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LAGUERRE POLYNOMIALS IN THE INVERSION OF MELLIN TRANSFORM

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INTRODUCTION

Let us call a complex-valued function $g(r)$ of the real variable r an inverse transform if:

i: $g(r)$ is defined in $(-\infty, \infty)$ and absolutely integrable in every finite interval,
ii: $g(r) = 0$ for all $r < 0$,
iii: $g(r)$ satisfies the conditions:

$$|g(r)| \leqq M_1 r^{-c_1} \quad \text{for} \quad r \leqq 1\,,$$
$$|g(r)| \leqq M_2 r^{-c_2} \quad \text{for} \quad r > 0\,,$$

where M_i, c_i $(i = 1, 2)$ are constants.

The Mellin transform of the function $g(r)$ is an analytic function of the complex variable s defined by

$$G(s) = \mathcal{M}\{g(r); s\} = \int_0^\infty r^{s-1}\, g(r)\, \mathrm{d}r\,, \quad s = \sigma + i\tau\,. \tag{1}$$

The integral (1) is absolutely convergent for all values of s such that $c_1 < \operatorname{Re} s = \sigma < c_2$.

The Mellin transform technique is a powerful method for the resolution of various boundary-value problems mainly in the plane elasticity (see, for example, Sneddon [1], Bogy [2] etc).

The main difficulty in applying Mellin transform techniques is the determination of the original function $g(r)$ from its transform $G(s)$.

This problem, formally solved by the inversion formula

$$g(r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-s}\, G(s)\, \mathrm{d}s\,, \quad c_1 < c < c_2\,, \tag{2}$$

cannot be solved, in most cases, analytically. Therefore the only possibility of inverting the transformed functions is by numerical means. Up to now only a few methods have been proposed for this numerical inversion.

Thus, an attempt was made by Tsamasphyros and Theocaris [3] to develop a method for the numerical inversion of the Mellin transform, which was based on the expansion of the original function in a series of Laguerre polynomials and determination of the coefficients of the expansion by means of a collocation on the real axis of the transformed plane. In the same paper the coefficients of the expansion are given in terms of forward differences of the transformed function. An error estimate of the numerical inversion procedure has also been given. But these algorithms have been found to be rather instanble numerically.

Evidently there is no escape from ill-conditioning, which, after all, only reflects the fact that the original inversion problem is not well posed. Nevertheless, in the present paper we present a more stable algorithm.

Another possibility for the numerical inversion of Mellin transform, is the numerical calculation of the complex inversion integral (2). But one could ascertain from tables given in [4] that the "nodes" in any low-order quadrature formula are mostly lying outside the strip of regularity $c_1 < \text{Re}s < c_2$ for a large class of transform functions (see [14]). The main disadvantage of this method is that it may become a time consuming procedure, if the inverse transform is required for a large number of values of the independent variable.

In the present paper we use the well known representation of the Mellin transform as a combination of two Laplace transforms:

$$\mathcal{M}\{g(r); s\} = \mathcal{L}\{g(e^{-t}); s\} + \mathcal{L}\{g(e^t); -s\} ; \quad t = |lnr| . \tag{3}$$

In this way we can take advantage of the methods of numerical inversion of the Laplace transform. In particular, we use the expansion of

$$\begin{aligned} f_1(t) &= g(e^t) , \\ f_2(t) &= g(e^t) , \end{aligned} \qquad t = |lnr| , \tag{4}$$

in a series of Laguerre polynomials, which is a method going back to Tricomi [5] and has been used by several authors [7], [8], [9], [10] for the numerical inversion of the Laplace transform.

The Mellin transform of these two series can be written as a Laurent series. Thus, the coefficients may be obtained by means of a simple collocation procedure. The resulting algorithm proved to be quite suitable for automatic computation.

STATEMENT OF THE PROBLEM

Obviously, using notation (4) we can write any original function $g(r)$ in the form

$$g(r) = f_1(t)\, H(1 - r) + f_2(t)\, H(r - 1) , \quad t = |lnr| , \tag{5}$$

where $H(x)$ is the Heaviside step function.

