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APPLICATION OF THE NORMAL DISTRIBUTION TO QUALITY CONTROL

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0. INTRODUCTION

Let the quality of products be classified by means of the departure of their measurement X from a given constant μ_0. A product is said to be one of the i-th class iff

$$X \in C_i = \{x : k_{i-1}\Delta \leqq |x - \mu_0| < k_i\Delta\}\,, \quad i = 1, \ldots, m\,,$$

where

$$\Delta > 0,\, k_0 < 0 < k_1 < k_2 < \ldots < k_m = \infty$$

are given constants.

Denote

$$B_i = \bigcup_{j=i}^{m} C_j = \{x : |x - \mu_0| \geqq k_{i-1}\Delta\}\,, \quad i = 2, \ldots, m\,.$$

Let given constants α_i, α_i^*, $i = 2, \ldots, m$ satisfy

$$0 < \alpha_m \leqq \alpha_{m-1} \leqq \ldots \leqq \alpha_2 < 1\,,$$

$$0 < \alpha_m^* \leqq \alpha_{m-1}^* \quad \ldots \leqq \alpha_2^* < 1\,,$$

$$\alpha_i < \alpha_i^*\,, \quad i = 2, \ldots, m\,.$$

Let X be normally distributed $N(\mu, \sigma^2)$, write $X \mathscr{L} N(\mu, \sigma^2)$. A large batch of inspected products should be accepted if

$$P(B_i) \leqq \alpha_i \quad \text{for all} \quad i = 2, \ldots, m\,, \tag{1}$$

while the adoption of them when

$$P(B_i) \geqq \alpha_i^* \quad \text{for at least one } i, \quad i = 2, \ldots, m\,, \tag{2}$$

would make the loss great. Therefore in this paper the optimal tests are formed in order to accept the inspected products if (1) is satisfied and to reject them if (2) holds, with probabilities larger than given numbers respectively, as well as to make the loss minimum.

We shall denote $\varphi(x) = (1/\sqrt{(2\pi)}) \exp(-x^2/2)$, and $\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,dt$ the density and distribution function of the normal distribution $N(0, 1)$. The following two Theorems will change the conditions (1) and (2) to conditions for μ and σ^2.

Theorem 1. (1) *is satisfied if and only if*

$$\sigma \leqq \sigma_0 = \min\{k_{i-1}\Delta/u_{\alpha_i} : 2 \leqq i \leqq m\}, \tag{3}$$

and

$$\left|\frac{\mu - \mu_0}{\sigma}\right| \leqq b = \min\{b_i : 2 \leqq i \leqq m\}, \tag{4}$$

where b_i is uniquely determined from

$$\Phi(k_{i-1}\Delta/\sigma + b_i) - \Phi(-k_{i-1}\Delta/\sigma + b_i) = 1 - \alpha_i, \quad i = 2, \ldots, m,$$

and u_α by $\Phi(u_\alpha) = 1 - \alpha/2$, i.e. $P(|N(0, 1)| \geqq u_\alpha) = \alpha$, $0 \leqq \alpha \leqq 1$.

Theorem 1 follows from the following Lemma.

Lemma 1. *Let $X \mathscr{L} N(\mu, \sigma^2)$. For*

$$P(|X - \mu_0| \leqq a) \geqq 1 - \alpha, a > 0, 0 < \alpha < 1, \tag{5}$$

it is necessary and sufficient that

$$\sigma \leqq a/u_\alpha \tag{6}$$

and

$$\left|\frac{\mu - \mu_0}{\sigma}\right| \leqq b, \tag{7}$$

where $b \geqq 0$ is uniquely determined by

$$\Phi(a/\sigma + b) - \Phi(-a/\sigma + b) = 1 - \alpha.$$

Proof.

$$P(|X - \mu_0| \leqq a) = P\left(-\frac{a}{\sigma} + \frac{\mu_0 - \mu}{\sigma} \leqq \frac{X - \mu}{\sigma} \leqq \frac{a}{\sigma} + \frac{\mu_0 - \mu}{\sigma}\right) =$$

$$= \Phi\left(\frac{a}{\sigma} + \frac{\mu_0 - \mu}{\sigma}\right) - \Phi\left(-\frac{a}{\sigma} + \frac{\mu_0 - \mu}{\sigma}\right) = G\left(\frac{\mu_0 - \mu}{\sigma}\right),$$

say, where $G(x) = G(|x|)$ and decreases strictly from $\Phi(a/\sigma) - \Phi(-a/\sigma)$ to 0 when $|x|$ increases from 0 to $+\infty$. Therefore (5) is satisfied iff $\Phi(a/\sigma) - \Phi(-a/\sigma) \geqq 1 - \alpha$, i.e. $a/\sigma \geqq u_\alpha$, and $|(\mu - \mu_0)/\sigma| \leqq b$ where $b \geqq 0$ is a solution of $G(b) = 1 - \alpha$.

