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# PARAMETRIC TEST FOR CHANGE IN A PARAMETER OCCURRING IN THE DENSITY OF ONE-PARAMETER EXPONENTIAL FAMILY

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## 1. INTRODUCTION

Let $X_1, \ldots, X_N$ be independent random variables where $X_i$ has the one-parameter exponential density with respect to a $\sigma$-finite measure $\mu$ of the form:

$$f(x, \theta_i) = h(x) \exp\left(\psi_1(\theta_i) U(x) + \psi_2(\theta_i)\right), \quad i = 1, 2, \ldots, N . \tag{1}$$

Let us consider the problem of testing $H_0$ against a class of alternatives $K = \{K_1, \ldots, K_s\}$ defined by

$$H_0 : \theta_1 = \ldots = \theta_N = \theta_0 \tag{2}$$

with $\theta_0$ known,

$$K_i : \theta_1 = \theta_0 + \Delta C_{i1}; \ldots; \theta_N = \theta_0 + \Delta C_{iN}, \quad i = 1, 2, \ldots, s,$$

where $\Delta$ is unknown, and $C_{ij}$ are so-called regression constants. $K_i$ is called the regression alternative.

A special case of this problem where

$$C_{i1} = \ldots = C_{ii} = 0 ; \quad C_{i,i+1} = \ldots = C_{iN} = 1 \tag{3}$$

for $i = 1, \ldots, N - 1$, has been investigated by Kander and Zacks [2].

## 2. LOCALLY AVERAGE MOST POWERFUL (LAMP) TEST

**Theorem 1.** *Suppose that $\psi_1(\theta)$ is increasing, and $\psi_1(\theta)$, $\psi_2(\theta)$ have finite first order derivatives $\psi_1'(\theta)$, $\psi_2'(\theta)$ on $\Omega$ – the parametric space.*

*For testing $H_0$ against $\{K_1, \ldots, K_s\}$ let us consider the test defined by the critical function*

$$\Phi(\mathbf{X}) = 1, \gamma, 0 \quad \text{if} \quad T_{Np}(U) >, =, < C_\alpha \tag{4}$$

*where*

$$T_{Np}(U) = \sum_{j=1}^{N} C_j(\mathbf{p})\, U(X_j)\,, \quad C_j(\mathbf{p}) = \sum_{m=1}^{s} C_{mj} p_m\,, \tag{5}$$

*and $\gamma$, $C_\alpha$ are defined so that the test has the level of significance $\alpha$, $\mathbf{p} = (p_1, \ldots, p_s)$, $\sum_{m=1}^{s} p_m = 1$, are the weights associated to the alternatives $K_1, \ldots, K_s$. Then there exists an $\varepsilon > 0$ such that for all $0 < \Delta \leqq \varepsilon$, the sum $\sum_{m=1}^{s} p_m E_m\, \Phi'(\mathbf{X})$ attains the maximum value at $\Phi$ within the class $\{\Phi'\}$ of all possible $\alpha$-level tests where $E_i$ denotes the expectation with respect to $K_i$.*

Proof. Put

$$f_p(\mathbf{x}, \theta_0, \Delta) = \sum_{i=1}^{s} p_i \prod_{j=1}^{N} h(x_j) \exp\Big( \sum_{j=1}^{N} \psi_1(\theta_0 + \Delta C_{ij})\, U(x_j) + \psi_2(\theta_0 + \Delta C_{ij})\Big)\,;$$

then $f_p(\mathbf{x}, \theta_0, 0)$ is the joint density of $\mathbf{X} = (X_1, \ldots, X_N)$ under $H_0$. Let $\Phi'(\mathbf{x})$ be any test of $H_0$. We have

$$\sum_{m=1}^{s} p_m E_m\, \Phi'(\mathbf{X}) = \int \Phi'(\mathbf{x})\, f_p(\mathbf{x}, \theta_0, \Delta)\, \mathrm{d}\mu(\mathbf{x})\,, \tag{6}$$

$$E_0\, \Phi'(\mathbf{X}) = \int \Phi'(\mathbf{x})\, f(\mathbf{x}, \theta_0)\, \mathrm{d}\mu(\mathbf{x})\,, \tag{7}$$

