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1. INTRODUCTION

In this paper we discuss quadratic homogeneous polynomials in n complex variables with real coefficients, shortly quadratic forms. The following notation is used: $\mathbf{a}$, $\mathbf{b}$, $\mathbf{x}$, $\mathbf{y}$ are real column vectors $\mathbf{x} = (x, x_2, \ldots, x_n)^{\mathsf{T}}$, x_i, $i = 1, 2, \ldots, n$, are real numbers, $^{\mathsf{T}}$ denotes transposition. Complex vectors are $\mathbf{z} = \mathbf{x} + \mathrm{j}\mathbf{y}$, $\mathrm{j}^2 = -1$; $\bar{\mathbf{z}} = \mathbf{x} - \mathrm{j}\mathbf{y}$ is the complex conjugate of $\mathbf{z}$. One single complex variable will be denoted by p. Γ will denote the open right half-plane, $\Gamma = \{p \text{ complex}, \operatorname{Re} p > 0\}$. The set of complex vectors $\mathbf{z} = (z_1, z_2, \ldots, z_n)^{\mathsf{T}}$ such that $\operatorname{Re} z_i > 0$ for all $i = 1, 2, \ldots, n$ will be $\Gamma^{(n)}$; $\Gamma^{(n)} = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n$. Capitals denote real symmetric matrices of order n, e.g. $\mathbf{A} = [a_{ij}]$; $\mathrm{r}(\mathbf{A})$, $\sigma(\mathbf{A})$, $\operatorname{Tr} \mathbf{A}$ respectively denote the rank, signature and trace of matrix $\mathbf{A}$; $\mathbf{a}_{.j}$ will be its j-th column, $\mathbf{a}_{i.}^{\mathsf{T}}$ its i-th row. Relations $>$, $\geqq$, when used for vectors or matrices, are treated elementwise; thus $\mathbf{Q} \geqq \mathbf{O}$ means that all elements of the matrix $\mathbf{Q}$ are nonnegative numbers. We shall tacitly assume that in such case $\mathbf{Q} \neq \mathbf{O}$. The usual scalar product notation $(.,.)$ will be used for real vectors: $(\mathbf{x}, \mathbf{y}) = \sum_i x_i y_i$. The quadratic form $(\mathbf{A}\mathbf{x}, \mathbf{x})$ will be denoted by $f_{\mathbf{A}}$ or $f_{\mathbf{A}}^{(n)}$, similarly $\mathbf{z}^{\mathsf{T}}\mathbf{A}\mathbf{z} = f_{\mathbf{A}}^{(n)}(\mathbf{z})$ will stand for the homogeneous quadratic polynomial of n complex variables, which is uniquely determined by the real matrix $\mathbf{A}$; $f_{\mathbf{A}}(\mathbf{z})$ will be also called a quadratic form.

Any complex vector $\mathbf{z}_0$ satisfying the condition $\mathbf{z}_0^{\mathsf{T}}\mathbf{A}\mathbf{z}_0 = f_{\mathrm{A}}(\mathrm{z}_0) = 0$ will be called a zero point or a root of the quadratic form $f_{\mathbf{A}}$. Evidently, if $\mathbf{z}_0$ is a root of $f_{\mathbf{A}}$, then the vectors $\bar{\mathbf{z}}_0$, $-\mathbf{z}_0$, $-\bar{\mathbf{z}}_0$ are also roots of $f_{\mathbf{A}}$.

Definition 1. A quadratic form $f_{\mathbf{A}}$ is said to be a *Hurwitz form* ($f_{\mathbf{A}} \in H$ or $\mathbf{A} \in \mathscr{H}$) iff $f_{\mathbf{A}}(\mathbf{z}) = \mathbf{z}^{\mathsf{T}}\mathbf{A}\mathbf{z} \neq 0$ for all $\mathbf{z} \in \Gamma^{(n)}$, i.e. iff $f_{\mathbf{A}}$ has no roots in the cartesian product of the open right half-planes Γ_i, $i = 1, 2, \ldots, n$.

The aim of Section 2 of this paper is to obtain necessary and sufficient conditions for $\mathbf{A} \in \mathscr{H}$. Various types of these conditions show that considering quadratic forms as homogeneous polynomials in n complex (rather than real) variables may explain some of their properties. In Section 3 we are discussing some linear transforms of Hurwitz forms. Here, the resuls as well as some methods of proofs will anticipate the main purpose of this paper, which is to extend the well established theory of one-variable Hurwitz polynomials to the multivariable case. Section 4 of this paper will show that Hurwitz forms, like more general Hurwitz polynomials in many variables, are closely related to the theory of multivariable positive real functions. In recent years considerable attention has been given to this class of functions mainly because of their significance in electrical network analysis and synthesis, in multidimensional digital filtering and related problems (see [1]).

Electrical networks, consisting of lumped passive variable elements or consisting of lumped elements (such as resistors and capacitors) together with distributed RC lines or lossless transmission lines, can be uniquely described in terms of multivariable positive real functions. The synthesis of such networks from a given positive real function is of great practical significance, nevertheless, as has been recently pointed out in a survey [1], its theory is far from being complete. Any rational positive real function in several variables is a ratio of two Hurwitz polynomials, but no results similar to the well-known Routh-Hurwitz criterion in the one variable case are known. Some numerical procedures developed so far [1] give results for polynomials with numerically given coefficients only and do not allow to discuss their possible changes. In practical cases restrictions imposed on the degree of a polynomial may be more acceptable than restrictions on the number of variables and therefore quadratic polynomials in n variables are the simplest nontrivial object of investigation.