Q.E.D.

Remark 1 (to Theorem 1). If $\sigma < \sigma_0$, then $b > 0$, and if $\sigma = \sigma_0$, then $b = 0$.

Theorem 2. (2) *holds iff either*

$$\sigma \geqq \sigma^* = \min \left\{ \frac{k_{i-1}\Delta}{\mu_{\alpha^*_i}} : i = 2, \ldots, m \right\} \tag{8}$$

or

$$\sigma < \sigma^* , \tag{9}$$

and

$$\left| \frac{\mu - \mu_0}{\sigma} \right| \geqq b^* = \min \{ b_i^* : i = 2, \ldots, m \} , \tag{10}$$

where $b_i^* > 0$ is a unique solution of

$$\Phi\left(\frac{k_{i-1}\Delta}{\sigma} + b_i^* \right) - \Phi\left(- \frac{k_{i-1}\Delta}{\sigma} + b_i^* \right) = 1 - \alpha_i^* , \quad i = 2, \ldots, m .$$

Proof. Note that Lemma 1 remains true if inequalities (5)–(7) are strict. Thus Theorem 1 with inequalities (1), (3), (4) being strict is also true. Hence the opposite statement to Theorem 1 appropriately modified affirms Theorem 2. Q.E.D.

Remark 2. Clearly $\sigma^* > \sigma_0$. Moreover $b_i^* > b_i$, $2 \leqq i \leqq m$ and then $b^* > b \geqq 0$ if $\sigma \leqq \sigma_0$.

Let us consider the test of $\mathscr{H}$: (1) is satisfied, against $\mathscr{K}$: (1) is not satisfied. We assume throughout the paper that σ^2 is known. In view of Theorem 1 there are two possibilities in the testing problem $(\mathscr{H}, \mathscr{K})$:

1) If $\sigma^2 > \sigma_0^2$ we accept $\mathscr{K}$ without any observation and without any loss,

2) If $\sigma^2 \leqq \sigma_0^2$ we have to test $\mathscr{H} : 0 \leqq c_n \leqq b$, i.e. (4) holds, against $\mathscr{K} : c_\mu < b$, where $c_\mu = |(\mu - \mu_0)/\sigma|$.

2. THE TESTS

Let $X_1, \ldots, X_n$ be a simple sample from X, where $X \mathscr{L} N(\mu, \sigma^2)$ with $\sigma^2 \leqq \sigma_0^2$. Denote

$$\overline{X} = (1/n) \sum_{i=1}^{n} X_i .$$

For $\mu \in \mathscr{H}$ and arbitrary probability e we denote by D the event

$$\left\{ \left| \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \right| \leqq u_e \right\} = \left\{ - u_e \leqq \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \leqq u_e \right\} \subset \left\{ -u_e - b\sqrt{n} \leqq \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} + \right.$$

$$\left. + \frac{\mu - \mu_0}{\sigma/\sqrt{n}} \leqq u_e + b\sqrt{n} \right\} = \left\{ \left| \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \right| \leqq u_e + b\sqrt{n} \right\} = D .$$

Therefore

$$\bar{D} = \left\{\left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| > u_e + b\sqrt{n}\right\} \subset \left\{\left|\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right| > u_e\right\},$$

and

(11) $$P(\bar{D}) = 1 - [\Phi(u_e + (c_\mu + b)\sqrt{n}) - \Phi(-u_e + (c_\mu - b)\sqrt{n})]$$

for each $\mu \in \mathscr{H}$. Since

$$P(|(\bar{X} - \mu_0)/(\sigma/\sqrt{n})| > x) = 1 - [\Phi(x + c_\mu\sqrt{n}) - \Phi(-x + c_\mu\sqrt{n})]$$

is increasing in $c_\mu \geqq 0$ for each $x \in R$, then $|(\bar{X} - \mu_0)/(\sigma/\sqrt{n})|$ is stochastically increasing in $c_\mu \geqq 0$ (see [1], chapter II. 7). Therefore the critical region of the form $\{|(\bar{X} - \mu_0)/(\sigma/\sqrt{n})| > \lambda\}$ is meaningful in our testing problem.

Theorem 3. *In testing $(\mathscr{H}, \mathscr{K})$ above, the test defined by the critical region*

(12) $$\bar{D} = \left\{\left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| > u_e + b\sqrt{n}\right\},$$

has the level $\delta \stackrel{\text{def}}{=} \sup\{P(\bar{D}) \mid \mathscr{H}\}$ *determined by*

(13) $$\delta = 1 - [\Phi(u_e + 2b\sqrt{n}) - \Phi(-u_e)] = e/2 + [1 - \Phi(u_e + 2b\sqrt{n})] =$$
$$= e - [\Phi(u_e + 2b\sqrt{(n)} - \Phi(u_e)],$$

and the probability of error of the first kind δ has the following properties.