with $f(\mathbf{x}, \theta_0) = f_p(\mathbf{x}, \theta_0, 0)$. It follows from (6), (7) that the problem of finding the test maximizing the average power $\sum_{m=1}^{s} p_m E_m\, \Phi'(\mathbf{X})$ within the class of all $\alpha$-level tests reduces to the problem of finding the most powerful test for testing $H_0$ against a simple alternative $f_p(\mathbf{x}, \theta_0, \Delta)$ with $\Delta$ fixed. The test, by Neyman-Pearson's Lemma, is defined by

$$\Phi(\mathbf{X}) = 1, \gamma, 0 \quad \text{if} \quad f_p(\mathbf{X}, \theta_0, \Delta) >, =, < C'_\alpha\, f(\mathbf{X}, \theta_0) \tag{8}$$

where $\gamma$, $C'_\alpha$ are constants chosen suitably. By some elementary calculation, it is easy to see that

$$f_p(\mathbf{X}, \theta_0, \Delta)/f(\mathbf{X}, \theta_0) = 1 + \Delta\, \psi'_1(\theta_0) \sum_{j=1}^{N} C_j(\mathbf{p})\, U(X_j) + \psi'_2(\theta_0) \sum_{j=1}^{N} C_j(\mathbf{p}) + 0(\Delta^2)\,. \tag{9}$$

Since $\psi_1'(\theta_0) > 0$, there exists an $\varepsilon > 0$ such that (8) is equivalent to (4) for each $\theta_0$ fixed and for all $0 < \Delta \leqq \varepsilon$. Q. E. D.

Remark. The test possessing the property defined in Theorem 1 is said to be LAMP. Suppose that the regression constants $C_{ij}$'s take on the form (3); then putting $p_1 = \ldots = p_{N-1} = 1/(N-1)$, we obtain from (4), (5)

$$\Phi(\mathbf{X}) = 1, \gamma, 0 \quad \text{if} \quad \sum_{j=2}^{N} (j-1)\, U(X_j) >, =, < C_\alpha . \tag{10}$$

This test was suggested by Kander and Zacks in [2].

The following theorem states that under some restrictions placed on $C_j(\mathbf{p})$ and $U(x)$ the test statistic given by (5) is asymptotically normal.

**Theorem 2.** *Assume that $X_1, X_2, \ldots, X_N, \ldots$ are any independent random variables possessing the distribution functions $F_1(x), F_2(x), \ldots, F_N(x), \ldots$, respectively. Further, suppose that $0 < M \leqq \operatorname{var} U(X_j) < \infty$ for all $j$, and that $U(x)$ is uniformly square integrable in $F_j(x)$, i.e. for any $\varepsilon > 0$ there exists an $A > 0$ depending only on $\varepsilon$ but not on $j$ such that $\int_{\{|x| \geqq A\}} U^2(x)\, \mathrm{d}F_j(x) < \varepsilon$ uniformly for all $j$. Then the test statistic $T_{Np}(U)$ given by (5) is asymptotically normal $N(\mu_{cp}, \sigma_{cp})$ where*

$$\mu_{cp} = E\, T_{Np}(U) = \sum_{j=1}^{N} C_j(\mathbf{p})\, E\, U(X_j) \tag{11}$$

$$\sigma_{cp}^2 = \operatorname{var} T_{Np}(U) = \sum_{j=1}^{N} C_j^2(\mathbf{p}) \operatorname{var}(U(X_j)), \tag{12}$$

*provided*

$$\sum_{j=1}^{N} C_j^2(\mathbf{p}) / \max_j C_j^2(\mathbf{p}) \to \infty . \tag{13}$$

Proof. Verifying the proof of Theorem V.1.2 in [5] we realize that the assertion of the theorem remains true under the conditions of Theorem 2.

The case where $\theta_0$ is unknown will be treated in the following examples.

