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# ON DETERMINATION OF EIGENVALUES AND EIGENVECTORS OF SELF-ADJOINT OPERATORS

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Two simple methods (1), (2) for approximate determination of eigenvalues and eigenvectors of linear bounded operators $A$ are considered in the following two cases: (i) the lower-upper bound $\lambda_1$ of the spectrum $\sigma(A)$ of $A$ is an isolated point of $\sigma(A)$; (ii) $\lambda_1$ (not necessarily an isolated point of $\sigma(A)$; with finite multiplicity) is an eigenvalue of $A$.

1. Let $X$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, $A : X \to X$ a linear self-adjoint operator on $X$. Since $A$ is self-adjoint and defined on $X$, $A$ is bounded by the closed graph theorem. Denote by $m$, $\lambda_1$ the exact spectral bounds of $A$, i.e. $m = \inf\{\langle Au, u \rangle : \|u\| = 1\}$, $\lambda_1 = \sup\{\langle Au, u \rangle : \|u\| = 1\}$. We shall investigate the Kellogg iteration method for calculation of eigenvalues and eigenvectors of $A$:

$$(1) \qquad u_{n+1} = \alpha_{n+1}^{-1} A u_n\,, \quad \alpha_{n+1} = \|Au_n\|\,, \quad (n = 0, 1, 2, \ldots)\,,$$

where the starting approximation $u_0 \in X$ is such that $u_0 \notin \ker A$, $\|u_0\| = 1$. By our assumption $\alpha_n > 0$ and $u_n \neq 0$ for each $n$.

Now we briefly describe the second method. Here, in addition, we assume that $A$ is positive on $X$, i.e. $\langle Au, u \rangle > 0$ for each $u \in X$, $u \neq 0$, and $\langle Au, u \rangle = 0$ implies $u = 0$. Then the spectrum $\sigma(A)$ of $A$ lies on the segment $[m, \lambda_1]$, where $m \geqq 0$. Let $R$ denote the set of all reals, $v_0 \in X$ an arbitrary (but fixed) non-zero element of $X$. Define a functional $f : R \times X \to X$ by $f(\mu, v) = \|Av - \mu v\|^2$, $\mu \in R$, $v \in X$ and let $\mu_1$ denote that value $\mu$ at which the function $\mu \to f(\mu, v_0)$ assumes its minimal value on $R$. The condition $f'_\mu(\mu_1, v_0) = 0$ implies that $\mu_1 = \langle Av_0, v_0 \rangle \,.\, \|v_0\|^{-2}$. Set $v_1 = \mu_1^{-1} A v_0$. Since $A$ is positive and self-adjoint and $v_0 \neq 0$, we obtain that $\mu_1 > 0$ and $v_1 \neq 0$. In general we get the following procedure for the construction of eigenvalues of $A$:

$$(2) \qquad v_{n+1} = \mu_{n+1}^{-1} A v_n\,, \quad \mu_{n+1} = \langle Av_n, v_n \rangle \,.\, \|v_n\|^{-2},$$

where $\mu_n > 0$ and $v_n \neq 0$ for each $n$. The method (2) is similar to that of Birger [2]:

$$(3) \qquad y_{n+1} = q_{n+1} A y_n\,, \quad q_{n+1} = \langle Ay_n, y_n \rangle \|Ay_n\|^{-2}\,,$$

who suggested it together with (1) without any mathematical justification. Nonetheless he found that in engineering problems his methods have some advantages in comparison with the older ones. The methods (2), (3) have been investigated by I. Marek [9], [10], W. V. Petryshyn [12] and the author [4–7], while the method (3) was studied by H. Bückner [3] for linear and nonlinear completely continuous operators having a certain decomposition property.

2. Recall that an operator $A : X \to X$ is said be nonnegative if $\langle Au, u\rangle \geqq 0$ for each $u \in X$. Let $\{E_\lambda\}$ denote the spectral family of a self-adjoint operator $A$. Let us remark that each isolated point of $\sigma(A)$ of a self-adjoint operator $A$ is an eigenvalue of $A$. In the sequel we assume that $A \neq 0$ and the starting approximations $u_0, v_0$ of (1), (2) satisfy the initial conditions: $\|u_0\| = 1$, $u_0 \notin \ker A$, $v_0 \neq 0$, respectively.

**Theorem 1.** *Let $X$ be a real Hilbert space, $A : X \to X$ a linear nonnegative self-adjoint operator. If the starting approximation $u_0 \in X$ of (1) is such that $E_\lambda u_0 \neq u_0$ for each $\lambda < \lambda_1$, then $\alpha_n \nearrow \lambda_1$, where $\alpha_n$ is defined by (1).*