(14) (i) $e/2 < \delta < e$,

(ii) $e - \delta$ *runs* $0 \uparrow [\Phi(2b\sqrt{n}) - 1/2]$ *as* e *runs* $0 \uparrow 1$,

(iii) δ *runs* $0 \uparrow 1 - [\Phi(2b\sqrt{n}) - 1/2]$ *as* e *runs* $0 \uparrow 1$,
and $\delta = \delta(e)$ *is convex,*

(iv) δ *runs* $e \downarrow e/2$ *as* n *runs* $0 \uparrow \infty$, *and* $\delta = \delta(n)$, *for* $n \in R^+$, *is convex.*

Proof. It is easy to see that $P(\bar{D})$ determined by (11) is increasing in c_μ and also $\sup\{P(\bar{D}) | \mathscr{H}\} = P\{\bar{D} | c_\mu = b\}$. Thus (13) follows from (11). (14) (i)–(ii) follow immediately from (13). (14)(iii)–(iv) follow from the derivations of (13) according to e and n respectively. In fact

$$\delta'_e = \tfrac{1}{2}\left[1 + \frac{\varphi(u_e + 2b\sqrt{n})}{\varphi(u_e)}\right] > 0,$$

$$\delta''_e = b\sqrt{(n)}\,\varphi(u_e + 2b\sqrt{n})/2\varphi^2(u_e) > 0,$$

and

$$\delta'_n = -b\varphi(u_e + 2b\sqrt{n})/\sqrt{n} < 0,$$

$$\delta''_n = (b/n)\,\varphi(u_e + 2b\sqrt{n})\,[b(u_e + 2b\sqrt{n}) + 1/2\sqrt{n}] > 0,$$

where we have used the relations

$$\Phi'(x) = \varphi(x)\,, \quad \varphi'(x) = -x\,\varphi(x)\,, \quad \Phi(u_e) = 1 - e/2$$

then $\mathrm{d}u_e/\mathrm{d}e = -1/2\,\varphi(u_e)$. Q.E.D.

The behaviour of δ is illustrated in Figures 1 and 2.

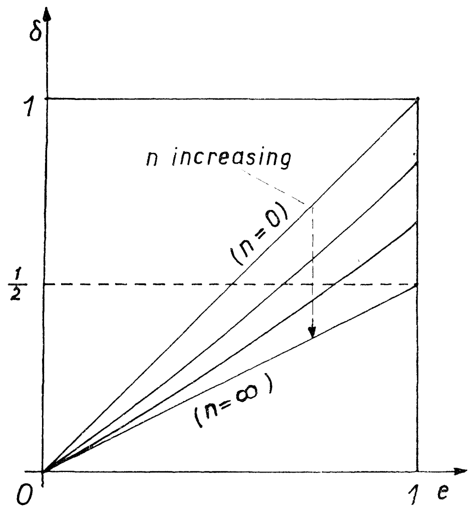

Fig. 1.

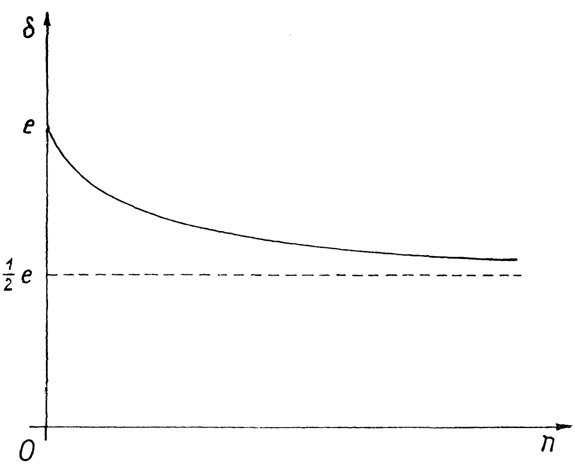

Fig. 2.

Now let us consider the error probability of the second kind

(15) $$\beta(\mu) = P(D \mid \mu) = \Phi[(c_\mu + b)\sqrt{(n)} + u_e] - \Phi[(c_\mu - b)\sqrt{(n)} - u_e]\,,$$

for $\mu \in \mathscr{K}$, i.e. $c_\mu > b$.

Theorem 4. *The test defined in Theorem 3 is unbiased, i.e.* $\gamma(\mu) = 1 - \beta(\mu) > \delta$, for $\mu \in \mathscr{K}$.