Example 1. Suppose that $X_j$, $j = 1, \ldots, N$, has the normal distribution $N(\theta_j, \sigma_j)$ with $\sigma_j$ known, namely $\sigma_j = 1$, $\theta_j$ being the unknown mean. Then

$$f(x, \theta_j) = (2\pi)^{-1/2} \exp\left((-1/2)(x - \theta_j)^2\right) =$$

$$= (2\pi)^{-1/2} \exp(-x^2/2) \exp(\theta_j x - \theta_j^2/2)$$

has the form (1) with $U(x) = x$, $h(x) = (2\pi)^{-1/2} \exp(-x^2/2)$, $\psi_1(\theta) = \theta$, $\psi_2(\theta) =$

$= -\theta^2/2$. For testing $H_0$ against $\{K_1, \ldots, K_s\}$ we can employ the test statistic

$$(14) \qquad T_{Np}(\mathbf{X}) = \sum_{j=1}^{N} C_j(\mathbf{p})(X_j - \theta_0), \quad \mathbf{X} = (X_1, \ldots, X_N),$$

which is equivalent to (5) if $\theta_0$ is known. On the contrary, when $\theta_0$ is unknown, we can expect that the test defined by the statistic obtained from (14) by replacing $\theta_0$ by $\bar{X} = \sum_{j=1}^{N} X_j/N$ will have some optimality property.

Assume that $\theta_0$ is unknown and admits a normal prior distribution $N(0, \tau)$. Then the density

$$f_m(\mathbf{x}, \theta_0, \Delta) = f_m(\mathbf{x}, \theta_1, \ldots, \theta_N) = (2\pi)^{-N/2} \exp\left(-\tfrac{1}{2}\sum_{j=1}^{N}(x_j - \theta_j)^2\right) =$$

$$= (2\pi)^{-N/2} \exp\left(-\tfrac{1}{2}\sum_{j=1}^{N}(x_j - \theta_0 - \Delta C_{mj})^2\right)$$

with $\mathbf{x} = (x_1, \ldots, x_N)$ may be considered as the conditional density of $X_1, \ldots, X_N$ under $K_m$.

The unconditional joint density of $\mathbf{X}$ under $K_m$ with respect to the prior distribution $N(0, \tau)$ of $\theta_0$ is given by

$$f_m(\mathbf{x}, \Delta) = \frac{1}{\tau\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mathbf{x}, \theta_0, \Delta) \exp(-\theta_0^2/2\tau^2)\, d\theta_0 =$$

$$= C(N, \tau) \exp\left\{-\tfrac{1}{2}\sum_{j=1}^{N}[x_j - \bar{x} - \Delta(C_{mj} - \bar{C}_m)]^2 -\right.$$

$$\left. - (N/2)(\bar{x} - \Delta\bar{C}_m)^2/(1 + N\tau^2)\right\}$$

where $\bar{C}_m = \sum_{j=1}^{N} C_{mj}/N$, and $C(N, \tau)$ is the constant depending only on $N$ and $\tau$. Note that

$$f_0(\mathbf{x}) = f_m(\mathbf{x}, 0) = C(N, \tau) \exp\left\{-\tfrac{1}{2}\sum_{j=1}^{N}(x_j - \bar{x}) - \tfrac{1}{2}N(\bar{x})^2/(1 + N\tau^2)\right\}$$

is the unconditional density of $\mathbf{X}$ under $H_0$.

Let $E_0$, $E_m$ be the expectations with respect to $f_0(\mathbf{x})$, $f_m(\mathbf{x}, \Delta)$ and let $\Phi'(\mathbf{X})$ be any test for testing $f_0(\mathbf{x})$ against $f_m(\mathbf{x}, \Delta)$, $m = 1, \ldots, s$, with $\Delta$ fixed. Then it is easy to see that the test maximizing $\sum_{m=1}^{s} p_m E_m \Phi'(\mathbf{X})$ within the class of all tests satifying $E_0 \Phi'(\mathbf{X}) \leqq \alpha$ is given by

$$(15) \qquad \Phi(\mathbf{X}) = 1, 0 \quad \text{if} \quad \sum_{m=1}^{s} p_m f_m(\mathbf{X}, \Delta)/f_0(\mathbf{X}) >, < C'_\alpha.$$

