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# ON THE COMPUTATIONAL COMPLEXITY OF CENTERS LOCATING IN A GRAPH

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## 1. INTRODUCTION

The purpose of this paper is to show that certain problems of centers locating in communication networks are very hard. This will be demonstrated by showing their NP-completeness.

We shall consider finite undirected graphs without loops or multiple edges. Given a graph $G$, $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. In general, each $v \in V(G)$ has a vertex weight $w(v)$ (measure of importance) and each edge has a weight (length). All the weights are positive integers. By an unweighted graph we mean one with all weights equal to 1. An edge joining vertices u and v is denoted by uv. The distance $d(v, S)$ of a vertex v from a set $S$ of points of $G$ is the minimum distance of v from a point of $S$.

All the problems we are going to study have many applications [1, 2]. Here we consider an interpretation in terms of communication networks. If $V(G)$ represents a collection of cities and an edge represents a communication link, then one may be interested in selecting a set $B$ of cities as sites for hospitals, stores, transmitting stations, etc. Various problems can arise. (1) The minimum $k$-basis problem: Given a number $k$, we are required to find a set $B$ of minimum cardinality such that $d(v, B) . w(v) \leqq k$ for every vertex $v$. (Note that a 1-basis is often called a dominating set.) (2) The $m$-center problem: Given a number $m$, find a set $B$ of cardinality $m$ such that $\max \{d(v, B)\, w(v) \mid v \in V(G)\}$ is minimum. (3) The $p$-median problem: Given a number $p$, find a set $B$ of cardinality $p$ such that $\sum d(v, B)\, w(v)$ is minimum (the sum is taken over all $v \in V(G)$). These are "basic" problems. We shall refer to "absolute" variants of the problems whenever $B$ can contain points on the edges (cf. [1, 2]).

All these problems are known from the literature to be very difficult. Therefore good algorithms for them are available only in some simple cases. See [16] for the minimum $k$-basis problem and for a generalization of it; for the $m$-center problem see [8, 9, 10]; the $p$-median problem can be found in [8, 9, 12]. Both the $m$-center and the $p$-median problems are discussed in the books [1, 2]; a generalization and further references can be found in [13]. The authors of [14] presented a branch-and-bound algorithm for solving the $p$-median problem; the bounds are obtained by solving the Lagrangian relaxation of the problem using the subgradient optimization method. They reported good computational experience; however, no theoretical time bounds that would support them are known.

We show that probably no good, i.e. polynomial time bounded (in the size of input) algorithms exist for these problems by proving their NP-completeness (see [11] for this notion). If good algorithms for these problems existed, there would be good algorithms for such problems as the traveling salesman, discrete multicommodity flow, Steiner tree, and many others [11]. Several of these problems remain NP-complete even when their domains are restricted: the traveling salesman problem in the plane with a natural discretized version of the Euclidean metric [3] or in the case of unweighted, cubic, planar graphs [7] (for others see e.g. [6]). On the other hand, for some "exact" problems also "near optimum" variants are NP-complete (e.g. determining the chromatic number of a graph [4]). We shall show that the $m$-center problem is of this kind. Some other problems (traveling salesman, discrete multicommodity flow and others), even in the case of a "weak" approximation, are NP-complete [15].

The usual method of showing that a problem is NP-complete is (i) to check that it is in NP, and then (ii) to show that a certain known NP-complete problem is reducible to it in polynomial time [11].

The first task is easy for all our problems. Indeed, the basic variants can be formulated as $0-1$ programming problems. Further, it is well-known that the problem of finding an absolute $p$-median reduces to the problem of finding a $p$-median only [9, 1, 2]. Finally, to find an absolute $m$-center, it is sufficient to consider only a certain (polynomial) number of local centers on every edge [8, 1, 2], consequently, the absolute $m$-center problem is in NP. The same arguments show that the absolute $k$-basis problem is in NP as well.

The second task will be accomplished below. It consists of a proper (polynomial) reduction of the following NP-complete cover problem [5] to a given problem.