Moreover

(16) (i) $\beta = \beta(\mu)$ *runs* $1 - \delta - \downarrow 0$ as c_μ *runs* $b + \uparrow \infty$,

(ii) *for each* $\mu \in \mathscr{K}$, *putting* $x = \sqrt{n} \in R^+$, $\beta = \beta(x) \uparrow$ *as* $x \uparrow x_0 = (-u_e + \sqrt{(u_e^2 + (2b/c_\mu)\log((c_\mu + b)/(c_\mu - b)))}/2b$, *and* $\beta(x) \downarrow 0$ *as* x *runs* $x_0 \uparrow \infty$,

(iii) *for each* $\mu \in \mathscr{K}, n \in N^+$,
$\beta = \beta(e)$ *runs* $1 \downarrow \Phi[(c_\mu + b)\sqrt{n}] - \Phi[(c_\mu - b)\sqrt{n}$ *as e runs* $0 \uparrow 1$,
$+$ *if* $b = 0$, $\beta(e)$ *is convex in* $e \in [0, 1]$,
$+$ *if* $b > 0$:

$\beta(e)$ *is convex for* $0 \leqq e \leqq 1$ *if* $c_\mu > \bar{c}$, *and it is convex for* $0 \leqq e < \tilde{e}$ *and concave for* $\tilde{e} \leqq e \leqq 1$ if $b < c_\mu \leqq \bar{c}$, *where* $\bar{c} = \bar{c}(n) > b$ *is a unique solution of* $n\bar{c}b = \frac{1}{2}\log((\bar{c} + b)/(\bar{c} - b))$, *and* $\tilde{e}$ *is determined by*
$u_{\tilde{e}} = -b\sqrt{(n)} + (1/(2\sqrt{(n)}\, c_\mu))\log((c_\mu + b)/(c_\mu - b))$.

Proof. From (13) and (15) we have $\beta(\mu) < 1 - \delta$ for $c_\mu > b$. (16) (i) follows from (15). The derivative of $\beta = \beta(x)$ is

$$\begin{aligned}\beta'(x) &= (c_\mu + b)\,\varphi[(c_\mu + b)\,x + u_e] - (c_\mu - b)\,\varphi[(c_\mu - b)\,x - u_e] \\ &= (c_\mu - b)\,\varphi[u_e + (c_\mu + b)\,x]\,H(x),\end{aligned}$$

where $H(x) = (c_\mu + b)/(c_\mu - b) - \exp(2bc_\mu x^2 + 2u_e c_\mu x)$ is decreasing from $H(0) = 2b/(c_\mu - b) > 0$ to $-\infty$ as x runs $0 \uparrow \infty$. So there is a unique positive solution

x_0 of $H(x) = 0 : x_0 = (1/(2b)(-u_e + \sqrt{(u_e^2 + (2b/c_\mu)\log((c_\mu + b)/(c_\mu - b)))})$.

Since the signs of $\beta'(x)$ and $H(x)$ are the same for $\mu \in \mathscr{K}(c_\mu > b)$, (16) (ii) is proved. For $\beta = \beta(e)$, we have

$$\begin{aligned}\beta'(e) &= (-1/2\,\varphi(u_e))\,[\varphi((c_\mu + b)\sqrt{(n)} + u_e) + \varphi((c_\mu - b)\sqrt{(n)} - u_e)] < 0\,, \\ \beta''(e) &= (-\sqrt{n}/4\,\varphi^2(u_e))\,[(c_\mu + b)\,\varphi((c_\mu + b)\sqrt{(n)} + u_e) - \\ &\quad - (c_\mu - b)\,\varphi((c_\mu - b)\sqrt{(n)} - u_e)] = \\ &= \sqrt{(n)}\,\varphi(-u_e + (c_\mu - b)\sqrt{n})/4\varphi^2(u_e)]\,h(e)\,,\end{aligned}$$

where

$$h(e) = c_\mu - b - (c_\mu + b)\exp\{-2\sqrt{(n)}\,c_\mu(u_e + b\sqrt{n})\}\,.$$

Since $h'(e) = -\sqrt{(n)}\,c_\mu(c_\mu + b)\exp\{-2\sqrt{(n)}\,c_\mu(u_e + b\sqrt{n})\}/\varphi(u_e) < 0$, then $h(e)\downarrow$. One has $h(o) = \lim_{u_e \to +\infty} h(e) = c_\mu - b > 0$, for $\mu \in \mathscr{K}$,

$$h(1) = h(e)|_{u_e = 0} = c_\mu - b - (c_\mu + b)\exp(-2bnc_\mu)\,.$$

Clearly for $c_\mu > b$ the signs of $h(1)$ and of $f(c_\mu) = nc_\mu b - (1/2)\log(c_\mu + b)/(c_\mu - b)$ are the same. Since

$$f'(c_\mu) = nb - (\tfrac{1}{2})(1/(c_\mu + b) - 1/(c_\mu - b)) = b(n + 1/(c_\mu^2 - b^2)) > 0$$

for $\mu \in \mathscr{K}$, and $b > 0$; and $f(b+) = -\infty$, $f(\infty) = \infty$, there is a unique $\bar{c} = \bar{c}(n)$ such that the signs of $f(c_\mu)$ and $c_\mu - \bar{c}$ are the same. Thus $h(1) \geqq 0$ if $c_\mu \geqq \bar{c}$, $h(1) < 0$

