in S \mid \langle \tau, \mathbf{e}(\mathbf{v}) \rangle = L(\mathbf{v}) \ \forall \mathbf{v} \in V\} ,$$

where

$$L(v) = \int_\Omega F_i v_i \, \mathrm{d}x + \int_{\Gamma_\sigma} g_i v_i \, \mathrm{d}s .$$

We introduce the set of plastically admissible stress tensors

$$B = \{\tau \in \mathbb{R}_\sigma \mid f(\tau) \leqq 1\} .$$

It is easy to see that B is convex and closed in $\mathbb{R}_\sigma$.

Finally, we define the set of plastically admissible stress fields

$$P = \{\tau \in S \mid \tau(\mathbf{x}) \in B \text{ a.e. in } \Omega\} .$$

The set P is convex and closed in S.

The Hencky's law can be stated in the following way (cf. [1], [2]). Introducing the projection $\Pi_B(\mathbf{x}) : \mathbb{R}_\sigma \to B$ onto the set B with respect to the scalar product $(c^{-1}(\mathbf{x})\,\sigma)_{ij}\,\tau_{ij}$, then

$$\sigma = \Pi_B(\mathbf{x})\, c\mathbf{e}\,. \tag{1.1}$$

Consider the actual strain tensor field $\mathbf{e}(u) \in S$,

$$e_{ij}(\boldsymbol{u}) = \tfrac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)\,,$$

and the actual stress tensor field $\sigma \in E(\mathbf{F}, \mathbf{g})$, where $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$ is the actual displacement field, $\boldsymbol{w} \in V$. (Suppose the existence of all these fields for the time being.) Moreover, let $\Pi : S \to P$ be the projection onto the set P with respect to the energy scalar product (σ, τ). Then

$$(\Pi\tau)(\mathbf{x}) = \Pi_B(\mathbf{x})\,\tau(\mathbf{x})$$

holds almost everywhere in Ω (see [2]). Hence we may write

$$\sigma = \Pi c\ \mathbf{e}(\boldsymbol{u})$$

and consequently, for any $\tau \in P$

$$(c\,\mathbf{e}(\boldsymbol{u}) - \sigma, \tau - \sigma) \leqq 0$$

i.e.,

$$\langle \mathbf{e}(\boldsymbol{u}_0) + \mathbf{e}(\boldsymbol{w}), \tau - \sigma\rangle - (\sigma, \tau - \sigma) \leqq 0\,. \tag{1.2}$$

Let us take

$$\tau \in E(\mathbf{F}, \mathbf{g}) \cap P\,.$$

Since $\tau - \sigma \in E(0, 0)$ and $\boldsymbol{w} \in V$,

$$\langle \mathbf{e}(\boldsymbol{w}), \tau - \sigma\rangle = 0\,.$$

Thus we obtain

$$(\sigma, \tau - \sigma) - \langle \mathbf{e}(\boldsymbol{u}_0),\ \tau - \sigma\rangle \geqq 0 \quad \forall \tau \in E(\mathbf{F}, \mathbf{g}) \cap P\,. \tag{1.3}$$

The inequality (1.3) is equivalent with the Haar-Kármán principle: *the actual stress field σ minimizes the functional of complementary energy*

$$\mathscr{S}(\tau) = \tfrac{1}{2}\|\tau\|^2 - \langle \mathbf{e}(\boldsymbol{u}_0), \tau\rangle \quad \textit{over} \quad E(\mathbf{F}, \mathbf{g}) \cap P\,.$$

In fact, both the functional $\mathscr{S}$ and the set $E(\mathbf{F}, \mathbf{g}) \cap P$ are convex and the equivalence follows easily.

Theorem 1.1. *Let the set $E(\mathbf{F}, \mathbf{g}) \cap P$ be non-empty. Then the Haar-Kármán principle has a unique solution σ.*

Proof. The sets $E(\mathbf{F}, \mathbf{g})$ and P are convex and closed in S, the functional $\mathscr{S}$ is quadratic and strictly convex. Hence the existence and uniqueness follows.