Any quadratic polynomial P^* in $(n - 1)$ variables may be "homogenized", i.e. a homogeneous quadratic polynomial P in n variables can be formed as follows:

$$P(z_1, z_2, \ldots, z_n) = z_n^2 P^* \left(\frac{z_1}{z_n}, \frac{z_2}{z_n}, \ldots, \frac{z_{n-1}}{z_n} \right).$$

The following implication is obvious:

$$\{P(\mathbf{z}) \neq 0 \quad \text{for all} \quad \mathbf{z} \in \Gamma^{(n)}\} \Rightarrow \{P^*(\mathbf{z}) \neq 0 \quad \text{for all} \quad \mathbf{z} \in \Gamma^{(n-1)}\}.$$

Hurwitz forms are thus important special cases of quadratic Hurwitz polynomials.

The main purpose of this paper is to contribute to the theory of multivariable positive real functions.

2. HURWITZ FORMS

If not stated otherwise we shall assume that all the quadratic forms have at least one positive coefficient. We shall start with a simple

Lemma 2. *A quadratic form in two variables is a Hurwitz form if and only if its determinant is nonpositive.*

Proof. Denote $f_{\mathbf{A}}^{(2)} = a_{11}z_1^2 + 2a_{12}z_1z_2 + a_{22}z_2^2$; if $f_{\mathbf{A}} \in H$, then no pair of coefficients can have opposite signs because in that case the polynomial $f_{\mathbf{A}}(z_1, 1)$ would not be a Hurwitz polynomial. Therefore $a_{ik} \geqq 0$ for $i, k = 1, 2$. Let now $\det \mathbf{A} = a_{11}a_{22} - a_{12}^2 = \Delta > 0$ so that $a_{11}a_{22} \neq 0$. If $f_{\mathbf{A}}(p_0, q_0) = 0$ then

$$a_{11} \operatorname{Re} p_0 = -a_{12} \operatorname{Re} q_0 \pm \operatorname{Im} q_0 \sqrt{\Delta}\,.$$

It follws that for any $\operatorname{Re} q_0 > 0$ there exists such a real number $\operatorname{Im} q_0$ that $f_{\mathbf{A}}(p_0, q_0) = 0$ and $\operatorname{Re} p_0 > 0$, which contradicts our assumption $f_{\mathbf{A}} \in H$. Therefore, $f_{\mathbf{A}} \in H$ implies $\det \mathbf{A} \leqq 0$. Let now $\det \mathbf{A} \leqq 0$ and $a_{ik} \geqq 0$. Then there exist nonnegative real numbers $\alpha, \beta, \gamma, \delta$ such that $f_{\mathbf{A}} = (\alpha z_1 + \beta z_2)(\gamma z_1 + \delta z_2)$ and therefore $f_{\mathbf{A}} \in H$. ∎

In the quadratic form $f_{\mathbf{A}}^{(n)} \in H$ let one variable, say z_k, be denoted by p; then

$$f_{\mathbf{A}}^{(n)} = \alpha p^2 + l^{(n-1)}p + g_{\mathbf{B}}^{(n-1)}\,, \tag{1}$$

where α is a nonnegative real number, $l^{(n-1)}$ and $g_{\mathbf{B}}^{(n-1)}$, respectively, is a linear and a quadratic form in the remaining $(n-1)$ variables z_i, $i \neq k$. Because any partial derivative of a Hurwitz polynomial is again a Hurwitz polynomial [4], and $f_{\mathbf{A}}^{(n)} \in H$ iff $\alpha + l^{(n-1)}p + g_{\mathbf{B}}^{(n-1)}p^2$ is a Hurwitz polynomial, we have

Lemma 3. *If $f_{\mathbf{A}}^{(n)} \in H$ in formula (1), then $g_{\mathbf{B}}^{(n-1)} \in H$.*

By induction we obtain

Corollary 4. *All coefficients of a Hurwitz form are nonnegative.*

The first form of a necessary and sufficient condition for $f_{\mathbf{A}} \in H$ is contained in the following theorem.

Theorem 5. *A quadratic form $f_{\mathbf{A}}$ with nonnegative coefficients is a Hurwitz form if and only if its matrix $\mathbf{A}$ satisfies the inequality*

$$(\mathbf{Ab}, \mathbf{a})^2 \geqq (\mathbf{Ab}, \mathbf{b})(\mathbf{Aa}, \mathbf{a}) \tag{2}$$

for all pairs of nonnegative vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} + \mathbf{b} > \mathbf{0}$.

The proof of this theorem is based on Lemmas 6 and 7 and will be performed later. Let us only mention here that inequalities similar to (2) ("converse Schwarz's inequality") did appear earlier in the analytical treatment of L_p spaces with $0 < p < 1$. By contradiction it immediately follows from Theorem 5 that if $f_{\mathbf{A}} \in H$ then the symmetric matrix $\mathbf{A}$ must be indefinite.