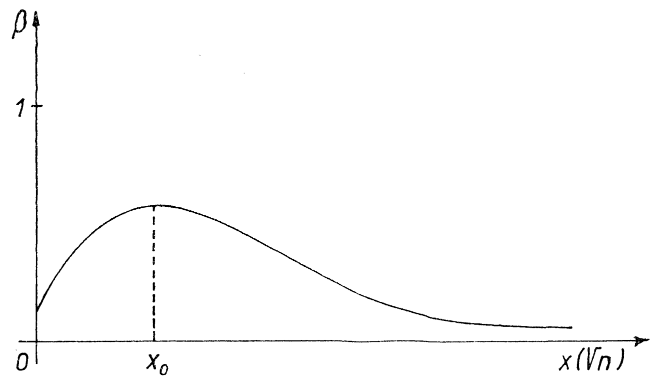

Fig. 3.

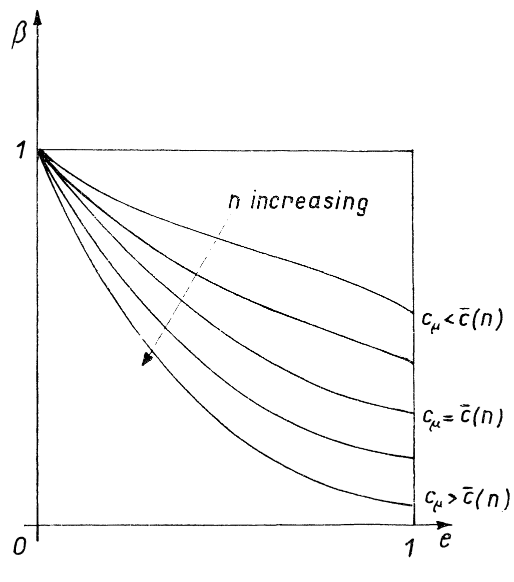

Fig. 4.

if $b < c_\mu < \bar{c}$, provided $b > 0$, i.e. $\sigma^2 < \sigma_0^2$, see Remark 1. If $b = 0$, i.e. $\sigma^2 = \sigma_0^2$, $h(1) \equiv 0$ for all $c_\mu > 0$. Since $\beta''(e)$ and $h(e)$ have the same sign we have finally:

In the case $b = 0$: $\beta''(e) \geqq 0$ and $h(e) \geqq 0$, $e \in [0, 1]$, for all $c_\mu > b$, In the case $b > 0 : \beta''(e) \geqq 0$ and $h(e) \geqq 0$, $e \in [0, 1]$, if $c_\mu \geqq \bar{c}$; and $\beta''(e)$ and $h(e)$ change their sign from positive to negative when e crosses the solution $\tilde{e}$ of $h(e) = 0$, if $b < c_\mu < \bar{c}$. That proves (16) (iii). Q.E.D.

Figures 3 and 4 illustrate the behaviour of β.

3. THE OPTIMAL INSPECTION PLANS

Suppose $X \mathscr{L} N(\mu, \sigma^2)$ with $\sigma^2 \leqq \sigma_0^2$ as in Section 2. Then $\sigma^2 < \sigma^{*2}$, by Remark 2. One can see from Theorem 2 and (16) (i) of Theorem 4 that $\beta(\mu^*)$, where $c_{\mu*} = b^*$, is the largest error probability we can make if we accept the inspected batch when (2) holds. So our purpose in this Section is to form the tests with the smallest sample size to make either 1) the error probabilities δ and $\beta(\mu^*)$ smaller than given numbers δ_0, and $\beta_0 \in (0, 1)$, or 2) the risk of the form $t = t_1\delta + t_2\,\beta(\mu^*) + t_3 n =$ minimum. Such tests will be called optimal tests in the first or second sense respectively.

a) The first case

The set of tests defined by the critical region $\bar{D}$ in (12) is equivalent to the set $\{(e, n) \in [0, 1] \times N^+\}$. Let G, $G \subset [0, 1] \times N^+$, be the set of all solutions of

$$\begin{aligned}(17) \qquad & \delta = (e/2) + [1 - \Phi(u_e + 2b\sqrt{n})] \leqq \delta_0\,,\\ & \beta(\mu^*) = \Phi[u_e + (b^* + b)\sqrt{n}] - \Phi[-u_e + (b^* - b)\sqrt{n}] \leqq \beta_0\,.\end{aligned}$$

Then $G_0 = G \cap \{[0, 1] \times \{n_0\}\}$, where $n_0 = \min\{n : (e, n) \in G\}$, is the set of all optimal solutions in the first sense.