Assume now that $f_1(t)$ and $f_2(t)$ can be expanded in a series of Laguerre polynomials $L_j(t)$ and let these series be truncated after N and $(N - 1)$ terms, respectively. Thus we can write

$$f_1(t) = e^{(c-1/2T)t} \sum_{j=0}^{N} a_j L_j\left(\frac{t}{T}\right) + e_{1N}(t), \tag{6}$$

$$f_2(t) = e^{-(c+1/2T)t} \sum_{j=0}^{N-1} b_j L_j\left(\frac{t}{T}\right) + e_{2N}(t), \tag{7}$$

where the truncation error e_{iN} $(i = 1, 2)$ is of the form

$$e_{iN}(t) = K_1 e^{(c-1/2T)t} L_{N+1}(t/T) \tag{8}$$

$$e_{2N}(t) = K_2 e^{-(c+1/2T)t} L_N(t/T) \tag{9}$$

Substituting relations (6) to (9) in (5) by termwise transformation (using (3)) we obtain [14, p. 175]

$$\begin{aligned} G(s) = \sum_{j=0}^{N} \frac{a_j}{s - c + 1/2T}\left(\frac{s - c - 1/2T}{s - c + 1/2T}\right)^j - \sum_{j=0}^{N-1} \frac{bj}{s - c - 1/2T}\left(\frac{s - c + 1/2T}{s - c - 1/2T}\right)^j + \\ \frac{K_1}{s - c + 1/2T}\left(\frac{s - c - 1/2T}{s - c + 1/2T}\right)^{N+1} - \frac{K_2}{s - c - 1/2T}\left(\frac{s - c + 1/2T}{s - c - 1/2T}\right)^N, \end{aligned} \tag{10}$$

where

$$c - \frac{1}{2T} < \operatorname{Re} s < c + \frac{1}{2T}.$$

Now we apply the fractional linear transformation:

$$s = c + \frac{1}{2T}\frac{z + 1}{1 - z}. \tag{11}$$

The half plane $\operatorname{Re} s \geqq c$ is thereby mapped onto the unit disc $|z| \leqq 1$. The points $s = c, (c + 1/2T), \infty$ are mapped onto the points $z = -1, 0, 1$, respectively, the half plane $c \leqq s \leqq \infty$ onto the diameter $-1 \leqq z \leqq 1$ and the line $\operatorname{Re} s = c$ onto the circle $|z| = 1$.

Now, using the transformation (11) and writing

$$h(z) = \frac{1}{T}\frac{z}{1 - z}\, G\left(c + \frac{1}{2T}\frac{1 + z}{1 - z}\right) \tag{12}$$

we obtain

$$h(z) = \sum_{j=0}^{N} a_j z^j + \sum_{j=0}^{N-1} (-b_j)\, z^{-(j+1)} + E(z). \tag{13}$$

The function $h(z)$ is regular in the annulus $1 - \varepsilon < |z| < 1 + \varepsilon$, thus the right hand side of the relation (13) is the truncated Laurent series expansion of $h(z)$, where the truncation error is given by

$$E(z) = K_1 z^{(N+1)} - K_2 z^{-(N+1)} . \tag{14}$$

Substituting also

$$z = e^{i\vartheta} \tag{15}$$

into the relation (13) we obtain:

$$h_1(\vartheta) = a_0 + \sum_{j=1}^{N} (a_j - b_{j-1}) \cos j\vartheta + E_1(\vartheta) , \tag{16}$$

$$h_2(\vartheta) = \sum_{j=1}^{N} (a_j + b_{j-1}) \sin j\vartheta + E_2(\vartheta) , \tag{17}$$

where

$$h_1(\vartheta) = \mathrm{Re}\,\{h(e^{i\vartheta})\} , \quad h_2(\vartheta) = \mathrm{Im}\,\{h(e^{i\vartheta})\} .$$

The right hand sides of (15) and (17) are respectively the cosine and sine Fourier approximations of $h_1(\vartheta)$ and $h_2(\vartheta)$ and the error terms $E_1(\vartheta)$ and $E_2(\vartheta)$ are given by

$$E_1(\vartheta) = (K_1 - K_2) \cos (N + 1)\, \vartheta , \tag{18}$$

$$E_2(\vartheta) = (K_1 + K_2) \sin (N + 1)\, \vartheta . \tag{19}$$

EVALUATION OF THE COEFFICIENTS: CONTINUOUS RANGE

Taking into consideration the fact that the right hand side of (13) is the truncated Laurent series expansion of $h(z)$ we can write:

$$a_j = \frac{h^{(j)}(0)}{j!} = \frac{1}{2\pi i} \oint_{|z|=1} h(z)\, z^{-(j+1)}\, \mathrm{d}z , \quad j = 0, 1, 2, \ldots, N , \tag{20}$$