The following lemma could be a starting point even for investigations of multivariable Hurwitz polynomials of degree greater than two.

Lemma 6. *A polynomial P in n complex variables with real coefficients and of degree k is nonzero in $\Gamma^{(n)}$ iff the one-variable polynomial*

$$R(p) = p^k P(\alpha_1 p + \beta_1/p, \alpha_2 p + \beta_2/p, \ldots, \alpha_n p + \beta_n/p)$$

of degree $2k$ has no roots in the open right half plane $\Gamma = \{p : \operatorname{Re} p > 0\}$ for any nonnegative values of α_i, β_i such that $\alpha_i + \beta_i > 0$, $i = 1, 2, \ldots, n$.

Proof. For any vector $\mathbf{z} \in \Gamma^{(n)}$ there exist a complex value $p \in \Gamma$ and n pairs of nonnegative numbers (α_i, β_i), $i = 1, 2, \ldots, n$, such that

$$z_i = \alpha_i p + \beta_i/p \quad i = 1, 2, \ldots, n\,; \quad \alpha_i + \beta_i > 0\,.$$

It is sufficient to choose p so that $\arg p = \max\limits_i |\arg z_i|$ and the rest is obvious. Because $\alpha_i \geqq 0$, $\beta_i \geqq 0$, $\alpha_i + \beta_i > 0$, we have

$$\operatorname{Re} p > 0 \Leftrightarrow \operatorname{Re} z_i > 0 \quad \text{for all} \quad 1 \leqq i \leqq n\,.$$

Therefore $R(p) \neq 0$ for all $p \in \Gamma$ iff $P(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \Gamma^{(n)}$. ∎

Lemma 7. *The polynomial*

$$T(p) = \alpha p^4 + 2\beta p^2 + \gamma, \quad \alpha, \beta, \gamma \geqq 0, \alpha + \beta + \gamma > 0$$

is nonzero for all $p \in \Gamma$ iff $\beta^2 - \alpha\gamma \geqq 0$.

Proof. In the condition $\beta^2 - \alpha\gamma \geqq 0$ the equality sign holds iff $T(p) = (\alpha p + \gamma)^2$, which is a Hurwitz polynomial. In proving Lemma 7 we may therefore assume that $\beta^2 - \alpha\gamma > 0$, which implies $\beta \neq 0$. The polynomial T is nonzero in the open right half-plane iff the continued fraction expansion of the ratio T'/T has positive coefficients (see e.g. [5]). By repeated division we get for $\alpha > 0$, $\gamma > 0$:

$$\frac{T'(p)}{4T(p)} = \cfrac{1}{p + \cfrac{1}{\frac{\alpha}{\beta} p + \cfrac{1}{\frac{\beta^2}{\beta^2 - \alpha\gamma} p + \cfrac{1}{\frac{\beta^2 - \alpha\gamma}{\gamma\beta} p}}}}$$

and our statement follows. If $\alpha = 0$ or $\gamma = 0$ then the lemma is trivial. ∎

Proof of Theorem 5. Let two vectors $\boldsymbol{a} \geqq \boldsymbol{0}$, $\boldsymbol{b} \geqq \boldsymbol{0}$, $\boldsymbol{a} + \boldsymbol{b} > \boldsymbol{0}$, be chosen arbitrarily. The polynomial R from Lemma 6 with the same quadratic form as the polynomial P can be formed as follows:

$$R(p) = p^2(\boldsymbol{a}^\mathsf{T} p + \boldsymbol{b}^\mathsf{T}/p)\, \boldsymbol{A}(\boldsymbol{a} p + \boldsymbol{b}/p) =$$
$$= (\boldsymbol{Aa}, \boldsymbol{a})\, p^4 + [(\boldsymbol{Ab}, \boldsymbol{a}) + (\boldsymbol{Aa}, \boldsymbol{b})]\, p^2 + (\boldsymbol{Ab}, \boldsymbol{b})\,.$$

The symmetry of A implies $(Ab, a) = (Aa, b)$ and Lemma 7 proves condition (2) to be necessary and sufficient for $f_A \in H$. ∎

A simple continuity argument shows that inequality (2) holds true for all nonnegative vectors a, b iff it holds true for all such vectors satisfying $a + b > 0$.

It is worth mentioning that in inequality (2) the equality sign holds for all pairs of nonnegative vectors a, b iff A is of rank 1 ($r(A) = 1$) and inequality (2) holds without the equality sign for all linearly independent pairs of nonnegative vectors iff $r(A) = n$. As for the first statement, let us first suppose that $(Ab, a)^2 = (Aa, a) . . (Ab, b)$. Denote by $\mathbf{e}^{(i)}$ the vector with its i-th coordinate equal to 1 and with all other coordinates equal to zero. For $i \neq j$ the vectors $\mathbf{e}^{(i)}$, $\mathbf{e}^{(j)}$ form a linearly independent pair of nonnegative vectors and therefore

$$(A\mathbf{e}^{(i)}, \mathbf{e}^{(j)})^2 = (A\mathbf{e}^{(i)}, \mathbf{e}^{(i)})\,(A\mathbf{e}^{(j)}, \mathbf{e}^{(j)})$$