Put for $(e, x) \in [0, 1] \times [0, \infty)$:

$$\delta(e, x) = (e/2) + [1 - \Phi(u_e + 2b\sqrt{x})]\,,$$

$$\beta(e, x) = \Phi[u_e + (b^* + b)\sqrt{x}] - \Phi[-u_e + (b^* - b)\sqrt{x}]\,.$$

Denote $\mathrm{G}_{\delta\beta}$ to be the set of all (e, s) such that

$$\delta(e, x) \leqq \delta_0\,,$$

$$\beta(e, x) \leqq \beta_0\,.$$

Let $\tilde{x} = \min\{x : (e, x) \in G_{\delta\beta}\}$.

Denote $\tilde{G}_{\delta\beta} = G_{\delta\beta} \cap \{[0, 1] \times \{\tilde{x}\}\}$.

Theorem 5. *Suppose* $0 < 2\delta_0 < 1 - \beta_0 < 1$. *Then*

(i) $\tilde{G}_{\delta\beta}$ *consists of a single point* $(\tilde{e}, \tilde{x})$, and $\delta_0 < \tilde{e} < 2\delta_0$.

(ii) *The optimal test* (e_0, n_0) *exists uniquely iff* $\tilde{x} \in N^+$. *In this case* $(e_0, n_0) = (\tilde{e}, \tilde{x})$.

(iii) *If* $\tilde{x} \notin N^+$, *the set of optimal tests is* $G_0 = [e_1, e_2] \times \{[\tilde{x}] + 1\}$, *where* $e_1 < e_2$

are determined uniquely by

$$\beta(e_1, [\tilde{x}] + 1) = \beta_0 ,$$
$$\delta(e_2, [\tilde{x}] + 1) = \delta_0 ,$$

where $[\tilde{x}]$ *stands for the integer part of* $\tilde{x}$.

Proof. Clearly $\delta(e, x) < \delta_0$ if $0 \leqq e \leqq \delta_0$, by (14)(i); $\delta(e, x) > \delta_0$ if $2\delta_0 \leqq e \leqq 1$, and $\delta(e, x) = \delta_0$ has a unique solution $x = x_e$ increasing strictly from 0 to $+\infty$ as e increases from δ_0 to $2\delta_0$, by (14)(iii–iv). Put

$$x = x(e) = \begin{matrix} 0, 0 \leqq e \leqq \delta_0 , \\ x_e, \delta_0 < e < 2\delta_0 . \end{matrix}$$

The set of all solutions of $\delta(e, x) \leqq \delta_0$ is

$$G_\delta = \{(e, x) : e \in [0, 2\delta_0), x \geqq x(e)\} ,$$

the equation of the lower boundary $\tilde{G}_\delta$ of which is $x = x(e)$. Similarly, by (16)(ii–iii), for $e \in [0, 1 - \beta_0)$ the equation $\beta(e, x) = \beta_0$ has a unique solution x_e^* decreasing from $+\infty$. (For $e \geqq 1 - \beta_0$ there is another solution $x_e^{**}(<x_e^*)$ increasing from 0. If $1 > \hat{e} > 1 - \beta_0$ such that $x_{\hat{e}}^* = x_{\hat{e}}^{**}$ then for $e > \hat{e}$ the equation $\beta(e, x) = \beta_0$ has no solution. But we do not need this fact.) Thus the set of solutions of $\beta(e, x) \leqq \beta_0$ for $e \in [0, 1 - \beta_0)$ is

$$G_\beta = \{(e, x) : e \in [0, 1 - \beta_0), x \geqq x_e^*\} ,$$

and $x = x(e) = x_e^*$ represents the lower boundary $\tilde{G}_\beta$ of G_β. From that one can easily get the results of Theorem 5, noting that $G_{\delta\beta} = G_\delta \cap G_\beta$ and $\tilde{x} = \tilde{G}_\delta \cap \tilde{G}_\beta$. Q.E.D.