$$b_j = \frac{1}{2\pi i} \oint_{|z|=1} h(z)\, z^j\, \mathrm{d}z , \quad j = 0, 1, 2, \ldots, (N - 1) , \tag{21}$$

$$K_1 = \frac{h^{(N+1)}(\zeta_1)}{(N + 1)!} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{h(z)}{(z - \zeta_1)^{N+1}}\, \mathrm{d}z , \quad \zeta_1 = e^{i\vartheta_1} , \tag{22}$$

$$K_2 = -\frac{1}{2\pi i} \oint_{|z|=1} h(z)\, (z - \zeta_2)^N\, \mathrm{d}z , \quad \zeta_2 = e^{i\vartheta_2}\, 2 , \tag{23}$$

where ϑ_k $(k = 1, 2)$ are points in the interval $(0, \pi)$.

Now, using the relations (16) and (17) we obtain the Fourier coefficients:

$$a_j = \frac{1}{\pi}\int_0^\pi \{h_1(\vartheta)\cos j\vartheta + h_2(\vartheta)\sin j\vartheta\}\,d\vartheta\,, \quad j = 0, 1, 2, \ldots, N\,, \tag{24}$$

$$b_j = \frac{1}{\pi}\int_0^\pi \{h_2(\vartheta)\sin(j+1)\,\vartheta - h_1(\vartheta)\cos(j+1)\,\vartheta\}\,d\vartheta\,, \tag{25}$$

$$j = 0, 1, 2, \ldots, (N-1)\,.$$

These results can also be obtained by directly substituting (15) into the equations (20) and (21).

We note also that we can arrive at analogous results by truncating the series expansion of $f_1(t)$ after $(N-1)$ terms and the series expansion of $f_2(t)$ after N terms. In this case the function

$$\hat{h}(z) = \frac{1}{T}\,\frac{1}{1-z}\,G\left(c + \frac{1}{2T}\,\frac{1+z}{1-z}\right)$$

must be used instead of $h(z)$.

EVALUATION OF THE COEFFICIENTS: DISCRETE RANGE

Evidently, if the Mellin transform of $g(r)$ is known, the coefficients a_j and b_j of the series expansions in Eqs. (6) and (7) can be calculated by means of Eqs. (20) (21) or (24) and (25). But these equations are not suited to numerical calculations. Thus a discretization is necessary. If the equispaced sampling points

$$\vartheta_k = \frac{2k+1}{2(N+1)}\,\pi \quad (k = 0, 1, \ldots, N) \tag{26}$$

are chosen and the mid-point rule applied to the integrals in (24) and (25), the following approximation is obtained:

$$a_j \approx \alpha_j = \frac{1}{N+1}\sum_{k=0}^{N}\{h_1(\vartheta_k)\cos j\vartheta_k + h_2(\vartheta_k)\sin j\vartheta_k\}\,, \tag{27}$$

$$j = 0, 1, 2, \ldots, N\,,$$

$$b_j \approx \beta_j = \frac{1}{N+1}\sum_{k=0}^{N}\{h_2(\vartheta_k)\sin(j+1)\,\vartheta_k - h_1(\vartheta_k)\cos(j+1)\,\vartheta_k\}\,, \tag{28}$$

$$j = 0, 1, 2, \ldots, (N-1)\,.$$

If the trapezoidal rule rather than the mid-point rule is used at the equispaced points

$$\varphi_k = \frac{k\pi}{N} \quad (k = 0, 1, \ldots, N) \tag{29}$$

one obtains:

$$(30)\qquad a_j \approx \alpha'_j = \frac{1}{N}\sum_{k=0}^{N}\{h_1(\varphi_k)\cos j\varphi_k + h_2(\varphi_k)\sin j\varphi_k\} \quad j = 0, 1, 2, \ldots, N\,,$$

$$(31)\qquad b_j \approx \beta'_j = \frac{1}{N}\sum_{k=0}^{N}\{h_2(\varphi_k)\sin(j+1)\,\varphi_k - h_1(\varphi_k)\cos(j+1)\,\varphi_k\}$$

$$j = 0, 1, 2, \ldots, (N-1)\,.$$

Obviously, for the calculation of the oscillatory integrals appearing in (24) and (25) it is possible to use special quadrature formulae for such integrals ([13], [14]), but the formulae (27), (28) or (30), (31) give an exact answer because both results are the discrete least squares approximations of $h_1(\vartheta)$ and $h_2(\vartheta)$ by the series expansions (16) and (17).