or (with $A = [a_{ij}]$) equivalently $a_{ij}^2 = a_{ii}a_{jj}$ for all $1 \leqq i, j \leqq n, i \neq j$. All the second order principal minors of the matrix A are equal to zero. Any row of the matrix A is therefore a multiple of $(\sqrt{a_{11}}, \sqrt{a_{22}}, \ldots, \sqrt{a_{nn}})$, which implies $r(A) = 1$. Conversely, let $r(A) = 1$. Then there exists a vector, say λ, such that $(Aa, b) = (\lambda, a) . . (a_{1\cdot}, b)$. The symmetry of A implies $(Ab, a) = (\lambda, b)\,(a_{1\cdot}, a)$ and therefore

$$(Aa, b)^2 - (Aa, a)\,(Ab, b) =$$
$$= (\lambda, a)\,(\lambda, b)\,(a_{1\cdot}, b)\,(a_{1\cdot}, a) - (\lambda, a)\,(a_{1\cdot}, a)\,(\lambda, b)\,(a_{1\cdot}, b) = 0\,.$$

Hence, the equality sign holds in formula (2). The statement concerning $r(A) = n$ will be proved below.

Theorem 8. [2]. *Let A be a real, symmetric nonnegative matrix of order greater than* 1. *The following statements* i) *and* ii) *are equivalent*:

i) *Whenever a, b are nonnegative vectors, then*

$$(Aa, a)\,(Ab, b) \leqq (Ab, a)^2\,.$$

ii) *A is a semielliptic matrix, i.e., it has exactly one simple positive eigenvalue.*

Proof. Let i) be satisfied and let a be a fixed positive vector. We shall first prove that

$$(Ay, y)\,(Aa. a) \leqq (Aa, y)^2 \tag{3}$$

for any vector y.

In fact, let y be such a vector. If $(Ay, y) = 0$ then (3) holds. It is therefore sufficient to consider $(Ay, y) \neq 0$. Because a is positive, there exists such a real number $\varepsilon > 0$ that $b = a + \varepsilon y$ is positive. For the vectors a, b inequality (2) holds and therefore the quadratic equation

$$(A(a + \xi b), a + \xi b) = 0$$

with unknown ξ, i.e. the quadratic equation

$$\xi^2(\boldsymbol{A}\boldsymbol{b}, \boldsymbol{b}) + 2\xi(\boldsymbol{A}\boldsymbol{b}, \boldsymbol{a}) + (\boldsymbol{A}\boldsymbol{a}, \boldsymbol{a}) = 0\,,$$

has two not necessarily distinct real roots ξ_1, ξ_2, none of which equals to -1. Because

$$\boldsymbol{a} + \xi\boldsymbol{b} = \boldsymbol{a} + \xi(\boldsymbol{a} + \varepsilon\boldsymbol{y}) = (1 + \xi)\,\boldsymbol{a} + \varepsilon\xi\boldsymbol{y}\,,$$

the equation

$$(\boldsymbol{A}(\boldsymbol{a} + \eta\boldsymbol{y}), \boldsymbol{a} + \eta\boldsymbol{y}) = 0$$

with unknown η again has real roots $\eta_i = \varepsilon\xi_i/(1 + \xi_i)$, $i = 1, 2$. This means that its discriminant is nonnegative, which is equivalent to (3).

Because $(\boldsymbol{A}\boldsymbol{a}, \boldsymbol{a}) > 0$ it follows from (3) that the quadratic form $(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{x})$ is negative semidefinite on the linear space $(\boldsymbol{A}\boldsymbol{a}, \boldsymbol{x}) = 0$ of dimension $n - 1$. Therefore the matrix $\boldsymbol{A}$ has $n - 1$ nonpositive eigenvalues. But $(\boldsymbol{A}\boldsymbol{a}, \boldsymbol{a}) > 0$, hence $\boldsymbol{A}$ is neither negative definite nor negative semidefinite and $\boldsymbol{A}$ has at least one positive eigenvalue. We may conclude that i) implies ii).

To prove the converse let ii) hold. Then the quadratic form $(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{x})$ is on any linear subspace either negative semidefinite or semielliptic. Let now $\boldsymbol{a}$, $\boldsymbol{b}$ nonnegative vectors. If they are linearly dependent, equality in i) holds. Let us thus assume that they are linearly independent. Then they define a linear subspace on which the quadratic form $(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{x})$ is either negative semidefinite or semielliptic. The same holds true also for the matrix

$$\boldsymbol{M} = \begin{bmatrix} (\boldsymbol{A}\boldsymbol{a}, \boldsymbol{a}); & (\boldsymbol{A}\boldsymbol{a}, \boldsymbol{b}) \\ (\boldsymbol{A}\boldsymbol{b}, \boldsymbol{a}); & (\boldsymbol{A}\boldsymbol{b}, \boldsymbol{b}) \end{bmatrix}.$$