Proof. First we prove that $(\alpha_n)$ is an increasing monotone sequence. Since $u_0 \notin \ker A$, $\|u_0\| = 1$, then $\|u_n\| = 1$ for each $n$ and $\alpha_n^2 = \alpha_n^2\|u_n\|^2 = \alpha_n\langle Au_{n-1}, u_n\rangle = \alpha_n\langle u_{n-1}, Au_n\rangle = \alpha_n\,\alpha_{n+1}\langle u_{n-1}, u_{n+1}\rangle = \alpha_n\alpha_{n+1}$. Hence $\alpha_n \leqq \alpha_{n+1}$ for each $n$. Since $(\alpha_n)$ is bounded, there exists $\lim \alpha_n = \alpha$ and $0 < \alpha \leqq \lambda_1$. We have to prove that $\alpha = \lambda_1$. Suppose $\alpha < \lambda_1$ and put $a = \frac{1}{2}(\alpha + \lambda_1)$. Then $0 < \alpha_n \leqq \alpha < a + \lambda_1$ for all $n$. Set $\beta = [a, \lambda_1]$, $b = a \,.\, \alpha^{-1}$. Then

$$\|E_\beta u_{n+1}\|^2 = \langle E_\beta u_{n+1}, u_{n+1}\rangle = \|u_{n+1}\|^2 - \langle E_a u_{n+1}, u_{n+1}\rangle =$$

$$= \int_m^{\lambda_1} \mathrm{d}\langle E_\lambda u_{n+1}, u_{n+1}\rangle - \int_m^{a} \mathrm{d}\langle E_\lambda u_{n+1}, u_{n+1}\rangle = \int_a^{\lambda_1} \mathrm{d}\langle E_\lambda u_{n+1}, u_{n+1}\rangle$$

for all $n$ ($n = 0, 1, 2, \ldots$). Using (1) and the properties of $\{E_\lambda\}$, we obtain

$$\int_a^{\lambda_1} \mathrm{d}\langle E_\lambda u_{n+1}, u_{n+1}\rangle = \alpha_{n+1}^{-2} \int_a^{\lambda_1} \mathrm{d}\langle E_\lambda A^2 u_n, u_n\rangle =$$

$$= \alpha_{n+1}^{-2} \int_a^{\lambda_1} \mathrm{d}\langle A^2 E_\lambda u_n, u_n\rangle = \alpha_{n+1}^{-2} \int_a^{\lambda_1} \lambda^2 \,\mathrm{d}\langle E_\lambda u_n, u_n\rangle \geqq$$

$$\geqq b^2 \,.\, \int_a^{\lambda_1} \mathrm{d}\langle E_\lambda u_n, u_n\rangle = b^2 \|E_\beta u_n\|^2 \,.$$

Hence

$$\|E_\beta u_{n+1}\| \geqq b\|E_\beta u_n\|$$

for each $n$. Continuing this process, we get

$$\|E_\beta u_n\| \geqq b^n \|E_\beta u_0\|$$

for all $n \geqq 1$. Since $E(\beta)\,u_0 = (E_{\lambda_1} - E_a)\,u_0 = u_0 - E_a u_0 \neq 0$ by our hypothesis and $b < 1$, we obtain that $\|E_\beta u_n\| \to +\infty$ as $n \to \infty$, which contradicts the fact that

$$\|E_\beta u_n\| \leqq \|u_n\| = 1\,.$$

Hence $\alpha_n \nearrow \lambda_1$ and the theorem is proved.

**Lemma 1** (*Compare* [12]). *Let $X$ be a real Hilbert space, $A : X \to X$ a linear self-adjoint operator, $\lambda_1$ and eigenvalue of $A$. Assume that the starting approximation $u_0$ of* (1) *is not orthogonal to* $\ker(A - \lambda_1 I)$.

*Then the sequence $(u_n)$ defined by* (1) *is of the form $u_n = a_n e_0 + z_n$, where $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$, $z_n \in \ker(A - \lambda_1 I)^\perp$ and $a_n > 0$ for each $n$.*

*Assume, in addition, that $z_n \neq 0$ for each $n$ in the last representation of $u_n$. Rewrite $u_0 = a_0 e_0 + z_0$ as $u_0 = a_0 e_0 + b_0 r_0$, where $b_0 = \|z_0\|$, $r_0 = z_0 \|z_0\|^{-1}$.*

*Then each $u_n$ is of the form $u_n = a_n e_0 + b_n r_n$. where $a_n = \lambda_1 \alpha_n^{-1} a_{n-1}$, $b_n = \alpha_n^{-1} b_{n-1} \|A r_{n-1}\|$, $r_n \notin \ker A$ and $r_n = A r_{n-1} \|A r_{n-1}\|^{-1}$ for each $n$.*

*Similar assertions are also valid for $(v_n)$, where $v_n$ is defined by* (2).

Proof. Since $u_0 \notin \ker(A - \lambda_1 I)^\perp$, $u_0$ is of the form $u_0 = a_0 e_0 + z_0$, where $a_0 > 0$, $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = l$, $z_0 \in \ker(A - \lambda_1 I)^\perp$. Assume that the representation of $(u_n)$ is valid for $n = i$, i.e. $u_i = a_i e_0 + z_i$, where $z_i \in \ker(A - \lambda_1 I)^\perp$, $a_i > 0$. Then $u_{i+1} = \alpha_{i+1}^{-1} A u_i = a_{i+1} e_0 + z_{i+1}$, where $a_{i+1} = \alpha_{i+1}^{-1} \lambda_1 a_i$, $z_{i+1} = \alpha_{i+1}^{-1} A z_i$.