For the evaluation of Eqs. (27), (28) or (30), (31) the fast Fourier transform can be applied, but often the number of terms needed in Eqs. (6) and (7) is not large, so that the use of the fast Fourier transform is unnecessary.

We note also that the relation (27) is the Chebyshev interpolation of the relation (16) and then the corresponding truncation error $E_1(\vartheta)$ is

$$(32)\qquad E_1(\vartheta) = \frac{1}{2^N(N+1)!}\cos(N+1)\,\vartheta D^N\, h_1(\eta_1)\,,$$

where h_1 is a point in the interval $(0, \pi)$ and D is the operator

$$(33)\qquad D = -\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\,.$$

COMPUTATIONAL PROCEDURE

In the inversion procedure the constant c and the scale factor T were introduced. The convergence in the mean of the series of Eqs. (6) and (7) was established for any positive value of T and for any value of c such that c lies within the strip of regularity. However, it has been found in practice that the series converge faster for some particular values of c and T.

Empirically it has been found that, for an approximation of $g(r)$ in the interval $r_{\min} \leqq r \leqq r_{\max}(r_{\min} > 0)$, a satisfactory choice of c is obtained by putting

$$(34)\qquad c = c_0 \pm \frac{1}{m}\,,$$

where:

(i) c_0 is the finite end of the strip of regularity,

(ii) the minus sign is selected if c_0 coincides with c_1, i.e. the right end of the strip of regularity, and

(iii)

(35) $$m = \operatorname{Max}\left(|\log r_{\min}|, |\log r_{\max}|\right).$$

If x_n is the largest zero of $L_n(x)$, then it satisfies the inequality (see [8], p. 121)

(36) $$x_n < 2n + 1 + \sqrt{((2n + 1)^2 + 1/4)} \simeq 4n + 2\,.$$

Therefore, $L_n(x)$ oscillates in the interval

(37) $$0 < x < 4n + 2$$

and approaches zero monotonically for $x < 4n + 2$. It can be seen that a linear combination of the form (6) or (7) yields a good approximation only in the interval

(38) $$0 < \frac{t}{T} < 4N + 2\,.$$

It has been empirically found that a satisfactory choice of the parameter T is obtained by putting

(39) $$T = \frac{m}{N}\,.$$

The next parameter to define is the number of terms N to be used in the series expansion of $f_1(t)$. In most of the problems examined by the authors N takes values between 20 and 50.

Concerning the truncation error $e_{iN}(t)$ $(i = 1, 2)$ we now observe:

(i) The Laguerre polynomial (L_{N+1} or L_N) oscillates in the interval $[r_{\min}, r_{\max}]$.

(ii) The coefficients K_1 and K_2 are bounded, if the derivatives of h_1 and h_2 with respect to ϑ up to the $(N + 1)$ th order are bounded.

(iii) Taking into consideration the relations (23) and (22) we affirm that e_{iN} decreases very quickly for sufficiently regular functions $h_1(\vartheta)$ and $h_2(\vartheta)$, and

(iv) In order that the error e_{iN} be bounded for $r \to 0$ it is also necessary that

$$c - \frac{1}{2T} > 0\,.$$

Finally, we note that the rate of decrase of the coefficients a_j and b_j should give an idea of how good the approximations expressed in the relations (6) and (7) are.

We present now the following six examples [9] to illustrate the method described above. For the calculation of coefficients a_j, b_j we used the formulae (27) and (28). The first five examples were chosen because the inverse functions present a jump discontinuity. In addition, the second, the third and the fourth functions have different inverse functions, according to whether a belongs to the one or the other strip of regularity. The fifth and the sixth functions present a finite strip of regularity and their extremities are singular points of the functions. In addition, the inverse of the fifth function has a pole at $x = 1$.

The numerical results for each of these examples appear in the corresponding table. In each table the first column gives the values of the variable r of the original function $g(r)$, the second column gives the exact values of $g(r)$. The next columns give the difference between these exact values and the approximate ones, the latter being calculated for $N = 10$, 20, 30 and 50.