If $(\boldsymbol{A}\boldsymbol{a}, \boldsymbol{a}) = 0$ then (2) is satisfied. Therefore, let $(\boldsymbol{A}\boldsymbol{a}, \boldsymbol{a}) > 0$. If, in addition, det M were positive, then the matrix $\boldsymbol{M}$ would be positive definite, which is impossible. Hence, $\det \boldsymbol{M} \leqq 0$, which is exactly inequality (2). Therefore ii) implies i) and the proof is now complete. We may add that $\boldsymbol{A}$ has only nonzero eigenvalues exactly when (2) holds without the equality sign for linearly independent vectors $\boldsymbol{a}$, $\boldsymbol{b}$. This justifies the second part of our remark made after the proof of Theorem 5. ■

Corollary 9. *The positive eigenvalue of any Hurwitz matrix (matrix of any Hurwitz form) cannot be less than the sum of absolute values of all other eigenvalues of this matrix.*

This follows immediately from the fact that $\boldsymbol{A} \in \mathscr{H}$ implies $\operatorname{Tr} \boldsymbol{A} = \sum_{i=1}^{n} \lambda_i > 0$, where λ_i are the eigenvalues of A.

Corollary 10. *Let k be a positive integer, $1 \leqq k \leqq n$, $\boldsymbol{I} = (i_1, i_2, \ldots, i_k)$, i_j integers, $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$ and let $\boldsymbol{Q}_I^{(k)}$ be the submatrix of order k of a matrix $\boldsymbol{Q}$ contatning exactly its rows and columns numbered $i_1, i_2, \ldots, i_k$. If $\boldsymbol{Q} \in \mathscr{H}$, then all matrices $\boldsymbol{Q}_I^{(k)}$ are semielliptic and*

$$(-1)^k \det \boldsymbol{D}_I^{(k)} \geqq 0\,. \tag{4}$$

From Lemma 3 we easily get the semiellipticity of all matrices $\mathbf{Q}_l^{(k)}$. The remaining part of Corollary 10 follows when considering the eigenvalues of semielliptic matrices. With minor additions to the proof of Theorem 8 the following statement can be formulated:

Theorem 11. [2] *Under the same assumptions as in Theorem 8, the following statements are equivalent*:

i) *Whenever* $\mathbf{a}$, $\mathbf{b}$ *are linearly independent nonnegative vectors, then*

$$(2') \qquad (\mathbf{A}\mathbf{a}, \mathbf{a})(\mathbf{A}\mathbf{b}, \mathbf{b}) < (\mathbf{A}\mathbf{a}, \mathbf{b})^2 .$$

ii) $\mathbf{A}$ *is an elliptic matrix, i.e. it has exactly one simple positive eigenvalue and all other eigenvalues are negative.*

The proof can be given similarly as for Theorem 8. In proving the inequality

$$(3') \qquad (\mathbf{A}\mathbf{y}, \mathbf{y})(\mathbf{A}\mathbf{a}, \mathbf{a}) < (\mathbf{A}\mathbf{a}, \mathbf{y})^2$$

only the case $(\mathbf{A}\mathbf{y}, \mathbf{y}) = 0$ must be discussed. However, if the positive vector $\mathbf{b} = \mathbf{a} + \varepsilon\mathbf{y}$ is used in inequality (2′), the validity of (3′) can be checked upon easily. According to (3′) the form $(\mathbf{A}\mathbf{x}, \mathbf{x})$ becomes negative definite on the subspace $(\mathbf{A}\mathbf{a}, \mathbf{x}) = 0$ and therefore i) implies ii). Conversely, we may follow step by step the proof of Theorem 8 with minor rewording. Only the case $(\mathbf{A}\mathbf{a}, \mathbf{a}) = 0$ and the case $\det \mathbf{M} = 0$ has to be discussed separately, but it would be superfluous to go into details here. ∎

Let now $\mathbf{A}$ be a symmetric matrix and let its leading principal submatrices of orders $1, 2, \ldots, n$ be considered. As usual, we shall denote their determinants by $D_1, D_2, \ldots, D_n$. The following statement holds.

Theorem 12. *Let* $\mathbf{A}$ *be a nonnegative symmetric matrix and let* $D_i \neq 0$ *for* $i = 1, 2, \ldots, n$. *Then* $\mathbf{A}$ *is a Hurwitz matrix iff*

$$(5) \qquad D_1 > 0\,, \quad D_2 < 0\,, \quad D_3 > 0\,, \ldots, (-1)^{n+1} D_n > 0\,.$$

Proof. According to Corollary 10, condition (5) is necessary for $\mathbf{A}$ to be a Hurtwitz matrix. The sufficiency follows from the well known results on signature and rank. Denoting by π, ν, respectively, the number of positive and negative eigenvalues we have $\pi + \nu = r(\mathbf{A})$, $\pi - \nu = \sigma(\mathbf{A})$. Here, $r(\mathbf{A}) = n$ and $\sigma(\mathbf{A}) = 2 - n$. Therefore $\mathbf{A}$ has exactly one positive eigenvalue and according to *Theorem* 11, $\mathbf{A} \in \mathscr{H}$. ∎

If $D_n \neq 0$ then the assumptions in the last theorem can be weakened; this problem will be dealt with in Section 4.

Interchanging arbitrary rows and the corresponding columns cannot affect the eigenvalues; hence if the assumptions of Theorem 12 are fulfilled for any nested set of principal submatrices of order $1, 2, \ldots, n$, then its conclusion remains to be true.