Note that $\sum_{m=1}^{s} p_m f_m/f_0$ may be expanded in the form:

$$\sum_{m=1}^{s} p_m f_m(\mathbf{x}, \Delta)/f_0(\mathbf{x}) = \sum_{m=1}^{s} p_m \exp\{\Delta \sum_{j=1}^{N} (x_j - \bar{x})(C_{mj} - \bar{C}_m) +$$
$$+ N\Delta\bar{C}_m\bar{x}/(1 + N\tau^2) + 0(\Delta^2)\} =$$
$$= 1 + \Delta \sum_{j=1}^{N} (x_j - \bar{x})(C_j(\mathbf{p}) - \bar{C}(\mathbf{p})) +$$
$$+ \frac{N\Delta}{1 + N\tau^2} \bar{C}(\mathbf{p})\,\bar{x} + 0(\Delta^2).$$

Consequently, when $\Delta$ is small enough and $\tau \to \infty$, $\sum_{m=1}^{s} p_m f_m(\mathbf{x}, \Delta)/f_0(\mathbf{x})$ is strictly increasing function of $T'_{Np}(\mathbf{x})$ where

(16) $$T'_{Np}(\mathbf{x}) = \sum_{j=1}^{N} (C_j(\mathbf{p}) - \bar{C}(\mathbf{p}))(x_j - \bar{x}) = \sum_{j=1}^{N} (C_j(\mathbf{p}) - \bar{C}(\mathbf{p}))\, x_j\,,$$

and (15) is equivalent to

(17) $$\Phi(\mathbf{X}) = 1, 0 \quad \text{if} \quad T'_{Np}(\mathbf{X}) >, < C_\alpha$$

for all $\Delta$ small enough. Then $\Phi(\mathbf{X})$ may be regarded as a locally Bayesian solution with respect to the normal prior distribution $N(0, \tau)$ of $\theta_0$ when $\tau \to \infty$ for the problem of testing $H_0$ against $\{K_1, \ldots, K_s\}$ concerning the mean of a normal distribution.

Remark. If the regression constants $C_{ij}$'s take on the form (3) and putting $p_1 = \ldots = p_{N-1} = 1/(N-1)$, then (16), (17) reduce to

(18) $$T'_N(\mathbf{X}) = \sum_{j=2}^{N} (j-1)(X_j - \bar{X}),$$

(19) $$\Phi(\mathbf{X}) = 1, 0 \quad \text{if} \quad T'_N(\mathbf{X}) >, < C_\alpha,$$

which have been considered by Chernoff and Zacks in [1].

Example 2. Suppose that $X_1, \ldots, X_N$ are independent and $X_j$ is normally distributed $N(\mu_j, \theta_j)$ where $\mu_j$ is known, namely $\mu_j = 0$ for all $j$, and $\theta_j$ is an unknown parameter.

Consider the problem of testing $H_0$ against $\{K_1, \ldots, K_s\}$ where

(20) $$H_0 : \theta_1 = \ldots = \theta_N = \theta_0\,,$$
$$K_i : \theta_1^2 = \theta_0^2(1 + \Delta C_{i1}), \ldots, \theta_N^2 = \theta_0^2(1 + \Delta C_{iN})$$
$$\text{for } i = 1, \ldots, s\,.$$

The density of $X_j$ under $H_0$ and $K_i$'s takes on the form

$$f(x, \theta) = (2\pi\,\theta^2)^{-1/2} \exp\left(-x^2/2\theta^2\right) = (2\pi)^{-1/2} \exp\left(-x^2/2\theta^2 - \tfrac{1}{2}\log\theta^2\right)$$

which has the form (1) with $U(x) = x^2$, $\psi_1(\theta) = -1/2\theta$, $\psi_2(\theta) = -\frac{1}{2}\log\theta^2$. Consequently, for testing $H_0$ against $\{K_1, \ldots, K_s\}$ the test given by (4), (5) reduces to

$$\Phi(\mathbf{X}) = 1, \gamma, 0 \quad \text{if} \quad T_{Np}(\mathbf{X}) = \sum_{j=1}^{N} C_j(\mathbf{p})\, X_j^2/\theta_0^2 > \,,\, = \,,\, < C_\alpha \tag{21}$$

provided $\theta_0$ is known.