Our assertion will be proved if we show that $\langle u, z_{i+1} \rangle = 0$ for each $u \in \ker . (A - \lambda_1 I)$. But this immediately follows from the assumption that $z_i \in \ker(A - \lambda_1 I)^\perp$ and the fact that $\ker(A - \lambda_1 I)^\perp$ is an invariant subspace with respect to $A$. The rest can be proved quite analogously.

**Theorem 2.** *Let $X$ be a real Hilbert space, $A : X \to X$ a linear nonnegative and self-adjoint operator on $X$. Assume that $\lambda_1$ (not necessarily an isolated point of $\sigma(A)$ with finite multiplicity) is an eigenvalue of $A$ and that the starting approximation $u_0$ of* (1) *is not orthogonal to* $\ker(A - \lambda_1 I)$.

*Then $\alpha_n \nearrow \lambda_1$. Moreover, if $\lambda_1$ is an isolated point of $\sigma(A)$, then $\lim_{n\to\infty} \|u_n - e_0\| = 0$, where $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$ and $(\alpha_n)$, $(u_n)$ are defined by* (1).

Proof. First of all we prove that $\alpha_n \nearrow \lambda_1$. Since $u_0 \notin \ker(A - \lambda_1 I)^\perp$ and this subspace is invariant with respect to $A$, then according to (1) $u_n \notin \ker(A - \lambda_1 I)^\perp$ for each $n$. By Lemma 1 each $u_n$ is of the form $u_n = a_n e_0 + z_n$, where $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$, $z_n \in \ker(A - \lambda_1 I)^\perp$ and $a_n > 0$. Using (1) and the fact that $\lambda_1$ is an eigenvalue of $A$ we get $a_n = \langle u_n, e_0 \rangle = \alpha_n^{-1} \langle A u_{n-1}, e_0 \rangle = \alpha_n^{-1} \lambda_1 \,.\, a_{n-1}$. Because $(\alpha_n)$ is a monotone increasing sequence (see the first part of the proof of Theorem 1) and $0 < \alpha_n \leqq \lambda_1$, we have that $a_n \geqq a_{n-1}$ for each $n$ $(n = 1, 2, \ldots)$. Moreover, $(a_n)$ is bounded. Passing to the limit in the above equality, we obtain that $\alpha_n \nearrow \lambda_1$.

To prove the second assertion, take $\lambda$ such that $m < \lambda < \lambda_1$. Then

$$\lambda_1^2 - \alpha_{n+1}^2 \geqq (\lambda_1^2 - \lambda^2) \|E_\lambda u_n\|^2 .$$

Indeed,

$$\lambda_1^2 - \alpha_{n+1}^2 = \lambda_1^2 - \|Au_n\|^2 = \langle(\lambda_1^2 - A^2) u_n, u_n\rangle =$$

$$= \int_m^{\lambda_1} (\lambda_1^2 - t^2) \, d\langle E_\lambda u_n, u_n\rangle \geqq \int_m^{\lambda} (\lambda_1^2 - t^2) \, d\langle E_t u_n, u_n\rangle \geqq$$

$$\geqq (\lambda_1^2 - \lambda^2) \|E_\lambda u_n\|^2 .$$

Since $\alpha_n \nearrow \lambda_1$, we conclude that $\|E_\lambda u_n\| \to 0$ and $\|(I - E_\lambda) u_n\| \to 1$ as $n \to \infty$ because $\|u_n\| = 1$ for each $n$. Put $P_0 = I - E_{\lambda_1 - 0}$, then $P_0$ is a projection of $X$ onto $\ker(A - \lambda_1 I)$. Since $u_n$ is of the form $u_n = a_n e_0 + z_n$, where $z_n \in \ker(A - \lambda_1 I)^\perp$, then $P_0 u_0 = a_n e_0$ and $a_n^2 = \|P_0 u_n\|^2$. Since $\lambda_1$ is an isolated point of $\sigma(A)$, there exists a constant $M > 0$ such that $\sigma(A) - \{\lambda_1\} \subset [m, M]$. Moreover, $\lambda_1$ is an eigenvalue of $A$ and $E_\lambda \to E_{\lambda_1 - 0}$ as $\lambda \to \lambda_1 - 0$ in the strong point operator topology of $(X \to X)$. By our hypothesis the segment $(M, \lambda_1)$ belongs to the resolvent set of $A$ and hence the family $\{E_\lambda\}$ is constant on $(M, \lambda_1)$. Taking $\lambda$ such that $M < \lambda < \lambda_1$, we obtain

$$|a_n - 1| = |\|P_0 u_n\| - 1| \leqq |\|P_0 u_0\| - \|(I - E_\lambda) u_n\|) +$$

$$+ |\|(I - E_\lambda) u_n\| - 1| = |\|(I - E_\lambda) u_n\| - 1| .$$

Hence $a_n \to 1$. The equality $\|u_n - e_0\|^2 = 2(1 - a_n)$ completes the proof.