3. LINEAR TRANFORMATIONS OF HURWITZ FORMS

In this section we shall deal with such linear transforms of forms, which preserve the Hurwitz property. After any orthogonal transform the eigenvalues of a matrix remain unchanged and therefore, in view of Theorem 8, we have to find such orthogonal transforms which preserve nonnegativity of the transformed matrix. For further use we shall say that a matrix $\boldsymbol{A}$ is a nonnegative bordering of a matrix $\boldsymbol{B}$ if in formula (1) $k = 1$, $\alpha = 0$, and $1^{(n-1)}$ is a linear form with nonnegative coefficients.

Lemma 13. *If $\boldsymbol{A} \in \mathscr{H}$ then there exists such an orthogonal matrix $\boldsymbol{T}$ that the matrix $\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T}$ is a nonnegative bordering of a certain matrix $\boldsymbol{B} \in \mathscr{H}$.*

Proof. Let $\boldsymbol{A} = [a_{ij}]$ be a matrix of order n in $\mathscr{H}$. Since $a_{ik} \geqq 0$ for all i, k, we may assume that $a_{ii} > 0$ for all $i = 1, 2, \ldots, n$; otherwise it sufficies to rearrange the rows and columns. Then not only the diagonal elements, but all elements must be positive; if a_{ik} were zero for some $i \neq k$, $1 \leqq i, k \leqq n$, then at least one of the second order principal submatrices would be positive in contradiction with the assumption $\boldsymbol{A} \in \mathscr{H}$ (see Corollary 10). Let now numbers c_{ik}, $i, k = 1, 2, \ldots, n$, be defined by

$$c_{ik} = a_{ii}a_{kk}/a_{ik}^2 .$$

The matrix with the elements c_{ik} is real, symmetric and all its off-diagonal elements are not greater than one:

$$0 < c_{ik} \leqq 1 .$$

Rows and columns of the matrix $\boldsymbol{A}$ can be rearranged so that $c_{12} = \min c_{ik}$. After such a rearranging the matrix $\boldsymbol{A}$ remains to be a Hurwitz matrix and therefore we may assume that $\boldsymbol{A}$ was given so that $c_{12} = \min c_{ik}$.
Let now the matrix

$$\boldsymbol{T}_{12} = \frac{1}{\sqrt{(\lambda^2 + a_{22}^2)}} \begin{bmatrix} -a_{22} & \lambda \\ \lambda & a_{22} \end{bmatrix}$$

be considered with $\lambda = a_{12} + \Delta$, $\Delta = \sqrt{(a_{12}^2 - a_{11}a_{22})}$. $\boldsymbol{T}_{12}$ is a symmetric matrix satisfying the condition $\boldsymbol{T}_{12}^{-1} = \boldsymbol{T}_{12}$ and therefore $\boldsymbol{T}_{12}$ as well as the matrix

$$\boldsymbol{T} = \left[\begin{array}{c|c} \boldsymbol{T}_{12} & \boldsymbol{0} \\ \hline \boldsymbol{0} & \boldsymbol{E} \end{array}\right]$$

are orthogonal matrices of order 2 and n, respectively ($\boldsymbol{E}$ is the unit matrix of order $n - 2$). Partitioning the matrix $\boldsymbol{A}$ conformally we have

$$\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \boldsymbol{T}^{-1} \left[\begin{array}{c|c} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \hline \boldsymbol{A}_{12} & \boldsymbol{A}_{22} \end{array}\right] \boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}_{12}^{-1}\boldsymbol{A}_{11}\boldsymbol{T}_{12} & \boldsymbol{T}_{12}\boldsymbol{A}_{12} \\ \boldsymbol{A}_{12}\boldsymbol{T}_{12} & \boldsymbol{A}_{22} \end{bmatrix}$$

and further

$$T_{12}^{-1}A_{11}T_{12} = \frac{1}{\lambda^2 + a_{22}^2}\begin{bmatrix} 0; \ \lambda(a_{22}^2 - a_{11}a_{22}) + a_{12}(\lambda^2 - a_{22}^2) \\ *; \ \lambda^2 a_{11} + 2\lambda a_{12}a_{22} + a_{22}^3 \end{bmatrix},$$

where $*$ indicates such a value that the matrix is symmetric. Denoting it for the moment by x_{12}, we get

$$x_{12} = (a_{22}^2 - a_{11}a_{22})(a_{12} + \Delta) + a_{12}(a_{12}^2 + 2a_{12}\Delta + a_{12}^2 -$$

$$- a_{11}a_{22} - a_{22}^2) = 2a_{12}^3 - 2a_{12}a_{11}a_{22} + (2a_{12}^2 + a_{22}^2 - a_{11}a_{22})\,\Delta =$$

$$= 2\Delta^2 a_{12} + (\Delta^2 + a_{12}^2 + a_{22}^2)\,\Delta > 0\,.$$

The matrix $T_{12}^{-1}A_{11}T_{12}$ is therefore nonnegative with zero in the "upper left corner". Let $T_{12}A_{12}$ be calculated:

$$T_{12}A_{12}\sqrt{(a_{22}^2 + \lambda^2)} = \begin{bmatrix} \lambda a_{23} - a_{13}a_{22}; \ \lambda a_{24} - a_{14}a_{22}; \ \ldots \\ \lambda a_{13} + a_{22}a_{23}; \ \lambda a_{14} + a_{22}a_{24}; \ \ldots \end{bmatrix}.$$