Let us consider the case where $\theta_0$ is unknown.

Assume that $u = 1/\theta_0^2$ is an exponentially distributed random variable with an unknown parameter $\lambda$, i.e. the density of $u$ is given by: $g_\lambda(u) = \lambda\exp(-\lambda u)$ for $u, \lambda > 0$. Thus the function

$$f_i(\mathbf{x}, u, \Delta) = (2\pi)^{-N/2} \prod_{j=1}^{N} (1 + \Delta C_{ij})^{-1/2}\, u^{N/2} \exp\left[-(u/2)\sum_{j=1}^{N} x_j^2/(1 + \Delta C_{ij})\right]$$

may be regarded as the conditional density of $X_1, \ldots, X_N$ under $K_i$ when $u$ is given.

The unconditional density of $X_1, \ldots, X_N$ under $K_i$ is defined by

$$f_i(\mathbf{x}, \Delta) = \lambda \int_0^\infty f_i(\mathbf{x}, u, \Delta) \exp(-\lambda u)\, \mathrm{d}u =$$

$$= C_N(\lambda) \prod_{j=1}^{N} (1 + \Delta C_{ij})^{-1/2} \left[2\lambda + \sum_{j=1}^{N} x_j^2/(1 + \Delta C_{ij})\right]^{-(N/2)-1}$$

where $C_N(\lambda)$ is the constant depending only on $N, \lambda$. Note that $f_0(\mathbf{x}) = f_i(\mathbf{x}, 0) = C_N(\lambda)\left[2\lambda + \sum_{j=1}^{N} x_j^2\right]^{-(N/2)-1}$ is the unconditional density of $X_1, \ldots, X_N$ under $H_0$. Let $\Phi'(\mathbf{X})$ be any test of the hypothesis $f_0(\mathbf{x})$ against the alternatives $\{f_1(\mathbf{x}, \Delta), \ldots, f_s(\mathbf{x}, \Delta)\}$ with $\Delta$ fixed. Let $E_0$, $E_i$, $i = 1, \ldots, s$, be the expectations with respect to the densities $f_0(\mathbf{x})$ and $f_i(\mathbf{x}, \Delta)$. Then the test, which maximizes $\sum_{i=1}^{s} p_i E_i\, \Phi'(\mathbf{X})$ within the class of all $\alpha$-level tests is defined by

$$\Phi(\mathbf{X}) = 1, \gamma, 0 \quad \text{if} \quad \sum_{i=1}^{s} p_i\, f_i(\mathbf{X}, \Delta)/f_0(\mathbf{X}) > \,,\, = \,,\, < C'_\alpha\,. \tag{22}$$

By some elementary calculation we easily obtain:

$$\sum_{i=1}^{s} p_i\, f_i(\mathbf{X}, \Delta)/f_0(\mathbf{X}) =$$

$$= 1 - (N/2)\,\bar{C}(\mathbf{p}) + \Delta(1 + N/2) \sum_{j=1}^{N} C_j(\mathbf{p})\, X_j^2/(2\lambda + \sum_{j=1}^{N} X_j^2) + 0(\Delta^2)\,.$$

If $\lambda \to 0$, then $\sum_{i=1}^{s} p_i f_i(\mathbf{X}, \Delta)/f_0(\mathbf{X})$ is a strictly increasing function of $\sum_{j=1}^{N} C_j(\mathbf{p}) X_j^2 / \sum_{j=1}^{N} X_j^2$, or, equivalently, of

$$T''_{Np}(\mathbf{X}) = N^{-1} \sum_{j=1}^{N} (C_j(\mathbf{p}) - \bar{C}(\mathbf{p}))\, X_j^2/S^2 \tag{23}$$

where $S^2 = N^{-1} \sum_{j=1}^{N} X_j^2$, and (22) is equivalent to

$$\Phi(\mathbf{X}) = 1, \gamma, 0 \quad \text{if} \quad T''_{Np}(\mathbf{X}) > , = , < C_\alpha \tag{24}$$

for all $0 < \Delta$ small enough.