Planar vertex cover with vertex degree at most 3 (PVC-3): Given a planar unweighted graph $G$ with vertex degrees at most 3, find a minimum cardinality set $S \subset V(G)$ such that every edge of $G$ is incident with a vertex of $S$.

## 2. THE MINIMUM $k$-BASIS PROBLEM

**Theorem 2.1.** *Both the basic and the absolute variants of the $k$-basis problem are* NP-*complete even in the case of unweighted planar graphs with the maximum degree* 3 *and* $k = 1$.

Proof. Using an idea from [5], we reduce the PVC-3 problem to our problem. Given a planar graph $G$ with the maximum degree 3 (the degree constraint will not be used, however), we construct a planar graph $G$ with no vertex degree exceeding 3 such that any minimum (absolute) 1-basis $B$ of $\hat{G}$ gives a minimum vertex cover $S$ of $G$.

The construction begins with a fixed planar representation of $G$. (The reader can well understand the proof with the aid of the example in Fig. 1.) In this diagram we first replace each vertex $v_i$ of $G$ with a cycle $C_i = v_i(1)\ x_i(1)\ y_i(1)\ v_i(2)\ x_i(2)\ y_i(2) \ldots v_i(d_i)\ x_i(d_i)\ y_i(d_i)\ v_i(1)$, where $d_i$ is the degree of $v_i$. (It does not matter whether $C_i$ is clockwise or conversely ordered.) If $v_iv_j \in E(G)$ and $i < j$, then a vertex $v_i(t_{ij})$ of $C_i$ and a vertex $v_j(t_{ji})$ of $C_j$ will be joined by an edge. Here $t_{ij}$ and $t_{ji}$ are

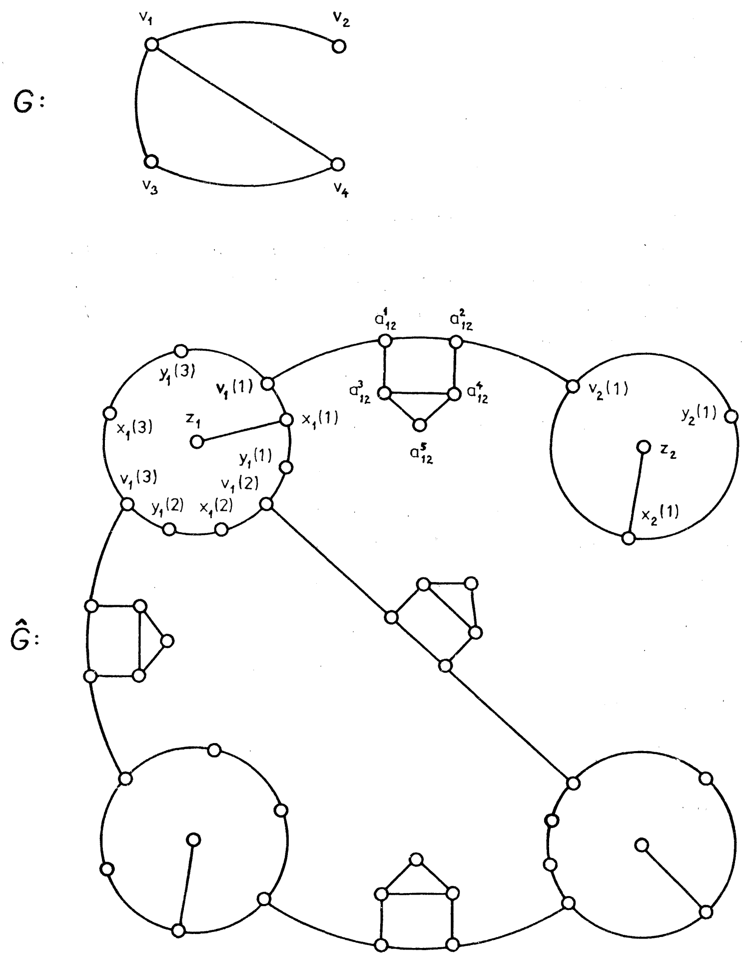


Fig. 1.