One can similarly prove the following theorem which extends the corresponding result of [6].

**Theorem 3.** *Assume that $A$ is positive and that the starting approximation $v_0$ of (2) is not orthogonal to $\ker(A - \lambda_1 I)$. Under the same conditions of Theorem 2 on $X$, $A$ and $\lambda_1$ we have that $\mu_n \nearrow \lambda_1$ and $\|v_n - Ne_0\| \to 0$, where $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$ and $N = \sup_n \|v_n\| < \infty$.*

Remark 1. The methods (1), (2) can be used for an approximate determination of eigenvalues and eigenvectors of linear bounded operators. Indeed, if $T$ is an arbitrary linear bounded operator, then $A = T^*T$ is self-adjoint and nonnegative.

**Theorem 4.** *Let $X$ be a real Hilbert space, $A : X \to X$ a linear nonnegative self-adjoint operator on $X$. Suppose that $\lambda_1$ is an isolated point of $\sigma(A)$ of $A$ (i.e. there exists a constant $M$ such that $\sigma(A) - \{\lambda_1\} \subset [m, M]$).*

*Then*

(i) $$\alpha_{n+1}^2 \leqq \lambda_1^2 \leqq \alpha_{n+1}^2 + \left(\frac{\alpha_n}{M}\right)^2 (\lambda_1^2 - \alpha_n^2), \quad n \geqq 1 .$$

*If the starting approximateion $u_0$ of* (1) *is not orthogonal to* $\ker(A - \lambda_1 I)$, *then*

(ii)
$$\|\mu_n - \langle \mu_n, e_0\rangle e_0\|^2 \leqq \frac{\lambda_1^2 - \alpha_n^2}{\lambda_1^2 - M^2}.$$

*Moreover, if the process* $(u_n)$ *is not finite, then*

(iii)
$$\|u_n - e_0\|^2 \leqq 2\left(\frac{M}{\alpha_n}\right)^2 \cdot \frac{\lambda_1 - m}{\lambda_1} \|u_{n-1} - e_0\|,$$

*where* $(u_n)$, $(\alpha_n)$ *are defined by* (1), $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$.

Proof. We prove (i). According to (1) we have that

$$0 \leqq \lambda_1^2 - \alpha_{n+1}^2 = \lambda_1^2 - \|Au_n\|^2 = \langle(\lambda_1^2 - A^2)\, u_n, u_n\rangle =$$
$$= \langle \alpha_n^{-2}(\lambda_1^2 - A^2)\, A^2 u_{n-1}, u_{n-1}\rangle =$$
$$= \alpha_n^{-2} \int_m^{\lambda_1} (\lambda_1^2 - \lambda^2)\, \lambda^2\, \mathrm{d}\langle E_\lambda u_{n-1}, u_{n-1}\rangle \leqq$$
$$\leqq \left(\frac{M}{\alpha_n}\right)^2 \int_m^M (\lambda_1^2 - \lambda^2)\, \mathrm{d}\langle E_\lambda u_{n-1}, u_{n-1}\rangle = \left(\frac{M}{\alpha_n}\right)^2 (\lambda_1^2 - \alpha_n^2),$$

for the family $\{E_\lambda\}$ is constant on the interval $(M, \lambda_1)$.

(ii) Since $\lambda_1$ is an isolated point of $\sigma(A)$ of $A$, $\lambda_1$ is an eigenvalue of $A$. Because $u_0 \notin \ker(A - \lambda_1 I)^\perp$, then according to Lemma 1 each $u_n$ is of the form $u_n = a_n e_0 + z_n$ where $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$, $z_n \in \ker(A - \lambda_1 I)^\perp$, $a_n > 0$, $n \geqq 1$. Then $\|u_n\|^2 = a_n^2 + \|z_n\|^2 = 1$ and

$$\|Au_n\|^2 = \langle A^2 u_n, u_n\rangle = \lambda_1^2 a_n^2 + \langle A^2 z_n, z_n\rangle,$$

because $\ker(A - \lambda_1 I)$, $\ker(A - \lambda_1 I)^\perp$ are invariant subspaces with respect to $A$. Then

$$\lambda_1^2 - \alpha_{n+1}^2 = \lambda_1^2 - \|Au_n\|^2 = \lambda_1^2(a_n^2 + \|z_n\|^2) - \lambda_1^2 a_n^2 - \langle A^2 z_n, z_n\rangle =$$
$$= \lambda_1^2 \|z_n\|^2 - \langle A^2 z_n, z_n\rangle = \langle(\lambda_1^2 - A^2)\, z_n, z_n\rangle.$$

Since the segment $J = (M, \lambda_1)$ belongs to the resolvent set of $A$, the family $\{E_\lambda\}$ is constant on J. Hence

$$\langle(\lambda_1^2 - A^2)\, z_n, z_n\rangle = \int_m^{\lambda_1} (\lambda_1^2 - \lambda^2)\, \mathrm{d}\langle E_\lambda z_n, z_n\rangle =$$
$$= \int_m^M (\lambda_1^2 - \lambda^2)\, \mathrm{d}\langle E_\lambda z_n, z_n\rangle \geqq (\lambda_1^2 - M^2)\, \|z_n\|^2.$$

Since $\|u_n - a_n e_0\| = \|u_n - \langle u_n, e_0\rangle e_0\| = \|z_n\|$, we obtain the desired estimation at once from the last inequality.