To prove that $T_{12}A_{12} \geqq 0$ it is sufficient to show that $\lambda a_{2k} - a_{1k}a_{22} \geqq 0$ for all $k = 3, 4, \ldots, n$. Suppose that for some k there is $a_{2k} < a_{1k}a_{22}/\lambda$. From the assumption $A \in \mathscr{H}$ it follows that $a_{2k} \geqq \sqrt{(a_{22}a_{kk})}$ and therefore $a_{22}a_{kk} < a_{1k}^2 a_{22}^2/\lambda^2$. Denoting $\mu = a_{12} - \Delta$, we have evidently $\mu > 0$ and $\lambda\mu = a_{11}a_{22}$. From the above inequality we get the estimate

$$\frac{\mu}{\lambda} > \frac{a_{11}a_{kk}}{a_{1k}^2} = c_{1k}$$

and on the other hand

$$\frac{\mu}{\lambda} = \frac{1 - \sqrt{(1 - c_{12})}}{1 + \sqrt{(1 - c_{12})}} < c_{12}\,,$$

which can be easily checked by analyzing the behaviour of the function $(1 - \sqrt{(1-x)}) : (1 + \sqrt{(1 - x)})$ for $x \in (0; 1]$. The last two estimates show that $c_{1k} < c_{12}$ for some $k = 3, 4, \ldots, n$, which contradicts our assumption $c_{12} = \min c_{ik}$. Therefore $T_{12}A_{12} \geqq 0$, which remained to be proved. ■

Successive application of the preceding lemma shows that any Hurwitz matrix can be given a form in which all diagonal elements, with at most one exception, are equal to zero.

Another type of conclusion is given by

Lemma 14. *Let* A *be a square, symmetric nonnegative matrix of order* n *and* B *a positive matrix of type* $n \times m$. *If* $A \in \mathscr{H}^{(n)}$, *then* ${}^{T}BAB \in \mathscr{H}^{(m)}$; *if* ${}^{T}BAB \notin \mathscr{H}^{(m)}$ *for some* $B > 0$ *then* $A \notin \mathscr{H}^{(n)}$.

Proof. Let $\mathbf{z} \in \Gamma^{(m)}$, then $\mathbf{y} = \mathbf{Bz} \in \Gamma^{(n)}$. If $\mathbf{A} \in \mathscr{H}^{(n)}$, then $(\mathbf{Ay}, \mathbf{y}) \neq 0$ for all $\mathbf{y} \in \Gamma^{(n)}$ so that $(\mathbf{ABz}, \mathbf{Bz}) = ({}^{\mathsf{T}}\mathbf{BABz}, \mathbf{z}) \neq 0$ for all $\mathbf{z} \in \Gamma^{(m)}$ and ${}^{\mathsf{T}}\mathbf{BAB} \in \mathscr{H}$. The other part of the lemma follows immediately. ■

These transformations will be made use of in the next section.

4. POSITIVE REAL FUNCTIONS AND HURWITZ FORMS

A function $f : C^n \to C^1$ is called *positive real* if it is analytic in $\Gamma^{(n)}$, $f(\Gamma^{(n)}) \subset \Gamma$ and for any real and positive vector $\mathbf{x}$ the value $f(\mathbf{x})$ is real and positive. Rational positive real functions form a basic mathematical tool in the analysis and synthesis of passive electrical networks. The first survey of basic properties of these functions was given by Koga [4]. Evidently, if f is positive real then so is $1/f$. Therefore positive real functions are nonzero in $\Gamma^{(n)}$ and, in the rational case, their numerators and denominators are Hurwitz polynomials. The ratio of any two Hurwitz polynomials, however, need not be a positive real function. In this section we shall investigate mainly homogeneous rational positive real functions with quadratic numerators. Two results of [4] will be referred to below:

Lemma 15. i) *If P is a Hurtwitz polynomial of n variables with real coefficients, then $f(z) = 1/P \cdot \partial P/\partial z_i$ is positive real for all $i = 1, 2, \ldots, n$.*

ii) *The ratio M/N of two Hurwitz polynomials in n variables is a positive real function if and only if the polynomial $M(z) + z_{n+1} N(z)$ in $n + 1$ variables is a Hurwitz polynomial.*

By means of i) of the previous lemma and a reasoning similar to that of Theorem 5 one result of Koga has been generalized and simplified [3] as follows:

Lemma 16. *A rational function $f : C^n \to C^1$, odd and holomorphic in $\Gamma^{(n)}$, is positive real iff all its first partial derivatives are positive on the set*

$$M = \{z \in C^{(n)},\ \operatorname{Re} z = 0\}\ .$$

From these and some further results the following theorem can be proved.

Theorem 17. *Let $\mathbf{A}$ be a Hurwitz matrix of order n. Then all matrices*

$$\mathbf{B}_{ij} = [b_{kl}]_{k,l=1}^{n} = [a_{ik}a_{jl} + a_{il}a_{jk} - a_{ij}a_{kl}]\,,$$

$i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n$, are symmetric and positive definite.