Thus the test defined by (24) is a locally Bayesian solution of the problem of testing $H_0$ against $\{K_1, \ldots, K_s\}$ with respect to the exponential prior distribution of $u = 1/\theta_0^2$ with the parameter $\lambda$, which has been supposed to tend to zero.

Remark. The distributions of $T'_{Np}(\mathbf{X})$ given by (16) and $T''_{Np}(\mathbf{X})$ given by (23) do not depend on $\theta_0$.

## 3. THE ASYMPTOTIC RELATIVE EFFICIENCY

The definition of the asymptotic relative efficiency was given in [3].

Let us now consider the asymptotic relative efficiency of the rank tests considered in [3] with respect to the parametric tests given by (17), (24) for testing hypotheses on the mean and on the variance of a normal distribution.

We say that an $\alpha$-level test is based on the test statistic $T$ if its critical region takes on the form $\{T > C_\alpha\}$.

Example 1. Let $X_1, \ldots, X_N$ be independent random variables possessing the normal distributions $N(\theta_1, 1), \ldots, N(\theta_N, 1)$, respectively. Consider the problem of testing $H_0$ against $\{K_1, \ldots, K_s\}$ defined by (2) with $\theta_1, \ldots, \theta_N$ being the means of the normal distributions. For testing $H_0$ against $\{K_1, \ldots, K_s\}$ we can employ the parametric test based on $T'_{Np}(\mathbf{X})$ given by (16) and the rank test based on the rank test statistic $T^{(1)}_{Np}(\mathbf{R})$ given by

$$T^{(1)}_{Np}(\mathbf{R}) = \sum_{j=1}^{N} [C_j(\mathbf{p}) - \bar{C}(\mathbf{p})]\, a_N^{(1)}(R_j)$$

where $a_N^{(1)}(j) = EV^{(j)} = E\,\phi^{-1}(U^{(j)})$ with $V^{(1)} < \ldots < V^{(N)}$; $U^{(1)} < \ldots < U^{(N)}$ being the ordered samples from the standardized normal and from the uniform distribution, respectively. This test was obtained from Corollary 1 in [3].

Assume that the condition

$$\sum_{j=1}^{N} [C_j(\mathbf{p}) - \bar{C}(\mathbf{p})]^2 / \max_j\, [C_j(\mathbf{p}) - \bar{C}(\mathbf{p})]^2 \to \infty$$

is fulfilled. Consider the alternative $K$ defined by

$$K : \theta_1 = d_1, \ldots, \theta_N = d_N ,$$

and assume that

$$\sum_{j=1}^{N} (d_j - \bar{d})^2 \to b^2 > 0 , \quad \max_j (d_j - \bar{d})^2 \to 0$$

hold. We shall show that the asymptotic relative efficiency of the test based on $T_{Np}^{(1)}(\mathbf{R})$ with respect to the test based on $T'_{Np}(\mathbf{X})$, say $e[T_{Np}^{(1)}(\mathbf{R}) : T'_{Np}(\mathbf{X})]$, is equal to 1.

As a matter of fact, $T'_{Np}(\mathbf{X})$ is normally distributed $N(0, \sigma_{cp})$ under $H_0$, and $N(b_1, \sigma_{cp})$ under $K$, where

$$b_1 = \sum_{j=1}^{N} [C_j(\mathbf{p}) - \bar{C}(\mathbf{p})] (d_j - \bar{d}) ,$$

$$\sigma_{cp}^2 = \sum_{j=1}^{N} [C_j(\mathbf{p}) - \bar{C}(\mathbf{p})]^2 ;$$

hence the asymptotic power of the test based on $T'_{Np}(\mathbf{X})$ is equal to

$$1 - \phi(k_{1-\alpha} - b_1/\sigma_{cp}) \tag{25}$$

where $k_{1-\alpha}$ is the $100(1 - \alpha)$ percentage point of the standardized normal distribution function $\phi(x)$. On the other hand, by Theorem 5 and Remark 1 in [3], $T_{Np}^{(1)}(\mathbf{R})$ has the same asymptotic distribution as $T'_{Np}(\mathbf{X})$ both under $H_0$ and under $K$; hence the asymptotic power of the test based on $T_{Np}^{(1)}(\mathbf{R})$ is also given by (25), and, by the definition of the asymptotic relative efficiency, $e[T_{Np}^{(1)}(\mathbf{R}) : T'_{Np}(\mathbf{X})] = 1$.