Remark 1.1. The formulation in terms of displacements is much more difficult to handle (see [1], [2], [3]), as far as the existence and uniqueness is concerned.

2. APROXIMATIONS BY EQUILIBRIUM FINITE ELEMENT MODELS

Let us consider two-dimensional problems, i.e. let $\Omega \subset \mathbb{R}^2$. In order to discretize the problem, one has to replace the set $E(\mathbf{F}, \mathbf{g}) \cap P$ by a finite-dimensional approximation. The simplest possibility is to work with piecewise constant stress fields on triangulations of the domain Ω. An analysis of such a method has been given by Mercier and Falk in [2], [4], [5]. In the present paper, we employ piecewise linear stress fields on composite triangles (see Watwood and Hartz [6]).

First we recall some results on the composite triangular block-elements, obtained by C. Johnson and Mercier [7]. Let us consider a triangle K with vertices a_1, a_2, a_3. Joining the vertices with the center of gravity O, we obtain three subtriangles K_i, $i = 1, 2, 3$. Consider a triangulation $\mathscr{T}_h$ of Ω and define $S_h = \{\sigma \in S | \sigma|_{K_i} \in [P_1(K_i)]^4$, $\sigma . \nu$ is continuous when crossing any side Oa_i, $i = 1, 2, 3$ and $a_i a_j$, for all $K \in \mathscr{T}_h\}$, where ν denotes the unit normal with respect to the side under consideration.

In [7] a linear mapping

$$r_h : S \cap [H^1(\Omega)]^4 \to S_h$$

is defined through the following set of conditions:

$$\text{(i)} \qquad \int_l ((r_h\sigma) . \nu - \sigma . \nu) . \mathbf{v} \, \mathrm{d}s = 0\,, \quad \forall \mathbf{v} \in [P_1(l)]^2\,, \quad (r_h\sigma) . \nu \in [P_1(l)]^2$$

on every side $l \in \mathscr{T}_h$,

$$\text{(ii)} \qquad \int_K (r_h\sigma - \sigma) \, \mathrm{d}\mathbf{x} = 0 \quad \forall K \in \mathscr{T}_h\,.$$

If $\sigma \in S \cap [H^j(\Omega)]^4$, $j = 1, 2$, then

$$\|\sigma - r_h\sigma\|_0 \leqq Ch^j |\sigma|_{j,\Omega} \tag{2.1}$$

holds for any regular family $\{\mathscr{T}_h\}$, $0 < h \leqq h_0$, of triangulations, where h is the maximal length of all sides in $\mathscr{T}_h$ and $|\sigma|_{j,\Omega}$ is the seminorm consisting of all derivatives of the j-th order. C is a constant independent of h and σ. Although the estimate (2.1) has bee proven in [7] for $j = 2$ only, the same argument is applicable to the case $j = 1$.

Let us define external approximations E_h of the set $E(\mathbf{F}, \mathbf{g})$:

$$E_h = \{\sigma_h \in S_h \mid \langle \sigma_h, \mathbf{e}(\mathbf{v}_h) \rangle = L(\mathbf{v}_h) \ \forall \mathbf{v}_h \in V_h\}\,,$$

where

$$V_h = \{\mathbf{v}_h \in V \mid \mathbf{v}_h|_K \in [P_1(K)]^2 \ \forall K \in \mathscr{T}_h\} .$$

Introduce also an approximation P_h of the set P:

$$P_h = \left\{\tau_h \in S_h \,\middle|\, f\left(\frac{1}{\operatorname{mes} K} \int_K \tau_h \,\mathrm{d}x\right) \leqq 1 \ \ \forall K \in \mathscr{T}_h\right\} .$$

In other words, the condition $\tau_h \in B$ a.e. in Ω is replaced by a weaker condition, that the mean values of τ_h on every $K \in \mathscr{T}_h$ belong to B. It is obvious that $E_h \not\subset E(\mathbf{F}, \mathbf{g})$ and $P_h \not\subset P$, in general.