Table 1

r	Exact $g(r)$	Error for			
		$N = 10$	$N = 20$	$N = 30$	$N = 50$
·10	·003909	·160 (−08)	−·189 (−12)	−·432 (−12)	·126 (−12)
·20	·019883	−·150 (−08)	−·124 (−12)	−·266 (−12)	−·174 (−11)
·30	·052327	−·136 (−08)	−·131 (−13)	−·154 (−12)	−·137 (−12)
·40	·104770	·306 (−08)	−·128 (−12)	−·545 (−12)	−·980 (−12)
·50	·180337	−·970 (−08)	·187 (−13)	−·365 (−12)	·901 (−12)
·60	·281897	−·301 (−08)	−·114 (−12)	·316 (−12)	·751 (−12)
·70	·412140	·527 (−09)	·355 (−13)	·197 (−12)	−·744 (−12)
·80	·573622	·309 (−08)	−·604 (−13)	−·416 (−12)	−·924 (−12)
·90	·768789	−·296 (−08)	0·	−·561 (−12)	·345 (−12)
1·00	0·000000	−·119 (−02)	·135 (−02)	·231 (−02)	·312 (−02)
2·00	0·000000	·134 (−03)	−·116 (−03)	·203 (−03)	−·245 (−03)
3·00	0·000000	−·139 (−03)	933 (−05)	− 143 (−03)	−·573 (−04)
4·00	0·000000	·749 (−04)	·317 (−06)	·845 (−04)	·158 (−03)
5·00	0·000000	·118 (−03)	·759 (−04)	−·514 (−04)	·839 (−04)
6·00	0·000000	·576 (−04)	−·618 (−04)	·103 (−03)	−·127 (−03)
7·00	0·000000	−·120 (−04)	−·680 (−04)	−·947 (−04)	·112 (−03)
8·00	0·000000	−·587 (−04)	·797 (−05)	−·481 (−04)	−·110 (−03)
9·00	0·000000	−·800 (−04)	·613 (−04)	·793 (−04)	·663 (−04)
10·00	0·000000	−·824 (−04)	·639 (−04)	·780 (−04)	·401 (−04)
SUM A		·17123305 (+00)	·17123305 (+00)	·17123305 (+00)	·17123305 (+00)
SUM B		·32525051 (−06)	·21048698 (−06)	·40917338 (−06)	·44795280 (−06)

At the two last lines of the tables we fournish for each value of N the quantities SUM A and SUM B, which are the sums of the squares of the coefficients a_j, b_j, respectively. The parameters SUM A and SUM B are of great importance for the numerical procedure, because, if they converge, they indicate that the series from the relations (6) and (7) also converge. We remark that in the first example for $N = 30$ the SUM B is greater than for $N = 20$ and thus the approximation is worse for $N = 30$ and $r < 1$.

The tests were performed on a CDC 6400 machine and we used a double precision arithmetic with about twenty-nine decimal digits for the major part of the computations.

The accuracy of the method described in this paper appears very satisfactory even if the Mellin transform has singularities at the end of the strip of regularity (as is the case of all the following examples).

Table 2

r	Exact $g(r)$	Error for			
		$N = 10$	$N = 20$	$N = 30$	$N = 50$
·10	·003162	·130 (−02)	−·266 (−10)	−·378 (−10)	−·568 (−10)
·20	·017889	−·295 (−03)	−·331 (−11)	−·445 (−11)	−·845 (−11)
·30	·049295	−·315 (−03)	−·670 (−12)	−·134 (−11)	−·226 (−11)
·40	·101193	·234 (−03)	−·472 (−12)	−·686 (−12)	−·106 (−11)
·50	·176777	·225 (−04)	−·142 (−12)	−·333 (−12)	−·415 (−12)
·60	·278855	−·917 (−04)	−·128 (−12)	−·123 (−12)	−·234 (−12)
·70	·409963	·139 (−05)	−·284 (−13)	−·728 (−13)	−·181 (−12)
·80	·572433	·495 (−04)	−·284 (−13)	−·568 (−13)	−·107 (−12)
·90	·768433	−·320 (−04)	·142 (−13)	−·249 (−13)	−·320 (−13)
1·00	0·000000	−·160 (−02)	−·286 (−03)	·195 (−03)	·602 (−03)
2·00	0·000000	·322 (−04)	·432 (−05)	·302 (−05)	−·835 (−05)
3·00	0·000000	−·119 (−04)	−·126 (−06)	−·772 (−06)	−·708 (−06)
4·00	0·000000	·301 (−05)	−·209 (−08)	·222 (−06)	·952 (−06)
5·00	0·000000	·278 (−05)	−·287 (−06)	−·776 (−07)	·290 (−06)
6·00	0·000000	·883 (−06)	·148 (−06)	·988 (−07)	−·278 (−06)
7·00	0·000000	−·105 (−06)	·111 (−06)	−·616 (−07)	·167 (−07)
8·00	0·000000	−·417 (−06)	−·930 (−08)	−·224 (−07)	−·117 (−06)
9·00	0·000000	−·430 (−06)	−·533 (−07)	·275 (−07)	·527 (−07)
10·00	0·000000	−·342 (−06)	−·427 (−07)	·208 (−07)	·244 (−07)
SUM A		·92008261 (−01)	·92008264 (−01)	·92008264 (−01)	·92008264 (−01)
SUM B		·55953065 (−06)	·93981168 (−08)	·29062758 (−08)	·16663161 (−07)