Example 2. Let $X_1, \ldots, X_N$ be independent random variables, which are normally distributed $N(0, \theta_1), \ldots, N(0, \theta_N)$, respectively. For testing $H_0$ against $\{K_1, \ldots, K_s\}$ defined by (20) with $\theta_0$ unknown we may employ the parametric test based on the test statistic $T''_{Np}(\mathbf{X})$ given by (23) and the test based on the rank test statistic

$$T_{Np}^{(2)}(\mathbf{R}) = \sum_{j=1}^{N} [C_j(\mathbf{p}) - \bar{C}(\mathbf{p})] \, a_N^{(2)}(R_j)$$

where $a_N^{(2)}(j) = E[V^{(j)}]^2 - 1 = E[\phi^{-1}(U^{(j)})]^2 - 1$ with $V^{(j)}$, $U^{(j)}$ being the same as in Example 1. This rank test was obtained from Corollary 2 in [3]. Let us now calculate the asymptotic relative efficiency of the test based on $T_{Np}^{(2)}(\mathbf{R})$ with respect to the test based on $T''_{Np}(\mathbf{X})$ under the alternative $K$ defined by

$$K : \theta_1^2 = \theta_0^2(1 + d_1), \ldots, \theta_N^2 = \theta_0^2(1 + d_N)$$

with $1 + d_j \geqq \delta > 0$ for all $j$.

Suppose that $\sum_{j=1}^{N} [C_j(\boldsymbol{p}) - \bar{C}(\boldsymbol{p})]^2 / \max_j [C_j(\boldsymbol{p}) - \bar{C}(\boldsymbol{p})]^2 \to \infty$, and that

$$\sum_{j=1}^{N} (d_j - \bar{d}_N)^2/(1 + \bar{d}_N)^2 \to b^{*2} > 0 , \quad \max_j (d_j - \bar{d}_N)^2/(1 + \bar{d}_n)^2 \to 0$$

with $\bar{d}_N = N^{-1} \sum_{j=1}^{N} d_j$.

Without loss of generality we may assume that $\theta_0 = 1$ since the distributions of $T_{Np}^{(2)}(\boldsymbol{R})$ and $T''_{Np}(\boldsymbol{X})$ do not depend on $\theta_0$ under $H_0$ and $K$. Under these assumptions we shall show that the asymptotic relative efficiency of the test based on $T_{Np}^{(2)}(\boldsymbol{R})$ with respect to the test based on $T''_{Np}(\boldsymbol{X})$ is equal to 1.

As a matter of fact, by Theorem 5 and Remark 3 in [3], the test statistic $T_{Np}^{(2)}(\boldsymbol{R})$ is asymptotically normal $N(0, \sigma_{cp})$ under $H_0$, and $N(b_2/(1 + \bar{d}_N), \sqrt{(2)}\, \sigma cp)$ under $K$ where

$$b_2 = \sum_{j=1}^{N} [C_j(\boldsymbol{p}) - \bar{C}(\boldsymbol{p})]\,(d_j - \bar{d}_N) , \quad \sigma_{cp}^2 = \sum_{j=1}^{N} [C_j(\boldsymbol{p}) - \bar{C}(\boldsymbol{p})]^2 ;$$

hence the asymptotic power of the test based on $T_{Np}^{(2)}(\boldsymbol{R})$ is equal to

$$1 - \phi(k_{1-\alpha} - b_2/\sigma_{cp}\sqrt{(2)}\,(1 + \bar{d}_N)) . \tag{26}$$

We shall now show that the test statistic $T''_{Np}(\boldsymbol{X})$ is asymptotically normal both under $H_0$ and under $K$ with the same asymptotic mean and variance as $T_{Np}^{(2)}(\boldsymbol{R})$.