We now define the approximate problem: to find $\sigma_h \in E_h \cap P_h$ such that

$$\mathscr{S}(\sigma_h) = \min \quad \text{over} \quad E_h \cap P_h . \tag{2.2}$$

Lemma 2.1. *If there exists a stress field*

$$\tau \in E(\mathbf{F}, \mathbf{g}) \cap P \cap [H^1(\Omega)]^4 ,$$

then the problem (2.2) *has a unique solution.*

Proof. Applying the mapping r_h to the stress field τ, we obtain

$$r_h \tau \in E_h . \tag{2.3}$$

In fact,

$$\mathbf{e}(\mathbf{v}_h)|_K \in [P_0(K)]^4 \quad \forall K \in \mathscr{T}_h , \quad \forall \mathbf{v}_h \in V_h .$$

Consequently, (ii) yields

$$\langle r_h \tau, \mathbf{e}(\mathbf{v}_h)\rangle = \langle \tau, \mathbf{e}(\mathbf{v}_h)\rangle = L(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h .$$

Furthermore,

$$r_h \tau \in P_h . \tag{2.4}$$

In fact, $\tau \in B$ a.e. and therefore

$$\frac{1}{\operatorname{mes} K} \int_K r_h \tau \,\mathrm{d}\mathbf{x} = \frac{1}{\operatorname{mes} K} \int_K \tau \,\mathrm{d}\mathbf{x} \in B \quad \forall K \in \mathscr{T}_h$$

follows from (ii) and the convexity of B.

Hence, the set $E_h \cap P_h \neq \emptyset$. E_h is convex and closed in S, being an affine hyperplane in the finite-dimensional space S_h. The set P_h is also convex and closed in S. To prove the closedness of P_h, we use that both the mean values on K and the yield function are continuous mappings of their arguments. The convexity of P_h follows from the convexity of f.

The rest of the proof is obvious.

Theorem 2.1. *Let the solution σ of the Haar-Kármán principle belong to $[H^1(\Omega)]^4$. Then*

$$\|\sigma - \sigma_h\| \to 0\,, \quad h \to 0 \tag{2.5}$$

holds for any regular family of triangulations.

Proof. We employ the following abstract proposition on the convergence of the Ritz-Galerkin approximations (see e.g. [8] – chapter 4).

Proposition 2.1. *Let u and u_h be the unique solutions of the problems*

$$\mathcal{F}(u) = \min \quad \textit{over} \quad \mathcal{K} \quad \textit{and}$$
$$\mathcal{F}(u_h) = \min \quad \textit{over} \quad \mathcal{K}_h\,,$$

respectively, where $\mathcal{F}$ is a quadratic functional in a real Hilbert space H, with positive definite second differential, $\mathcal{K} \subset H$ a closed convex set and $\mathcal{K}_h \subset H$ a closed convex subset for any h, $0 < h \leqq h_0$.

Assume that:

(H 1) *to every $h \in (0, h_0 >$ there exists an element $v_h \in \mathcal{K}_h$ such that*

$$\|u - v_h\| \to 0 \quad \textit{for} \quad h \to 0\,;$$

(H 2) *$v_h \in \mathcal{K}_h$, $u^* \in H$, $v_h \rightharpoonup u^*$ (weakly) for $h \to 0$ implies $u^* \in \mathcal{K}$.*

Then

$$\|u_h - u\| \to 0\,, \quad h \to 0\,. \qquad \square$$

We can apply the proposition with $\mathcal{F} \equiv \mathcal{S}$, $H \equiv S$, $\mathcal{K} = E(\mathbf{F}, \mathbf{g}) \cap P$, $\mathcal{K}_h = E_h \cap P_h$, $u \equiv \sigma$, $u_h \equiv \sigma_h$.

To verify the condition (H 1), we realize that

$$\|\sigma - r_h\sigma\| \leqq Ch|\sigma|_{1,\Omega}$$

by virtue of (2.1) and $r_h\sigma \in E_h \cap P_h$ – see (2.3), (2.4).