(iii) By the second part of Lemma 1 each $(u_n)$ is of the form $u_n = a_n e_0 + b_n r_n$ where $a_n = \alpha_n^{-1} a_{n-1} \lambda_1$, $b_n = \alpha_n^{-1} b_{n-1} \|Ar_{n-1}\|$, $b_0 = \|z_0\|$, $r_n = Ar_{n-1} \|Ar_{n-1}\|^{-1}$, $r_0 = z_0 \, . \, \|z_0\|^{-1}$ and $r_n \in \ker (A - \lambda_1 I)^{\perp}$, $\|r_n\| = 1$ for each $n$. Since $\|u_n\|^2 = a_n^2 + b_n^2 = 1$ and the segment $(M, \lambda_1)$ belongs to the resovent set of $A$, we obtain that

$$b_n^2 = \frac{b_{n-1}^2}{\alpha_n^2} \|Ar_{n-1}\|^2 = \frac{b_{n-1}^2}{\alpha_n^2} \int_m^{\lambda_1} \lambda^2 \, \mathrm{d}\langle E_\lambda r_{n-1}, r_{n-1}\rangle = \frac{b_{n-1}^2}{\alpha_n^2} \int_m^{M} \lambda^2 \, \mathrm{d}\langle E_\lambda r_{n-1}, r_{n-1}\rangle \leqq$$

$$\leqq b_{n-1}^2 \left(\frac{M}{\alpha_n}\right)^2 \int_m^M \mathrm{d}\langle E_\lambda r_{n-1}, r_{n-1}\rangle = b_{n-1}^2 \left(\frac{M}{\alpha_n}\right)^2 .$$

Hence

$$b_n^2 = 1 - a_n^2 \leqq \left(\frac{M}{\alpha_n}\right)^2 (1 - a_{n-1}^2) .$$

We show that the sequence $(a_n)$, where $a_n = \langle u_n, e_0\rangle$, is monotone increasing. Since $(\alpha_n)$ is monotone increasing, $0 < \alpha_n \leqq \lambda_1$, and $a_n = \langle u_n, e_0\rangle = \alpha_n^{-1} \, . \, \langle Au_{n-1}, e_0\rangle = \alpha_n^{-1} \lambda_1 \langle u_{n-1}, e_0\rangle = \alpha_n^{-1} \lambda_1 a_{n-1}$, we have that $a_{n-1} \leqq a_n$ for each $n \geqq 1$. Therefore $0 < 1 + a_{n-1} \leqq 1 + a_n$ and $1 - a_n \leqq M^2 \alpha_n^{-2} (1 - a_{n-1})$.

Furthermore,

$$\|u_n - e_0\|^2 = 2 - 2\langle u_n, e_0\rangle = 2(1 - a_n) \leqq 2 \left(\frac{M}{\alpha_n}\right)^2 (1 - a_{n-1}) .$$

On the other hand, $e_0 \in \ker (A - \lambda_1 I)$, $\|e_0\| = 1$ and the spectral theorem imply that

$$1 - a_{n-1} = \langle u_{n-1} - \lambda_1^{-1} A e_0, u_{n-1}\rangle =$$

$$= \lambda_1^{-1} \int_m^{\lambda_1} (\lambda_1 - \lambda) \, \mathrm{d}\langle E_\lambda (u_{n-1} - e_0), u_{n-1}\rangle \leqq$$

$$\leqq \frac{\lambda_1 - m}{\lambda_1} \int_m^{\lambda_1} \mathrm{d}\|E_\lambda (u_{n-1} - e_0)\| = \frac{\lambda_1 - m}{\lambda_1} \|u_{n-1} - e_0\| .$$

Hence the estimation (iii) at once follows from the last relation and the above inequality. The proof is complete.

**Theorem 5.** *Suppose that the conditions of Theorem* 4 *on $X$, $A$ and $\lambda_1$ are satisfied. If $A$ is positive, then*

(i)
$$0 \leqq \lambda_1 - \mu_{n+1} \leqq \left(\frac{M}{\mu_n}\right)^2 (\lambda_1 - \mu_n) .$$

*If $v_0 \notin \ker (A - \lambda_1 I)^{\perp}$, where $v_0$ is a starting approximation of* (2), *then*

(ii)
$$\|v_n - \langle v_n, e_0\rangle \, e_0\|^2 \leqq \frac{\lambda_1 - \mu_n}{\lambda_1 - M} \|v_n\|^2 .$$

*In case* $(v_n)$ *is not finite, then*

$$\text{(iii)} \qquad \|v_n - e_0\|v_n\| \| \leqq 2\left(\frac{M}{\mu_n}\right)^2 \|v_n\| \left(\|v_{n-1}\| - \langle v_{n-1}, e_0\rangle\right),$$

*where* $(\mu_n)$, $(v_n)$ *are defined by* (2) *and* $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$.