Remark 3. In applications both δ_0 and β_0 are often small and satisfy $2\delta_0 < 1 - \beta_0$. Thus $\tilde{G}_{\delta\beta}$ consists of a single point $(\tilde{e}, \tilde{x})$ with $\delta_0 < \tilde{e} < 2\delta_0$, and $(\tilde{e}, \tilde{x})$ may be found approximately as follows:

Put $e_1 = \delta_0 + \frac{1}{2}\delta_0$,

and

$$e_i = \begin{matrix} e_{i-1} + \delta_0/2^i , & \text{if} & x_{e_{i-1}} < x_{e_{i-1}}^* , \\ e_{i-1} - \delta_0/2^i , & \text{if} & x_{e_{i-1}} > x_{e_{i-1}}^* , \end{matrix}$$

$$i = 2, 3, \ldots, \quad \text{provided} \quad x_{e_j} \neq x_{e_j}^* , \quad j = 1, \ldots, i - 1 .$$

If $x_{e_i} = x_{e_i}^*$ then $(\tilde{e}, \tilde{x}) = (e_i, x_{e_i})$. If $x_{e_i} \neq x_{e_i}^*$, then

$$G_{\delta\beta} \ni (e_i, \max(x_{e_i}, x_{e_i}^*)) \approx (\tilde{e}, \tilde{x}) \quad \text{with} \quad |e_i - \tilde{e}| < \delta_0/2^i \to 0 ,$$

and

$$|\tilde{x} - \max(x_{e_i}, x_{e_i}^*)| < |x_{e_i} - x_{e_i}^*| \to 0 .$$

b) The second case

Let

$$t = t(u_e, n) = t_1\delta + t_2\beta(\mu^*) + t_3 n = \tag{18}$$
$$= t_1[\tfrac{1}{2}e + 1 - \Phi(u_e + 2b\sqrt{n})] +$$
$$+ t_2[\Phi(u_e + (b^* + b)\sqrt{n}) - \Phi(-u_e + (b^* - b)\sqrt{n})] + t_3 n$$

where t_1, t_2, t_3 are positive constants.

Let $T = \{(u_e, n) : t(u_e, n) = \min, e \in [0, 1], n \in N^+\}$. Then $(\tilde{u}_e, \tilde{n}) \in T$, with $\tilde{n} = \min\{n : (u_e, n) \in T\}$, determines the optimal test in the second sense.

As n is not continuous, we may first find, for given $n \in N^+$, the $\tilde{u}_e = \tilde{u}_e(n)$ which minimizes t.

Clearly

$$t'_{u_e} = -[\varphi(u_e) + \varphi(u_e + 2b\sqrt{n})]\, t_1 + [\varphi(u_e + (b^* + b)\sqrt{n}) + \tag{19}$$
$$+ \varphi(u_e - (b^* - b)\sqrt{n})]\, t_2\,,$$

and $t'_{u_e} = 0$ if and only if

$$k(u_e, n) \overset{\text{def}}{=} \frac{\varphi(u_e + (b^* + b)\sqrt{n}) + \varphi(u_e - (b^* - b)\sqrt{n})}{\varphi(u_e) + \varphi(u_e + 2b\sqrt{n})} = \frac{t_1}{t_2}\,. \tag{20}$$

Lemma 2. *For* $h > 0$,

$$\varphi(x)\,\varphi(y) - \varphi(x + h)\,\varphi(y + h) > 0 \quad \text{if} \quad x + y + h > 0\,,$$
$$= 0 \quad \text{if} \qquad = 0\,,$$
$$< 0 \quad \text{if} \qquad < 0\,.$$

Proof. Since

$$\sqrt{(2\pi)}\,[\varphi(x)\,\varphi(y) - \varphi(x + h)\,\varphi(y + h) =$$
$$= \exp[-\tfrac{1}{2}(x^2 + y^2)]\,\{1 - \exp[-h(h + x + y)]\}\,,$$

and for $h > 0$, $1 - \exp[-h(h + x + y)]$ has the same sign as $h + x + y$, the Lemma follows. Q.E.D.

Lemma 3. *Put*

$$k(x) = \frac{\varphi(x - a) + \varphi(x + a + c)}{\varphi(x) + \varphi(x + c)}\,, \quad a > 0\,, \quad c \geqq 0\,.$$

Then

$$\text{(i)}\ k'(x) > 0 \quad \text{if} \quad x + \tfrac{1}{2}c > 0\,,$$
$$= 0 \quad \text{if} \qquad = 0\,,$$
$$< 0 \quad \text{if} \qquad < 0\,,$$
$$\text{(ii)}\ \lim_{x \to \infty} k(x) = +\infty\,.$$

Proof. As $\varphi'(x) = -x\,\varphi(x)$,

$$[\varphi(x) + \varphi(x + c)]^2\, k'(x) =$$

$$= -[(x - a)\,\varphi(x - a) + (x + a + c)\,\varphi(x + a + c)]\,[\varphi(x) + \varphi(x + c)] +$$

$$+ [\varphi(x - a) + \varphi(x + a + c)]\,[x\,\varphi(x) + (x + c)\,\varphi(x + c)] =$$

$$= a[\varphi(x - a)\,\varphi(x) - \varphi(x + c)\,\varphi(x + a + c)] +$$

$$+ (a + c)\,[\varphi(x - a)\,\varphi(x + c) - \varphi(x)\,\varphi\,(x + a + c)]\,.$$

Applying Lemma 2, one gets (i).