Let us consider the condition (H 2). First we show that

$$\tau_h \in E_h\,, \quad \tau_h \rightharpoonup \tau \quad \text{in} \quad S \quad \text{(weakly) implies} \quad \tau \in E(\mathbf{F}, \mathbf{g})\,. \tag{2.6}$$

In fact, for any $\mathbf{v} \in V$ there exists a sequence $\{\mathbf{v}_h\}$, $\mathbf{v}_h \in V_h$, such that

$$\|\mathbf{v} - \mathbf{v}_h\|_{1,\Omega} \to 0\,, \quad h \to 0\,.$$

Consequently, $\mathbf{e}(\mathbf{v}_h) \to \mathbf{e}(\mathbf{v})$ in S and (2.6) follows from

$$\langle \tau_h, \mathbf{e}(\mathbf{v}_h)\rangle = L(\mathbf{v}_h)\,,$$

if we pass to the limit with h.

It remains to verify that

$$\tau_h \in P_h\,, \quad \tau_h \rightharpoonup \tau \quad \text{(weakly) in} \quad S \Rightarrow \tau \in P\,. \tag{2.7}$$

To this end we prove an auxiliary

Lemma 2.2. *Denote for any* $\omega \in S$

$$\psi_h(\omega) \in S$$

the tensor function such that

$$\psi_h(\omega)\big|_K = \frac{1}{\operatorname{mes} K}\int_K \omega \, \mathrm{d}\mathbf{x} \quad \forall K \in \mathscr{T}_h\,.$$

Then $\tau_h \rightharpoonup \tau$ *(weakly) in* S *for* $h \to 0$ *implies that*

$$\psi_h(\tau_h) \rightharpoonup \tau \quad \text{(weakly) in} \quad S\,.$$

Proof. For any $s \in S$ we may write

$$\left|\langle s, \psi_h(\tau_h) - \tau\rangle\right| \leqq \left|\langle s, \psi_h(\tau_h) - \psi_h(\tau)\rangle\right| + \left|\langle s, \psi_h(\tau) - \tau\rangle\right|. \tag{2.8}$$

It is well-known that:

$$\|\psi_h(\tau) - \tau\|_0 \to 0\,, \quad h \to 0\,, \quad \forall \tau \in S\,. \tag{2.9}$$

Furthermore, we have

$$\langle s, \psi_h(\tau_h - \tau)\rangle = \int_\Omega s_{ij}\, \psi_{hij}(\tau_h - \tau)\, \mathrm{d}\mathbf{x} =$$

$$= \sum_{K \in \mathscr{T}_h} \int_K s_{ij}\, \mathrm{d}x \int_K (\tau_h - \tau)_{ij} (\operatorname{mes} K)^{-1}\, \mathrm{d}x = \langle \psi_h(s), \tau_h - \tau\rangle\,.$$

Using (2.9), we conclude that both terms on the right-hand side of (2.8) tends to zero, which proves the lemma. □

Now we are able to verify (2.7). Recall that

$$\tau_h \in P_h \Leftrightarrow \psi_h(\tau_h) \in P\,, \quad \tau_h \in S_h$$

follows from the definition of P_h. By virtue of Lemma 2.2, we have

$$\psi_h(\tau_h) \rightharpoonup \tau \quad \text{in} \quad S\,.$$

Since P is weakly closed, $\tau \in P$. Q.E.D.

Next let us employ internal approximations of the set $E(\mathbf{F}, \mathbf{g})$. To this end, assume that the body forces $\mathbf{F}$ and the surface tractions $\mathbf{g}$ are piecewise constant and piecewise linear with respect to a fixed triangulation $\mathscr{T}_{h_0}$, respectively.

Then a stress field $\chi \in E(\mathbf{F}, \mathbf{g})$ exists, which is **piecewise linear** with respect to $\mathcal{T}_{h_0}$.