Proof. We sketch only the proof of (iii). Each $v_n$ of $(v_n)$ can be expressed in the form $v_n = p_n e_0 + q_n f_n$, where $p_n = \langle v_n, e_0\rangle > 0$, $q_n$ are constants, $e_0 \in \ker(A - \lambda_1 I)$, $\|e_0\| = 1$, $f_n \in \ker(A - \lambda_1 I)^{\perp}$, $\|f_n\| = 1$. Similarly as in the proof of Theorem 4 one can conclude that

$$\|v_n\|^2 - p_n^2 \leqq \left(\frac{M}{\mu_n}\right)^2 \left(\|v_{n-1}\|^2 - p_{n=1}^2\right).$$

Since the sequence $(v_n)$ is bounded and $\mu_n \nearrow \lambda_1$, $(p_n)$ is monotone increasing and bounded. Moreover, we have that

$$0 < \|v_{n-1}\| + p_{n-1} \leqq \|v_n\| + p_n.$$

Hence the above two inequalities imply that

$$\|v_n\| - p_n \leqq \left(\frac{M}{\mu_n}\right)^2 \left(\|v_{n-1}\| - p_{n-1}\right).$$

The equality

$$\|v_n - \| v_n\|e_0\|^2 = 2\|v_n\| \left(\|v_n\| - \langle v_n, e_0\rangle\right)$$

completes the proof.

Remark 2. Under the assumptions of Theorems 4, 5 there exist sufficiently large integers $n_0$, $n_1$ such that for each $p$ $(p = 1, 2, \ldots)$ we have

$$0 \leqq \lambda_1^2 - \alpha_{n_0+p}^2 \leqq \beta_{n_0+p-1}^2 \cdot \beta_{n_0+p-2}^2 \cdots \beta_{n_0}^2(\lambda_1^2 - \alpha_{n_0}^2),$$

$$0 \leqq \lambda_1 - \mu_{n_1+p} \leqq \gamma_{n_1+p-1}^2 \cdot \gamma_{n_1+p-2}^2 \cdots \gamma_{n_1}^2(\lambda_1 - \mu_{n_1}),$$

where $0 < \beta_{n_0+p-1} \leqq \beta_{n_0+p-2} \leqq \ldots \leqq \beta_{n_0} < 1$, $\gamma_{n_1+p-1} \leqq \gamma_{n_1+p-2} \leqq \ldots \leqq \gamma_{n_1} < 1$,

$$\beta_{n_0+i} = M\alpha_{n_0+i}^{-1}, \; \gamma_{n_1+i} = \mu_{n_1+i}^{-1}M, \; i = 0, 1, 2, \ldots, p - 1.$$

The estimations at once follow from Theorem 4, 5, the facts that $\alpha_n \nearrow \lambda_1$, $\mu_n \nearrow \lambda_1$ and the hypothesis that $\lambda_1$ is an isolated point of $\sigma(A)$.

The inequalities

$$\|u_n - e_0\| \leqq 2\left(\frac{M}{\alpha_n}\right)^2 \left(1 - \lambda_1^{n-1} \left(\prod_{k=1}^{n} \alpha_k\right)^{-1} \langle u_0, e_0\rangle\right),$$

$$\|v_n - \| v_n\|e_0\| \leqq 2\left(\frac{M}{\mu_n}\right)^2 \left(\|v_{n-1}\| - \lambda_1^{n-1} \left(\prod_{k=1}^{n} \mu_k\right)^{-1} \langle v_0, e_0\rangle\right)$$

which at once follow from Theorem 4, 5, respectively, provide further estimations for the methods (1) and (2).

Now we derive the error estimations under the general condition that $\lambda_1$ (not necessarily an isolated point of $\sigma(A)$ with finite multiplicity) is an eigenvalue of $A$. We do it for instance for the method (2); a similar result also holds for the procedure (1).

**Theorem 6.** *Let $X$ be a real Hilbert space, $A : X \to X$ a linear positive and self-adjoint operator on $X$. Assume that $\lambda_1$ (not necessarily an isolated point of $\sigma(A)$ with finite multiplicity) is an eigenvalue of $A$. Suppose that the starting approximation $v_0 \in X$ of (1) is not orthogonal to* $\ker(A - \lambda_1 I)$. *If $0 < \varepsilon < \lambda_1 - m$, then*

$$(4) \qquad (\lambda_1 - \varepsilon)\,\|P_\varepsilon w_n\|^2 \leqq \mu_n \leqq \lambda_1 - \varepsilon(1 - \|P_\varepsilon w_n\|^2)\,,$$

$$\lambda_1 \|P_0 w_n\|^2 \leqq \mu_n \leqq \lambda_1$$

*for each $n$, where $m, \lambda_1$ are the exact spectral bounds of $\sigma(A)$ of $A$, $P_\varepsilon = E_{\lambda_1} - E_{\lambda_1 - \varepsilon}$, $P_0 = E_{\lambda_1} - E_{\lambda_1 - 0}$, $w_n = v_n \|v_n\|^{-1}$, $v_n$ and $\mu_n$ are defined by (2) and $\|P_\varepsilon w_n\| \to l$ as $n \to \infty$.*