chosen in such a way that the resulting diagram is again planar, $t_{ij} \neq t_{ij'}$ whenever $j \neq j'$, $t_{ji} \neq t_{ji'}$ whenever $i \neq i'$, and after contracting each cycle $C_i$ to a single vertex $v_i$, we must obtain the original diagram of $G$. Further, for each $i$ we add a vertex $z_i$ and the edge $x_i(1)\, z_i$. Finally, every edge $v_i(t_{ij})\, v_j(t_{ji})$, $i < j$, is replaced by a subgraph $L_{ij}$ consisting of the edges: $v_i(t_{ij})\, a_{ij}^1$, $a_{ij}^1 a_{ij}^2$, $a_{ij}^2 v_j(t_{ji})$, $a_{ij}^1 a_{ij}^3$, $a_{ij}^2 a_{ij}^4$, $a_{ij}^3 a_{ij}^4$, $a_{ij}^3 a_{ij}^5$ and $a_{ij}^4 a_{ij}^5$. The obtained graph is denoted by $\hat{G}$. We see that if $G$ has $n$ vertices and $e$ edges, then $\hat{G}$ has $n + 3 \sum d_i + 5e = n + 11e$ vertices and $n + 3 \sum d_i + 8e = n + 14e$ edges. Clearly, $\hat{G}$ is planar with the maximum degree 3.

Now, let $B$ be a minimum (basic or absolute) 1-basis of $\hat{G}$. We can suppose that $B \subset V(\hat{G})$ (otherwise, replacing every point of $B$ by a closest vertex, we obtain a minimum 1-basis with the property; i.e., no absolute 1-basis is less than a minimum basic 1-basis). Further, we can obviously assume that no $z_i$ is in $B$, which implies that every $x_i(1) \in B$. To dominate the vertices $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^5$, we need at least two vertices from $L_{ij}$. As $v_i(t_{ij})$ and $a_{ij}^4$ or $a_{ij}^3$ and $v_j(t_{ji})$ suffice, we can assume that $B$ contains neither $a_{ij}^1$ nor $a_{ij}^2$ and that

$$\{v_i(t_{ij}), v_j(t_{ji})\} \cap B \neq \emptyset \,. \tag{2.1}$$

To dominate the vertices of $C_i$, at least one of the three vertices $v_i(t)$, $x_i(t)$, and $y_i(t)$ must occur in $B$ for every $t = 1, 2, \ldots, d_i$. Hence $B$ contains at least $d_i$ vertices from every $C_i$. Moreover, as $x_i(1) \in B$ and $\{a_{ij}^1, a_{ij}^2\} \cap B = \emptyset$, we see that $|B \cap C_i| = d_i$ only if $x_i(1), \ldots, x_i(d_i) \in B$. Thus, if there exists a $t$ with $v_i(t) \in B$, we must have $|B \cap C_i| \geqq d_i + 1$. Define the set

$$S = \{v_i \mid v_i \in V(G),\ |B \cap C_i| \geqq d_i + 1\} \,.$$

Since (2.1) holds, we see that $S$ covers all edges of $G$. Clearly,

$$|B| \geqq e + \sum d_i + |S| = 3e + |S| \,. \tag{2.2}$$

Now we are going to show that $S$ is a minimum vertex cover of $G$. Let $S^*$ be a minimum vertex cover of $G$. Put $I = \{i \mid v_i \in S^*\}$. One can easily verify that the set

$$\begin{aligned} B^* = {} & \{v_i(t) \mid v_i \in S^*,\ 1 \leqq t \leqq d_i\} \cup \{x_i(1) \mid 1 \leqq i \leqq n\} \cup \\ & \cup \{x_i(t) \mid v_i \notin S^*,\ 2 \leqq t \leqq d_i\} \cup \{a_{ij}^3 \mid v_i \notin S^*\} \cup \\ & \cup \{a_{ij}^4 \mid v_j \notin S^*\} \cup \{a_{ij}^5 \mid v_i \in S^*,\ v_j \in S^*\} \end{aligned}$$

is a 1-basis of $\hat{G}$ and that

$$|B^*| = \sum_{i \in I} d_i + n + \sum_{i \notin I} (d_i - 1) + e = 3e + |S^*| \,. \tag{2.3}$$

By the assumption on $B$ we have $|B^*| \geqq |B|$ and comparing (2.2) and (2.3), we obtain $|S^*| \geqq |S|$, as desired. Since $\hat{G}$ and $S$ can clearly be constructed in a time that is a polynomial in the size of $G$, the required NP-completeness is proved.