Inserting $\sigma = \chi + \tau$ into the Haar-Kármán principle, we obtain the following equivalent problem:

$$(2.10) \qquad J(\tau) = \min \quad \text{over} \quad \mathcal{K}\,,$$

where

$$J(\tau) = \tfrac{1}{2}\|\tau\|^2 + (\tau, \chi) - \langle \mathbf{e}(\mathbf{u}_0), \tau \rangle\,,$$
$$\mathcal{K} = E(0, 0) \cap (-\chi + P)\,.$$

Let us approximate the set $\mathcal{K}$ by the set

$$\mathcal{K}_h = \left\{ \tau_h \in E_h^0 \,\middle|\, f\left(\frac{1}{\operatorname{mes} K} \int_K (\chi + \tau_h)\,\mathrm{d}\mathbf{x}\right) \leqq 1 \;\; \forall K \in \mathcal{T}_h \right\},$$

where

$$E_h^0 = E(0, 0) \cap S_h\,, \quad \mathcal{T}_h \text{ is a refinement of } \mathcal{T}_{h_0}\,.$$

We define the approximate problem

$$(2.11) \qquad J(\tau_h) = \min \quad \text{over} \quad \mathcal{K}_h\,.$$

Lemma 2.3. *Let a $\sigma_0 \in E(\mathbf{F}, \mathbf{g}) \cap P$ exists such that $\sigma_0 - \chi \equiv \tau_0 \in [H^1(\Omega)]^4$. Then the problem (2.11) has a unique solution τ_h.*

Proof. We have $\tau_0 \in E(0, 0) \cap [H^1(\Omega)]^4$ and

$$r_h \tau_0 \in E_h^0$$

follows from [7] – (5.5) and Lemma 2. Moreover, since $\chi|_K \in [P_1(K)]^4$ for all $K \in \mathcal{T}_h$, we have

$$r_h \chi = \chi$$

and consequently

$$(2.12) \qquad r_h(\chi + \tau_0) = \chi + r_h \tau_0\,.$$

Since $\chi + \tau_0 \in B$ a.e. in Ω, the mean values of $\chi + r_h\tau_0$ in every triangle K belong to B, by virtue of (2.12), the condition (ii) for r_h and the convexity of B. Thus we conclude that $r_h \tau_0 \in \mathcal{K}_h$.

The set $\mathcal{K}_h$ is convex and closed in S (cf. an analogous assertion in the proof of Lemma 2.1). Hence the existence follows. The uniqueness is a consequence of the strict convexity of the functional J.

Theorem 2.2. *Assume that $\sigma - \chi \equiv \tau \in [H^1(\Omega)]^4$. Then*

$$\|\tau - \tau_h\| \to 0\,, \quad h \to 0\,,$$

holds for any regular family of triangulations, refining $\mathcal{T}_{h_0}$.

The proof is parallel to that of Theorem 2.1. We use also that $r_h \tau \in \mathscr{K}_h$ follows, like in proving Lemma 2.3.

Remark 2.1. In three-dimensional problems, we can employ piecewise linear stress fields on tetrahedral block-elements composed of four subtetrahedrons. Estimates parallel to (2.1) hold for an analogous mapping r_h (see the forthcoming paper [10]). Then the above results remain true.

3. TORSION PROBLEM

Let us consider a cylindrical bar subjected to a twisting moment at one end while keeping the other end fixed. Using the Saint-Venant theory of torsion and the Haar-Kármán principle, we are led to the following problem in terms of stresses ($p_i = C\tau_{i3}$, $i = 1, 2$, $C = \text{const}$):

$$\mathscr{S}(\boldsymbol{p}) = \tfrac{1}{2}\|\boldsymbol{p}\|^2 - (\varphi, \boldsymbol{p}) = \min \quad \text{over} \quad E \cap P\,, \tag{3.1}$$

where $\boldsymbol{p} \in [L_2(\Omega)]^2$, $\Omega \subset \mathbb{R}^2$ represents the cross section of the bar (multiply connected, in general); $(\cdot, \cdot)$ and $\|\cdot\|$ are the usual scalar product and the norm in $[L_2(\Omega)]^2$, respectively,

$$\varphi_1 = -C_0 x_2\,, \quad \varphi_2 = C_0 x_1\,, \quad C_0 = \text{const},$$

$$E = \{\boldsymbol{p} \in [L_2(\Omega)]^2 \mid (\boldsymbol{p}, \operatorname{grad} v) = 0 \ \forall v \in H^1(\Omega)\}\,,$$

$$P = \{\boldsymbol{p} \in [L_2(\Omega)]^2 \mid f(\boldsymbol{p}) \leqq 1 \text{ a.e. in } \Omega\}\,,$$

where f is a given continuous and convex function in $\mathbb{R}^2$, $f(0) < 1$.