Proof. Let $\varepsilon$ be an arbitrary number such that $0 < \varepsilon < \lambda_1 - m$, where $m, \lambda_1$ are the exact spectral bounds of $\sigma(A)$ of $A$. Denote by $R(E_{\lambda_1 - \varepsilon})$ the range of $E_{\lambda_1 - \varepsilon}$, i.e. $R(E_{\lambda_1 - \varepsilon}) = \{u \in X : u = E_{\lambda_1 - \varepsilon}(v),\ v \in X\}$. The properties of the spectral family imply that the closed subspaces $R(E_{\lambda 1 - \varepsilon})$, $R(E_{\lambda_1 - \varepsilon})^\perp$ are invariant with respect to $A$. Set $w_n = v_n \|v_n\|^{-1}$, $(n = 0, 1, 2, \ldots)$. Then each $w_n$ can be uniquely expressed in the form $w_n = a_n^{(\varepsilon)} g_n + b_n^{(\varepsilon)} \tilde{z}_n$, where $g_n \in R(E_{\lambda_1 - \varepsilon})^\perp$, $\tilde{z}_n \in R(E_{\lambda_1 - \varepsilon})$, $\|g_n\| = \|\tilde{z}_n\| = 1$ and $(a_n^{(\varepsilon)})^2 + (b_n^{(\varepsilon)})^2 = 1$. We show that $\lim_n (b_n^{(\varepsilon)})^2 = 0$. We have that

$$\lambda_1 = \langle Aw_n, w_n\rangle = \lambda_1 - \langle A(a_n^{(\varepsilon)} g_n + b_n^{(\varepsilon)} \tilde{z}_n), a_n^{(\varepsilon)} g_n + b_n^{(\varepsilon)} \tilde{z}_n\rangle =$$

$$= \lambda_1((a_n^{(\varepsilon)})^2 + (b_n^{(\varepsilon)})^2) - (a_n^{(\varepsilon)})^2 \langle Ag_n, g_n\rangle - (b_n^{(\varepsilon)})^2 \langle A\tilde{z}_n, \tilde{z}_n\rangle\,.$$

We estimate the products $\langle Ag_n, g_n\rangle$, $\langle A\tilde{z}_n, \tilde{z}_n\rangle$. Clearly, $\langle Ag_n, g_n\rangle \leqq \lambda_1 \cdot \|g_n\|^2 = \lambda_1$. Since $\tilde{z}_n \in R(E_{\lambda_1 - \varepsilon})$, there are $h_n \in X$ such that $\tilde{z}_n = E_{\lambda_1 - \varepsilon}(h_n)$. Hence

$$\langle A\tilde{z}_n, \tilde{z}_n\rangle = \langle AE_{\lambda_1 - \varepsilon} h_n, h_n\rangle = \int_u^{\lambda_1} \lambda \, \mathrm{d}\langle E_\lambda E_{\lambda_1 - \varepsilon} h_n, h_n\rangle =$$

$$= \int_m^{\lambda_1 - \varepsilon} \lambda \, \mathrm{d}\langle E_\lambda h_n, h_n\rangle \leqq (\lambda_1 - \varepsilon) \int_m^{\lambda_1 - \varepsilon} \mathrm{d}\|E_\lambda(h_n)\|^2 =$$

$$= (\lambda_1 - \varepsilon)\,\|E_{\lambda_1 - \varepsilon}(h_n)\|^2 = (\lambda_1 - \varepsilon)\,\|\tilde{z}_n\|^2 = \lambda_1 - \varepsilon\,.$$

Therefore $\lambda_1 - \langle Aw_n, w_n\rangle \geqq \varepsilon (b_n^{(\varepsilon)})^2$. However, Theorem 3 implies that $\lim_n (\lambda_1 - \langle Aw_n, w_n\rangle) = 0$, and hence $\lim_n (b_n^{(\varepsilon)})^2 = 0$. We obtain