Actually, first assume that $\bar{d}_N = \bar{d} = N^{-1} \sum_{j=1}^{N} d_j \to d_0 > -1$. Then $S^2$ converges with probability 1 to 1 under $H_0$, and to $1 + d_0$ under $K$. On the other hand, $N^{-1} \sum_{j=1}^{N} [C_j(\boldsymbol{p}) - \bar{C}(\boldsymbol{p})]\, X_j^2$ is, by Theorem 2 (the condition on uniform square integrability of $U(x) = x^2$ in the normal distribution functions $N(0, 1 + d_j)$ is satisfied by the assumption that $\bar{d}_N$ is bounded and $\max_j (d_j - \bar{d}_N) \to 0$), asymptotically normal $N(0, \sqrt{(2)}\, \sigma_{cp}/N)$ under $H_0$, and $N(b_2/N, \sqrt{(2)}\, \sigma'_{cp}/N)$ under $K$, where

$$\begin{aligned} \sigma'^2_{cp} &= \sum_{j=1}^{N} [C_j(\boldsymbol{p}) - \bar{C}(\boldsymbol{p})]^2\, [1 + d_j]^2 = \\ &= \sum_{j=1}^{N} [C_j(\boldsymbol{p}) - \bar{C}(\boldsymbol{p})]^2\, [1 + \bar{d}_N]^2\, [1 + 0(d_j - \bar{d}_N)] \sim \\ &\sim \sum_{j=1}^{N} [C_j(\boldsymbol{p}) - \bar{C}(\boldsymbol{p})]^2\, (1 + \bar{d}_N)^2 \sim (1 + d_0)^2\, \sigma_{cp}^2 \end{aligned}$$

since $\max_j (d_j - \bar{d}_N)^2 \to 0$. Consequently, by Proposition $X$, Chapter II, in [4], $T''_{Np}(\boldsymbol{X})$ is asymptotically normal $N(0, \sqrt{(2)}\, \sigma_{cp}/N)$ under $H_0$, and $N(b_2/N(1 + \bar{d}_N)$, $\sqrt{(2)}\, \sigma_{cp}/N)$ under $K$.

Further, the assertion about the asymptotic normality of $T''_{Np}(\mathbf{X})$ under $K$ remains true if we only assume that $\bar{d}_N$ is bounded.

As a matter of fact, assume, on the contrary, that $T''_{Np}(\mathbf{X})$ is not asymptotically normal $N(b_2/N(1 + \bar{d}_N), \sqrt{(2)}\,\sigma_{cp}/N)$ under $K$. Then there exists a sequence $\{N_\nu\}$ such that this assumption holds for every subsequence of $\{N_\nu\}$. Thus passing to a proper subsequence, if necessary, we may assume, without loss of generality, that $N_\nu \to \infty$ and $\bar{d}_{N_\nu} \to d_0$ since $\bar{d}_N$ is bounded. By the above argument, $T''_{Np}(\mathbf{X})$ is asymptotically normal $N(b_2/N_\nu(1 + \bar{d}_{N_\nu}), \sqrt{(2)}\,\sigma_{cp}/N_\nu)$ and this contradicts the above assumption. Finally, it follows that the asymptotic relative efficiency of the test based on $T^{(2)}_{Np}(\mathbf{R})$ with respect to the test based on $T''_{Np}(\mathbf{X})$ is equal to 1.

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## Souhrn

## PARAMETRICKÝ TEST PRO ZMĚNU PARAMETRU V HUSTOTĚ JEDNOPARAMETRICKÉ EXPONENCIÁLNÍ RODINY

NGUYEN-VAN-HUU


Vyšetřuje se problém testování hypotézy, že pozorování jsou nezávislá identicky rozložená, proti třídě alternativ regrese v parametru, a to pro jednoparametrickou exponenciální rodinu. Odvozuje se parametrický test pro tento problém a rovněž jeho relativní eficience vzhledem k pořadovému testu navrženému autorem v předcházející publikaci.



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