It is readily seen, that the problem (3.1) has a unique solution. In fact, $0 \in E \cap P$, $E \cap P$ is closed and convex in $[L_2(\Omega)]^2$ and $\mathscr{S}$ is strictly convex, quadratic.

To approximate the problem (3.1), we employ some finite element spaces, introduced by Raviart and Thomas in [11].

Let us assume that Ω is a bounded polygonal domain and consider regular family of triangulations $\mathscr{T}_h$ of Ω, $h \to 0$. We construct finite elements on any triangle $K \in \mathscr{T}_h$ by means of an affine invertible mapping

$$F_K : \hat{\boldsymbol{x}} \to F_K(\hat{\boldsymbol{x}}) = B_K \hat{\boldsymbol{x}} + \boldsymbol{b}_K\,,$$

such that $F_K(\hat{K}) = K$, where $\hat{K}$ is the unit right reference triangle in the (ξ, η)-plane. Introduce the linear space of vector-functions

$$\hat{Q} = \{q_1 = a_0 + a_1\xi + a_2\eta + a_3\xi(\xi + \eta),$$
$$q_2 = b_0 + b_1\xi + b_2\eta + b_3\eta(\xi + \eta)\}\,,$$

where $a_i, b_i \in \mathbb{R}$ are arbitrary coefficients.

Then we define

$$S_h = \{\boldsymbol{p} \in [L_2(\Omega)]^2 |\ \forall K \in \mathscr{T}_h\ \exists \hat{\boldsymbol{p}} \in \hat{Q} \quad \text{such that}$$
$$\boldsymbol{p}|_K = (\det B_K)^{-1}\ B_K \hat{\boldsymbol{p}} \circ F_K^{-1}\ ;$$

$\boldsymbol{p} \,.\, \nu$ is continuous, when crossing any side common to two adjacent triangles$\}$.

From [11] – proof of Theorem 3 – we conclude that a linear mapping

$$r_h : [H^1(\Omega)]^2 \to S_h$$

exists such that:

$$\int_K (r_h \boldsymbol{q} - \boldsymbol{q})\, \mathrm{d}\boldsymbol{x} = 0 \quad \forall K \in \mathscr{T}_h\,, \tag{3.2}$$

$$\|r_h \boldsymbol{q} - \boldsymbol{q}\| \leqq C h^j |\boldsymbol{q}|_{j,\Omega}\,, \quad j = 1, 2\,, \tag{3.3}$$

provided that $\boldsymbol{q}$ belongs to $[H^j(\Omega)]^2$.

We define

$$E_h = \{\boldsymbol{q}_h \in S_h \mid (\boldsymbol{q}_h, \operatorname{grad} v_h) = 0 \quad \forall v_h \in V_h\}\,,$$

where

$$V_h = \{v_h \in H^1(\Omega)\ |v_h|_K \in P_1(K)\ \forall K \in \mathscr{T}_h\}$$

is the standard finite element space; furthermore, we introduce

$$P_h = \left\{\boldsymbol{q}_h \in S_h \,\middle|\, f\left(\frac{1}{\operatorname{mes} K} \int_K \boldsymbol{q}_h\, \mathrm{d}\boldsymbol{x}\right) \leqq 1 \ \forall K \in \mathscr{T}_h\right\}.$$

The approximate problem will be defined as follows:

$$\mathscr{S}(\boldsymbol{p}_h) = \min \quad \text{over} \quad E_h \cap P_h\,. \tag{3.4}$$

Lemma 3.1. *The problem* (3.4) *has a unique solution.*

Proof. The set $E_h \cap P_h$ contains the zero element and is convex and closed. Hence the existence and uniqueness of the solution follows.