$$(5) \qquad \lambda_1 - \varepsilon(1 - (a_n^{(\varepsilon)})^2) \geqq \mu_n$$

and $(a_n^{(\varepsilon)})^2 \to 1$ as $n \to \infty$. Put $P_\varepsilon = I - E_{\lambda_1 - \varepsilon}$, $P_0 = I - E_{\lambda_1 - 0}$. Then $R(E_{\lambda_1 - \varepsilon})^\perp = X \ominus R(E_{\lambda_1 - \varepsilon}) = E_{\lambda_1}(X) \ominus E_{\lambda_1 - \varepsilon}(X) = P_\varepsilon(X)$, $P_\varepsilon \to P_0$ in the point norm topology of $(X \to X)$ as $\varepsilon \to 0_+$, and $P_\varepsilon$, $P_0$ are the projectors onto $R(E_{\lambda_1 - \varepsilon})^\perp$, $\ker(A - \lambda_1, I)$, respectively. Then $P_\varepsilon(w_n) = a_n^{(\varepsilon)} g_n$ and $\|P_\varepsilon(w_n)\|^2 = (a_n^{(\varepsilon)})^2 \to 1$ as $n \to \infty$. Furthermore, $\langle Ag_n, g_n \rangle \geqq \lambda_1 - \varepsilon$ for each $n \geqq 0$. Indeed, since $g_n \in R(E_{\lambda_1 - \varepsilon})^\perp$ and $P_\varepsilon(X) = R(E_{\lambda_1 - \varepsilon})^\perp$, there are $c_n \in X$ such that $g_n = P_\varepsilon(c_n)$. Then

$$\langle Ag_n, g_n \rangle = \langle AP_\varepsilon c_n, c_n \rangle = \int_m^{\lambda_1} \lambda \, \mathrm{d}\langle E_\lambda c_n, c_n \rangle - \int_m^{\lambda_1 - \varepsilon} \lambda \, \mathrm{d}\langle E_\lambda c_n, c_n \rangle =$$

$$= \int_{\lambda_1 - \varepsilon}^{\lambda_1} \lambda \, \mathrm{d}\langle E_\lambda c_n, c_n \rangle \geqq (\lambda_1 - \varepsilon) \int_{\lambda_1 - \varepsilon}^{\lambda_1} \mathrm{d}\langle E_\lambda c_n, c_n \rangle =$$

$$= (\lambda_1 - \varepsilon) \|g_n\|^2 = \lambda_1 - \varepsilon .$$

Since

$$\langle Aw_n, w_n \rangle = (a_n^{(\varepsilon)})^2 \langle Ag_n, g_n \rangle + (b_n^{(\varepsilon)})^2 \langle A\tilde{z}_n, \tilde{z}_n \rangle$$

and $(a_u^{(\varepsilon)})^2 = \|P_\varepsilon w_n\|^2$ for each $n$, we have that

$$\|P_\varepsilon w_n\|^2 (\lambda_1 - \varepsilon) \leqq (a_n^{(\varepsilon)})^2 \langle Ag_n, g_n \rangle \leqq \langle Aw_n, w_n \rangle = \mu_n \leqq \lambda_1 .$$

Now the first estimation at once follows from the last inequality and (5), while the second one is a consequence of (4) and thr fact that $P_\varepsilon \to P_0$ as $\varepsilon \to 0_+$ in the point norm topology of $(X \to X)$. Theorem 6 is proved.

Remark 3. The estimation

$$\mu_n \leqq \lambda_1 - \varepsilon \|E_{\lambda_1 - \varepsilon} w_n\|^2 \quad (n = 0, 1, 2, \ldots) .$$

holds, where $(\mu_n)$, $(v_n)$ are defined by (2) and $w_n = \|v_n\|^{-1} v_n$, $0 < \varepsilon < \lambda_1 - m$. Note that this estimate is rather worse that the corresponding one on the right hand side of (4).

**Theorem 7.** *Under all the other condition of Theorem 6 on $X$, $A$, assume only that $A$ is nonegative. Assume that the starting approximation $u_0$ of (1) is not orthogonal to* $\ker(A - \lambda_1 I)$. *If $0 < \varepsilon < \lambda_1 - m$, then*

$$(\lambda_1 - \varepsilon)^2 \|P_\varepsilon u_n\|^2 \leqq \alpha_n^2 \leqq \lambda_1^2 - \varepsilon(2\lambda_1 - \varepsilon)(1 - \|P_\varepsilon u_n\|^2) ,$$

$$\lambda_1 \|P_0 u_n\| \leqq \alpha_n \leqq \lambda_1$$

*for each $n$, where $m$, $\lambda_1$, $P_0$ have the same meaning as in Theorem 6 and $\|P_\varepsilon(u_n)\| \to 1$ as $n \to \infty$.*

Remark 4. Some results of this paper were communicated by the author at the IVth Conference on basic problems of numerical analysis, Plzeň, Czechoslovakia, September 4–8, 1978.

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## Souhrn

## K URČENÍ VLASTNÍCH ČÍSEL A VLASTNÍCH FUNKCÍ SAMOADJUNGOVANYCH OPERÁTORŮ

JOSEF KOLOMÝ

V článku jsou vyšetřeny jednoduché metody (1), (2) pro výpočet vlastních čísel a vlastních funkcí lineárních samoadjungovaných operátorů. Je ukázáno, že obě metody konvergují i v případě, kdy přesná horní hranice $\lambda_1$ spektra $\sigma(A)$ operátoru $A$ není isolovaným bodem spektra $\sigma(A)$ s konečnou násobností. Jsou odvozeny odhady chyb pro konvergenci obou metod a je ukázáno, že je lze též užít i pro výpočet vlastních čísel lineárních ohraničených operátorů.

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