Example 1

$$G(s) = \log \frac{s+\alpha}{s+\beta}, \quad \operatorname{Re} s < -\alpha, -\beta, \quad g(r) = \begin{cases} \dfrac{r^{\alpha} - r^{\beta}}{\log r}, & 0 < r < 1 \\ 0 \quad , & 1 < r < \infty \end{cases}$$

$$\alpha = 3.00, \quad \beta = 2.00$$

Example 2

$$G(s) = \frac{1}{s+\alpha}, \quad \operatorname{Re} s > -\alpha, \quad g(r) = \begin{cases} r^{\alpha}, & 0 < r < 1 \\ 0, & 1 < r < \infty \end{cases}$$

$$\alpha = 2.50$$

Example 3

$$G(s) = \frac{1}{s+\alpha}, \quad \operatorname{Re} s < -\alpha, \quad g(r) = \begin{cases} 0, & 0 < r < 1 \\ r^{\alpha}, & 1 < r < \infty \end{cases}$$

$$\alpha = 2.50$$

Table 3

r	Exact $g(r)$	Error for			
		$N = 10$	$N = 20$	$N = 30$	$N = 50$
·10	0·000000	−·345 (−13)	−·137 (−11)	−·152 (−11)	·631 (−13)
·20	0·000000	·592 (−12)	−·173 (−10)	·319 (−11)	−·446 (−10)
·30	0·000000	−·106 (−11)	·154 (−09)	·326 (−10)	−·248 (−10)
·40	0·000000	−·967 (−11)	−·439 (−09)	−·633 (−09)	−·514 (−08)
·50	0·000000	·169 (−10)	·109 (−08)	−·155 (−08)	·221 (−08)
·60	0·000000	·556 (−10)	−·256 (−08)	·388 (−08)	·385 (−08)
·70	0·000000	−·745 (−10)	·536 (−08)	·510 (−08)	−·627 (−08)
·80	0·000000	−·201 (−09)	−·911 (−08)	−·136 (−07)	−·132 (−07)
·90	0·000000	·453 (−09)	·142 (−08)	−·340 (−07)	·138 (−07)
1·00	− 1·000000	·213 (−01)	·185 (−01)	·174 (−01)	·164 (−01)
2·00	− 5·656854	−·545 (−01)	−·357 (−01)	·344 (−01)	−·292 (−01)
3·00	− 15·588457	·351 (+00)	·179 (−01)	−·151 (+00)	−·423 (−01)
4·00	− 32·000000	−·687 (+00)	·222 (−02)	·325 (+00)	·426 (+00)
5·00	− 55·901699	−·295 (+01)	·145 (+01)	−·541 (+00)	·618 (+00)
6·00	− 88·181631	−·328 (+01)	−·268 (+01)	·247 (+01)	−·213 (+01)
7·00	−129·641814	·136 (+01)	−·590 (+01)	−·452 (+01)	·376 (+01)
8·00	−181·019336	·122 (+02)	·126 (+01)	−·419 (+01)	−·669 (+01)
9·00	−243·000000	·283 (+02)	·165 (+02)	·117 (+02)	·687 (+01)
10·00	− 316·227766	·468 (+02)	·276 (+02)	·186 (+02)	·667 (+01)
SUM A		·13150941 (−17)	·54138785 (−14)	·20084387 (−13)	·43492747 (−13)
SUM B		·31944946 (+00)	·31938405 (+00)	·31936798 (+00)	·31935728 (+00)