Theorem 3.1. *Let the solution* $\boldsymbol{p}$ *of* (3.1) *belong to* $[H^1(\Omega)]^2$. *Then*

$$\|\boldsymbol{p} - \boldsymbol{p}_h\| \to 0\,, \quad h \to 0$$

holds for any regular family of triangulations.

Proof. We employ Proposition 2.1, setting $J = \mathscr{S}$, $H = [L_2(\Omega)]^2$, $\mathscr{K} = E \cap P$, $\mathscr{K}_h = E_h \cap P_h$, $u \equiv \boldsymbol{p}$, $u_h \equiv \boldsymbol{p}_h$.

To verify the condition (H 1), we use the estimate (3.3):

$$\|\boldsymbol{p} - r_h \boldsymbol{p}\| \leqq C h |\boldsymbol{p}|_{1,\Omega}$$

and prove that $r_h \boldsymbol{p} \in E_h \cap P_h$. In fact, for any $v_h \in V_h$ we may write

$$(r_h \boldsymbol{p}, \operatorname{grad} v_h) = (\boldsymbol{p}, \operatorname{grad} v_h) = 0\,,$$

by virtue of (3.2) and $v_h \in H^1(\Omega)$. Consequently, $r_h \boldsymbol{p} \in E_h$.

Second, $f(\boldsymbol{p}) \leqq 1$ a.e. in Ω and therefore

$$f\left(\frac{1}{\operatorname{mes} K} \int_K r_h \boldsymbol{p} \, \mathrm{d}\boldsymbol{x}\right) = f\left(\frac{1}{\operatorname{mes} K} \int_K \boldsymbol{p} \, \mathrm{d}\boldsymbol{x}\right) \leqq 1 \quad \forall K \in \mathscr{T}_h$$

follows from (3.2), the convexity and continuity of f. Thus $r_h \boldsymbol{p} \in P_h$.

Let us verify (H 2). We have

$$\boldsymbol{p}_h \in E_h\,, \quad \boldsymbol{p}_h \rightharpoonup \boldsymbol{p} \quad \text{(weakly) in} \quad H \Rightarrow \boldsymbol{p} \in E\,.$$

In fact, for any $v \in H^1(\Omega)$ there exists a sequence $\{v_h\}$, $v_h \in V_h$, $v_h \to v$ in $H^1(\Omega)$, $h \to 0$. Then

$$(\boldsymbol{p}_h, \operatorname{grad} v_h) = 0$$

and passing to the limit with $h \to 0$, we obtain

$$(\boldsymbol{p}, \operatorname{grad} v) = 0\,.$$

It remains to prove that

$$\boldsymbol{p}_h \in P_h\,, \quad \boldsymbol{p}_h \rightharpoonup \boldsymbol{p} \quad \text{in} \quad H \Rightarrow \boldsymbol{p} \in P\,.$$

We employ Lemma 2.2, where the space S is replaced by H. Thus we have

$$\boldsymbol{p}_h \in P_h \Leftrightarrow \psi_h(\boldsymbol{p}_h) \in P\,, \quad \boldsymbol{p}_h \in S_h\,,$$

by virtue of the definition of P_h. From Lemma 2.2,

$$\psi_h(\boldsymbol{p}_h) \rightharpoonup \boldsymbol{p} \quad \text{in} \quad H\,, \quad h \to 0\,.$$

Since P is weakly closed, $\boldsymbol{p} \in P$ follows. Q.E.D.

Remark. The regularity assumption $\boldsymbol{p} \in [H^1(\Omega)]^2$ is satisfied if Ω is convex – see Brezis and Stampacchia [12].