Remark 2.1. Our construction of $\hat{G}$ from $G$ is a modification of that presented in [5]. Conversely, this modification can be used also for the proof in [5]. However, our construction is simpler.

## 3. THE $m$ CENTER PROBLEM

This problem is related to the preceding one but now $m$ is fixed and the maximum distance from $B$ is minimized. Here we show that even finding a near-optimal $m$-center is difficult. Following the general scheme (see e.g. [15]), we can define also the $\varepsilon$-approximation (basic or absolute) $m$-center problem: Given a graph $G$, an integer $m$, and a real number $\varepsilon \geqq 0$, find a set $\tilde{B}$ of $m$ points from $G$ that

$$\left( \max_{v \in V(G)} d(v, \tilde{B})\, w(v) - d^* \right) / d^* \leqq \varepsilon \,,$$

where $d^* = \min \{ \max_{v \in V(G)} d(v, b)\, w(v) \mid B \subset G,\ |B| = m \}$. Hence a certain deviation from the minimum is allowed.

**Theorem 3.1.** *The following center problems are* NP-*complete.*

(i) *The $\varepsilon$-approximation basic m-center problem with $\varepsilon < 1$ for planar unweighted graphs with vertex degrees at most* 3.

(ii) *The $\varepsilon$-approximation absolute m-center problem with $\varepsilon < 1/2$ for planar unweighted graphs with vertex degrees at most* 3.

(iii) *The $\varepsilon$-approximation absolute m-center problem with $\varepsilon < 1$ for planar graphs with vertex degrees at most* 3, *vertex weights* 1 *or* 2, *and edge weights* 1.

Proof. (i) We reduce the minimum 1-basis problem (which is NP-complete by Th. 2.1) to our problem. Let $m$ be the minimum number such that there is an $m$-center $B$ of a given graph $G$ with

$$\max_{v \in V(G)} d(v, B) < 2 \,. \tag{3.1}$$

(Note that such an $m$ can be found by solving a separate $m$-center problem for each $m = 1, 2, \ldots, |V(G)|$.) As (3.1) is equivalent to

$$\max_{v \in V(G)} d(v, B) = 1 \,, \tag{3.2}$$

we immediately see that $B$ is a minimum 1-basis of $G$. The proof of (i) follows.

(ii) We proceed analogously. Let $m$ be the minimum number such that there is an absolute $m$-center $B$ of a given graph $G$ with

$$\max_{v \in V(G)} d(v, B) < 3/2 \,. \tag{3.3}$$

Then every point of $B$ can be replaced with a closest vertex of $G$ without violating (3.3). Consequently, we can suppose that $B \subset V(G)$. Then (3.3) is equivalent to

$$\max_{v \in V(G)} d(v, B) = 1 \tag{3.4}$$

and $B$ is a minimum 1-basis of $G$. This completes the proof of (ii).

(iii) Here we shall reduce the PVC-3 problem to our problem. Given a planar graph $G$ with the maximum degree 3, we construct a planar graph $\bar{G}$ with the maximum degree 3 by inserting a new vertex $y_{ij}$ into every edge $v_i v_j$ (see Fig. 2). More precisely, $V(\bar{G}) = V(G) \cup \{y_{ij} \mid v_i v_j \in E(G), i < j\}$ and $E(\bar{G}) = \{v_i y_{ij}, y_{ij} v_j \mid v_i v_j \in \in E(G), i < j\}$. We set the weights of all the edges and the old vertices $v_i$ (depicted


Fig. 2.