Example 4

$$G(s) = \frac{1}{(s+\alpha)^2 + \beta^2}, \quad \operatorname{Re}(s+\alpha) > 0, \quad g(r) = \begin{cases} -\dfrac{r^\alpha \sin(\beta \log(r))}{\beta}, & 0 < r < 1 \\ 0, & 1 < r < \infty \end{cases}$$

$$\alpha = 2.50, \quad \beta = 1.50$$

Example 5

$$G(s) = \frac{\pi}{s} \tan\left(\frac{\pi s}{2}\right), \; -1 < \operatorname{Re} s < 1, \quad g(r) = \log\left|\frac{1+r}{1-r}\right|$$

Example 6

$$G(s) = \frac{\pi \sin(\pi/n)}{na \sin(\pi s/n\alpha)\sin(\pi(s+2)/n\alpha)}, 0 < \operatorname{Re} s < (n-1)\alpha, \; g(r) = \frac{1-r^\alpha}{1-r^{n\alpha}}$$

$$\alpha = 2.00, \quad n = 4.00$$

Table 4

r	Exact $g(r)$	Error for $N=10$	$N=20$	$N=30$	$N=50$
·10	−·000648	−·265 (−02)	−·452 (−11)	−·639 (−11)	−·971 (−11)
·20	·007930	·513 (−03)	−·563 (−12)	−·773 (−12)	−·145 (−11)
·30	·031959	·242 (−03)	−·117 (−12)	−·229 (−12)	−·387 (−12)
·40	·066166	−·232 (−03)	−·790 (−13)	−·114 (−12)	−·178 (−12)
·50	·101619	−·112 (−07)	−·218 (−13)	−·542 (−13)	−·675 (−13)
·60	·128911	·829 (−04)	−·178 (−13)	−·169 (−13)	−·355 (−13)
·70	·139347	−·934 (−05)	−·888 (−15)	−·888 (−14)	−·275 (−13)
·80	·125363	−·408 (−04)	−·355 (−14)	−·799 (−14)	−·169 (−13)
·90	·080626	·289 (−04)	−·222 (−14)	−·844 (−14)	−·102 (−13)
1·00	0·000000	−·942 (−04)	·118 (−03)	·206 (−03)	·280 (−03)
2·00	0·000000	·154 (−05)	−·179 (−05)	·319 (−05)	−·388 (−05)
3·00	0·000000	−·667 (−06)	·523 (−07)	−·816 (−06)	−·329 (−06)
4·00	0·000000	·256 (−06)	·866 (−09)	·235 (−06)	·442 (−06)
5·00	0·000000	·178 (−06)	·119 (−06)	−·819 (−07)	·135 (−06)
6·00	0·000000	·403 (−07)	−·512 (−07)	·104 (−06)	−·129 (−06)
7·00	0·000000	−·211 (−07)	−·459 (−07)	−·651 (−07)	·777 (−07)
8·00	0·000000	−·358 (−07)	·385 (−08)	−·237 (−07)	−·543 (−07)
9·00	0·000000	−·322 (−07)	·220 (−07)	·291 (−07)	·244 (−07)
10·00	0·000000	−·237 (−07)	·177 (−07)	·220 (−07)	·114 (−07)
SUM A		·14475092 (−02)	·14475103 (−02)	·14475103 (−02)	·14475103 (−02)
SUM B		·30611176 (−08)	·16076245 (−08)	·32433056 (−08)	·35989359 (−08)