Finally, let us consider internal approximations of the set E, i.e. let us approximate the set $\mathscr{K} = E \cap P$ by the set

$$\mathscr{K}_h = E_h^0 \cap P_h\,,$$

where

$$E_h^0 = E \cap S_h\,.$$

It is not difficult to find that (cf. [11])

$$E_h^0 = \{\boldsymbol{p} \in S_h \mid \operatorname{div} \boldsymbol{p} = 0 \text{ for all } K \in \mathscr{T}_h \text{ and } \boldsymbol{p} \,.\, \boldsymbol{\nu} = 0 \text{ for the sides on } \partial\Omega\}\,.$$

We define the approximate problem

$$\mathscr{S}(\boldsymbol{p}_h) = \min \quad \text{over} \quad E_h^0 \cap P_h \,. \tag{3.5}$$

Lemma 3.2. *The problem* (3.5) *has a unique solution.*

Proof. The set $E_h^0 \cap P_h$ contains the zero element, being also closed and convex in H.

Theorem 3.2. *Assume that* $\boldsymbol{p} \in [H^1(\Omega)]^2$. *Then*

$$\|\boldsymbol{p} - \boldsymbol{p}_h\| \to 0 \,, \quad h \to 0$$

holds for the solution $\boldsymbol{p}_h$ *of the problem* (3.5) *and for any regular family of triangulations.*

The proof is parallel to that of Theorem 3.1. Note that

$$\boldsymbol{p} \in E \cap [H^1(\Omega)]^2 \Rightarrow r_h\boldsymbol{p} \in E_h^0$$

follows from [11] (see Lemma 2 and the proof of Theorem 3 there).

References

[1] *G. Duvaut, J. L. Lions:* Les inéquations en mécanique et en physique. Paris, Dunod 1972.

[2] *B. Mercier:* Sur la théorie et l'analyse numérique de problémes de plasticité. Thesis. Université Paris VI, 1977.

[3] *P. Suquet:* Existence and regularity of solutions for plasticity problems. (Preprint). Proc. IUTAM Congress in Evanston — 1978.

[4] *R. Falk, B. Mercier:* Estimation d'erreur en élastoplasticité. C. R. Acad. Sc. Paris, 282, A, (1976), 645—648.

[5] *R. Falk, B. Mercier:* Error estimates for elasto-plastic problems. R.A.I.R.O. Anal. Numér., 11 (1977), 135—144.

[6] *V. B. Watwood, B. J. Hartz:* An equilibrium stress field model for finite element solution of two-dimensional elastostatic problems. Inter. J. Solids Structures 4, (1968), 857—873.

[7] *C. Johnson, B. Mercier:* Some equilibrium finite element methods for two-dimensional elasticity problems. Numer. Math. 30 (1978), 103—116.

[8] *J. Céa:* Optimisation, théorie et algorithmes. Dunod, Paris 1971.

[9] *F. Brezzi, W. W. Hager, P. A. Raviart:* Error estimates for the finite element solution of variational inequalities. Part I. Primal Theory. Numer. Math. 28, (1977), 431—443.

[10] *M. Křížek:* An equilibrium finite element method in three-dimensional elasticity. Apl. Mat. 27 (1982).

[11] *P. A. Raviart, J. M. Thomas:* A mixed finite element method for 2-nd order elliptic problems. Math. Aspects of Fin. El. Meth. Rome 1975, Springer-Verlag 1977, 292—315.

[12] *H. Brezis, K. Stampacchia:* Sur la regularité de la solution d'inéquations elliptiques. Bull. Soc. Math. France 96, (1968), 153—180.

Souhrn

ANALÝZA PRUŽNĚ PLASTICKÝCH TĚLES PODLE HENCKYOVA MODELU METODOU KONEČNÝCH PRVKŮ

IVAN HLAVÁČEK

Na základě variační formulace v napětích – tzv. principu Haara-Kármána – jsou definovány po částech lineární aproximace pole napětí a dokazuje se jejich konvergence. Vzhledem k podmínkám rovnováhy aproximace jsou jak externí tak interní, vzhledem k podmínce plasticity však jen externí.

Podobně je studován také problém kroucené válcové tyče.

Author's address: Ing. *Ivan Hlaváček*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1, Czechoslovakia.