in Fig. 2 as circles) to be 1 and the weights of the new vertices $y_{ij}$ (squares in Fig. 2) to be 2 (i.e., $w(v_i) = 1$ and $w(y_{ij}) = 2$). Let $m$ be the minimum number such that an absolute $m$-center $B$ of $\bar{G}$ with

$$\max_{v \in V(\bar{G})} d(v, B)\, w(v) < 4 . \tag{3.5}$$

We shall suppose that $B \subset V(G)$ (otherwise we can replace each point of $B$ by a closest old vertex without violating (3.5)). Then (3.5) is equivalent to

$$\max_{v \in V(\bar{G}} d(v, B)\, w(v) = 2 \tag{3.6}$$

and we see that $B$ is a minimum vertex cover of $G$ (as any vertex cover $B$ of $G$ fulfils (3.6)). The proof of (iii) follows and the theorem is completely proved.

## 4. THE $p$-MEDIAN PROBLEM

**Theorem 4.1.** *Both the basic and the absolute $p$-median problems are* NP-*complete even if restricted to planar unweighted graphs with vertex degrees at most* 3.

Proof. We reduce the PVC-3 problem to our problem. Given a planar graph $G$ with the maximum degree 3, we form the graph $\bar{G}$ as in the proof of Theorem 3.1 (iii) (see Fig. 2) but now we let $\bar{G}$ unweighted (i.e. all its weights will be 1). Put $n = |V(G)|$ and $e = |E(G)|$. Let $p$ be the minimum number such that there is a (basic or absolute) $p$-median $M$ of $\bar{G}$ with

$$\sum_{v \in V(\bar{G})} d(v, M) \leqq 2(n - p) + e . \tag{4.1}$$

There is a well-known (polynomial) method which makes it possible to replace any absolute $p$-median by a basic $p$-median with the same value [9, 1, 2]. This method also enables us to replace every new vertex $y_{ij}$ of $M$ by an old vertex ($v_i$ or $v_j$). So we can suppose that $M \subset V(G)$. Further, we assert that $M$ is a minimum vertex cover of $G$. First, since the distance between two old vertices is at least 2 and the distance of any new vertex from any set $S \subset V(G)$ is at least 1, we can write

$$\sum_{v \in V(\bar{G})} d(v, S) \geqq 2(n - |S|) + e\,. \tag{4.2}$$

Secondly, the equality in (4.2) holds if and only if $d(v, S) = 2$ for every old vertex $v$ not in $S$ and $d(v, S) = 1$ for every new vertex $v$ (i.e., $S$ is a vertex cover of $G$). Our set $M$ fulfils (4.2) with equality (because of (4.1)) and thus $M$ is a vertex cover of $G$. The minimality of $|M|$ follows from the choice of $p$. This completes the proof.

## 5. CONCLUDING REMARKS

We have shown the NP-completeness of the problems considered. This indicates strong evidence for the impossibility of efficient algorithms for these problems. As the problems are easy for graphs with the maximum degree less than 3, our results are in this sense the best possible.

In practice, one is usually satisfied with a "near-optimal" solution. However, Theorem 3.1 indicates that there is also little hope to find an effective algorithm giving "sufficiently good" $m$-centers. On the other hand, we have no such results for the minimum $k$-basis or $p$-median problems (the NP-completeness is shown only for the "exact" variants). Note that the question of $\varepsilon$-approximation for the vertex cover problem is also open.

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## Súhrn

## O VÝPOČTOVEJ ZLOŽITOSTI LOKALIZÁCIE STREDÍSK V GRAFE

JÁN PLESNÍK

Práca je venovaná úlohám nasledovného druhu. Daná je cestná sieť spájajúca isté mestá. V tejto sieti treba vybrať množinu $B$ takých bodov, v ktorých sa majú postaviť strediská. Uvažujú sa 3 konkrétne problémy: (1) K danému číslu k treba vybrať $B$ s minimálnou mohutnosťou a tak, aby vzdialenosť z ľubovoľného mesta do $B$ nepresahovala $k$. (2) Pri danej mohutnosti množiny $B$ minimalizovať maximálnu vzdialenosť od $B$. (3) Pri danej mohutnosti množiny $B$ minimalizovať súčet všetkých vzdialeností od $B$. Dokazuje sa, že všetky tieto problémy sú NP — úplné a teda ťažké.

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