Proof. The function

$$F(\mathbf{z}) = \frac{1}{2}\,\frac{1}{(\mathbf{Az}, \mathbf{z})}\,\frac{\partial}{\partial z_i}(\mathbf{Az}, \mathbf{z}) = \frac{(\mathbf{a}_{i\cdot}^{\mathsf{T}}, \mathbf{z})}{(\mathbf{Az}, \mathbf{z})}$$

is positive real according to i) of Lemma 15. According to Lemma 16, its partial derivatives on the set $M = \{\mathbf{z} \in C^{(n)} : \operatorname{Re} \mathbf{z} = \mathbf{0}\}$ are positive and therefore for any real vector $\mathbf{x}$,

$$\text{(6)} \qquad \frac{\partial}{\partial z_j} \frac{(\mathbf{a}_{i\cdot}^{\mathrm{T}}, \mathbf{z})}{(\mathbf{A}\mathbf{z}, \mathbf{z})}\bigg|_M = \frac{-a_{ij}(\mathbf{A}\mathbf{x}, \mathbf{x}) + 2(\mathbf{a}_{i\cdot}^{\mathrm{T}}, \mathbf{x})(\mathbf{a}_{j\cdot}^{\mathrm{T}}, \mathbf{x})}{(\mathbf{A}\mathbf{x}, -\mathbf{x})^2} < 0\,.$$

The numerator in (6) is positive for all values $i, j \leqq n$ which has to be proved. ∎

It is a well known fact that the conclusion of Theorem 12 remains true if one of the determinants D_i is equal to zero. Making use of this, together with ii) of Lemma 15, we get the following

Theorem 18. *Let* $\mathbf{A}$ *be a Hurwitz matrix of order n and let* $\mathbf{l}$ *be any nonnegative n-dimensional vector. Then the function*

$$F(\mathbf{z}) = \frac{(\mathbf{A}\mathbf{z}, \mathbf{z})}{(\mathbf{l}, \mathbf{z})}$$

is positive real iff the bordering of the matrix $\mathbf{A}$ *by the vector* $\mathbf{l}$ *(as in Section 3) is a Hurwitz matrix.*

Theorem 19. *Let* $\mathbf{Q} = [q_{ik}]$ *be a nonnegative, square, symmetric matrix of order n such that* $q_{ii} = 0$, $i = 1, 2, \ldots, (n-1)$. *Then the positive definiteness of all the matrices* $\mathbf{B}_{ij}$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n)$ *is sufficient for* $\mathbf{Q}$ *to be a Hurwitz matrix.*

Proof. For any fixed i and j we shall say that the matrix $\mathbf{B}_{ij}$ belongs to the element a_{ij} of the matrix. $\mathbf{Q}$, Let us consider a principal submatrix $\mathbf{Q}^{(k)}$ of the matrix $\mathbf{Q}$ which contains the element a_{ij}, and let the matrix $\mathbf{B}_{ij}^{(k)}$ be constructed (it "belongs to the same elements a_{ij}" but involves elements of $\mathbf{Q}^{(k)}$ only). Then $\mathbf{B}_{ij}^{(k)}$ is a principal submatrix of $\mathbf{B}_{ij}$, hence $\mathbf{B}_{ij}^{(k)}$ is positive definite. It is easy to verify directly that for $n = 2$ our statement holds true. Let us suppose that it is true for some k, $2 \leqq k < n$. The k-th order matrix $\mathbf{Q}^{(k)}$ is therefore the matrix of a Hurwitz form, which may be considered to be the denominator of a rational function. Since its bordering satisfies the conditions of Corollary 18, all the partial derivatives (as in the proof of Theorem 17) of the function

$$F_k(\mathbf{z}) = \frac{(\mathbf{q}_{k+1\cdot}, \mathbf{z})}{(\mathbf{Q}^{(k)}\mathbf{z}, \mathbf{z})}$$

are positive and F_k is positive real. The matrix $\mathbf{Q}^{(k+1)}$ is therefore a Hurwitz matrix and our statement is proved by induction. ∎

The theorems proved so far show the Hurwitz forms to have some interesting special properties; some of these results have an immediate impact on problems of the multivariable positive real function theory and on the synthesis of electrical networks containing mixed types of elements. We intend to devote another paper to these applications.

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Souhrn

HURWITZOVY KVADRATICKÉ FORMY I

JIŘÍ GREGOR

V článku jsou formulovány nutné a postačující podmínky pro to, aby kvadratická forma v n komplexních proměnných a s reálnými koeficienty byla nenulová na kartézském součinu otevřených pravých polorovin, tj. na množině $\Gamma^{(n)} = \{z \in C^{(n)}, \operatorname{Re} z > 0\}$ a jsou studovány lineární transformace, které tuto vlastnost kvadratické formy zachovávají. Dokázaných podmínek je pak použito na nejjednodušší případy racionálních positivně reálných funkcí n komplexních proměnných s cílem jejich aplikace ve vícedimensionálním popisu pasivních elektrických obvodů se soustředěnými prvky.

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