In order to prove (ii), note that

$$k(x) = \{\exp(ax - a^2/2) + \exp[-\tfrac{1}{2}(a + c)^2 - (a + c)x]\} :$$

$$: \{1 + \exp(-c^2/2 - cx)\}\,. \qquad \text{Q.E.D.}$$

Theorem 6. *Let $k = k(u_e, n)$ be defined in (20).*

(i) *k increases strictly from $k(0, n)$ to ∞ as u_e from 0 to ∞ for each $n \in N^+$, where*

$$0 < k(0, n) = \frac{\varphi((b^* + b)\sqrt{n}) + \varphi((b^* - b)\sqrt{n})}{(1/\sqrt{(2\pi)}) + \varphi(2b\sqrt{n})} < 1\,. \tag{21}$$

(ii) *The equation (20) has a solution $u_e = u_e^*(n)$ if and only if*

$$t_1/t_2 \geqq k(0, n)\,. \tag{22}$$

In this case, $u_e^*(n)$ is unique, and

$$u_e^*(n) > 0 \quad \text{if} \quad t_1/t_2 > k(0, n)\,.$$

$$= 0 \quad \text{if} \qquad\quad = k(0, n)\,,$$

(iii) *t attains its minimum in $u_e = u_e^*(n)$ for n satisfying (22) and in $u_e = 0$ for n not satisfying (22).*

Proof. (i) follows from Lemma 3, putting $x = u_e$, $a = (b^* - b)\sqrt{n}$, $c = 2b\sqrt{n}$. (ii) is a consequence of (i). In order to prove (iii) note that for n satisfying (22), $k(u_e, n) - t_1/t_2$ and then $t'_{u_e} < 0$ or >0 if $u_e - u_e^*(n) < 0$ or >0; and for n not satisfying (22), $k(u_e, n) > t_1/t_2$ and then $t'_{u_e} > 0$ for $u_e \geqq 0$.

Let
$$\tilde{u}_e(n) = \begin{matrix} u_e^*(n) & \text{if} & k(0, n) < t_1/t_2\,, \\ 0 & \text{if} & \geqq t_1/t_2\,. \end{matrix}$$

Then from (ii) and (iii) t attains its minimum in a unique $u_e = \tilde{u}_e(n)$ for each $n \in N^+$.

Q.E.D.

Put

(23) $$t_0 = \min \{t(\tilde{u}_e(n), n) : 1 \leqq n \leqq [(t_1 + t_2)/t_3] + 1\} ,$$

(24) $$\tilde{n} = \min \{n : t(\tilde{u}_e(n), n) = t_0, 1 \leqq n \leqq [(t_1 + t_2)/t_3] + 1\} .$$

Theorem 7. *$(\tilde{u}_e(\tilde{n}), \tilde{n})$ is a unique optimal solution in the second sense. Especially, if $(t_1 + t_2)/t_3 < 1$, the optimal solution is $(\tilde{u}_e(1), 1)$.*

Proof. From (18) one has

$$t(\tilde{u}_e(n_1), n_1) = \min_{u_e \geqq 0} t(u_e, n_1) < t_1 + t_2 + t_3 n_1 ,$$

$$t(\tilde{u}_e(n_2), n_2) = \min_{u_e \geqq 0} t(u_e, n_2) > t_3 n_2 .$$

Thus $t(\tilde{u}_e(n_1), n_1)\ t(\tilde{u}_e(n_2), n_2)$ if $(t_1 + t_2)/t_3 \leqq n_2 - n_1$. Therefore t_0 defined in (23) is equal to $\min \{t(\tilde{u}_e(n), n) : n \in N^+\} = \min \{t(u_e, n) : (u_e, n) \in [0, 1] \times N^+\}$.

Q.E.D.

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Souhrn

APLIKACE NORMÁLNÍHO ROZLOŽENÍ NA KONTROLU JAKOSTI

NGUYEN VAN HO

Článek podává nové výsledky z teorie statistických přejímek měření za předpokladu normálního rozložení zkoumaného znaku jakosti při testování hypotézy o přípustném, resp. nepřípustném podílu výrobků v dávce se znakem mimo předepsanou toleranci. Jsou odvozeny optimální přejímací plány pro případ, kdy výrobky jsou klasifikovány do m tříd podle velikosti odchylek od předepsané střední hodnoty.

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