Table 5

r	Exact $g(r)$	Error for $N=10$	$N=20$	$N=30$	$N=50$
·10	·200671	—·761 (—02)	·407 (—02)	·297 (—02)	·177 (—02)
·20	·405465	·126 (—01)	·775 (—02)	—·541 (—02)	—·151 (—02)
·30	·619039	—·306 (—01)	·563 (—03)	—·776 (—02)	·463 (—02)
·40	·847298	·117 (—01)	—·723 (—02)	—·624 (—02)	—·563 (—02)
·50	1·098612	·476 (—01)	·109 (—01)	·121 (—01)	—·773 (—02)
·60	1·386294	—·252 (—01)	—·129 (—01)	·107 (—01)	·766 (—02)
·70	1·734601	—·805 (—01)	·162 (—01)	—·371 (—01)	·428 (—01)
·80	2·197225	·643 (—01)	—·331 (—01)	—·119 (—01)	·102 (+00)
·90	2·944439	·108 (+01)	·121 (+00)	·208 (+00)	·166 (+00)
2·00	1·098612	·425 (—01)	·205 (—01)	·180 (—01)	—·158 (—01)
3·00	·693147	—·107 (—01)	·221 (—01)	·409 (—02)	—·108 (—01)
4·00	·510826	—·541 (—01)	—·197 (—01)	·135 (—01)	·644 (—02)
5·00	·405465	—·143 (—01)	·120 (—01)	—·127 (—01)	—·337 (—03)
6·00	·336472	·275 (—01)	·121 (—01)	·107 (—01)	—·848 (—02)
7·00	·287682	·480 (—01)	—·970 (—02)	·568 (—03)	·931 (—02)
8·00	·251314	·495 (—01)	—·167 (—01)	—·117 (—01)	—·771 (—02)
9·00	·223144	·395 (—01)	—·773 (—02)	—·932 (—03)	·139 (—02)
10·00	·200671	·240 (—01)	·539 (—02)	·105 (—01)	·800 (—02)
SUM A		·51082877 (+01)	·50806763 (+01)	·50707592 (+01)	·50760485 (+01)
SUM B		·36384673 (+01)	·36159899 (+01)	·36067379 (+01)	·36120888 (+01)

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Table 6

r	Exact $g(r)$	Error for $N=10$	$N=20$	$N=30$	$N=50$
·10	·990000	−·518 (−02)	−·369 (−02)	−·199 (−02)	·846 (−03)
·20	·960002	·415 (−02)	−·172 (−02)	−·307 (−04)	−·271 (−02)
·30	·910060	·180 (−03)	·425 (−02)	·306 (−03)	·237 (−03)
·40	·840551	−·386 (−02)	−·314 (−02)	−·243 (−02)	−·130 (−02)
·50	·752941	·156 (−02)	·327 (−02)	−·177 (−02)	·250 (−02)
·60	·650933	·378 (−02)	−·287 (−02)	·234 (−02)	·211 (−02)
·70	·541199	−·151 (−02)	·340 (−02)	·170 (−02)	−·106 (−02)
·80	·432574	−·343 (−02)	−·276 (−02)	−·221 (−02)	−·142 (−02)
·90	·333607	·429 (−02)	·681 (−03)	−·326 (−02)	·135 (−02)
1·00	·250000	·107 (+00)	·125 (+00)	·132 (+00)	·137 (−00)
2·00	·011765	−·115 (−01)	−·108 (−01)	·115 (−01)	−·104 (−01)
3·00	·001220	·124 (−01)	·615 (−03)	−·814 (−02)	−·188 (−02)
4·00	·000229	−·696 (−02)	·696 (−03)	·472 (−02)	·728 (−02)
5·00	·000061	−·104 (−01)	·697 (−02)	−·231 (−02)	·424 (−02)
6·00	·000021	−·469 (−02)	−·579 (−02)	·594 (−02)	−·527 (−02)
7·00	·000008	·160 (−02)	−·586 (−02)	−·534 (−02)	·512 (−02)
8·00	·000004	·570 (−02)	·135 (−02)	−·217 (−02)	−·455 (−02)
9·00	·000002	·747 (−02)	·608 (−02)	·498 (−02)	·342 (−02)
10·00	·000001	·753 (−02)	·602 (−02)	·451 (−02)	·178 (−02)
SUM A		·68164016 (+00)	·68140071 (+00)	·68130111 (+00)	·68121388 (+00)
SUM B		·12816517 (−01)	·12068143 (−01)	·11580189 (−01)	·11078355 (−01)

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Souhrn

LAGUEROVY POLYNOMY A INVERZE MELLINOVY TRANSFORMACE

PERICLES S. THEOCARIS, G. J. TSAMASPHYROS

Aby bylo možno použít známou reprezentaci Mellinovy transformace jako kombinace dvou Laplaceových transformací, je inverzní funkce $g(r)$ vyjádřena ve tvaru rozvoje Laguerrových polynomů v proměnné $t = \ln r$. Mellinovu transformaci řady lze napsat jako Laurentovu řadu. Tak je možno odhadnout koeficienty rozvoje i chybu numerické inverze. Diskrétní aproximace metodou nejmenších čtverců poskytuje jinou možnost určení koeficientů rozvoje v řadu. Tato poslední metoda je ukázána na numerických příkladech.

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