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Milan Práger

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UNIVERSALLY OPTIMAL APPROXIMATION OF FUNCTIONALS

MILAN PRÁGER

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1. INTRODUCTION

The error of quadrature formulae is very often estimated by the norm of the error functional in a Hilbert space. This approach leads to a natural problem of optimization of quadrature formulae as minimization of the norm of the error functional. Let us mention here the comprehensive book of Sobolev [3]. Nevertheless, many other authors have considered the problem, too.

In [1] it was shown that this approach can lead to very different results if the optimization in different spaces is performed. In the same paper, fundamental concepts of universal asymptotical optimality and universal optimality in order were introduced. In addition to quadrature formulae, functionals of integral type in the space $C_{2\pi}$ of continuous 2π-periodical functions were considered. In [2] quadrature formulae utilizing the values of derivatives in addition to the function values were studied in spaces of periodic functions continuous with their derivatives up to the order n.

The present paper continues this investigation. We study a universal optimization of a general functional in the space of continuous functions as well as in the spaces of smoother functions. Our approximating formulae use only the function values.

Further, the connection of this theory with spline and trigonometric interpolations is demonstrated. One result on convergence in the whole space $C_{2\pi}$ is also contained. In the conclusion some possibilities of application of the results to the computation of values of singular integrals (not necessarily of periodic functions) are shown.

2. PERIODIC SPACES

In this section we shall introduce classes of Hilbert spaces fundamental for further investigations, and their properties. For the reader's convenience we repeat here some results of [1] and [2]. By a Hilbert space we understand a complete Hilbert space.

We denote by $C_{2\pi}$ the space of complex-valued continuous 2π-periodic functions with the usual maximum norm.

Definition 2.1. *We shall say that a Hilbert space H is a periodic space if it has the following properties*:

(i) $H \subset C_{2\pi}$,

(ii) *H is dense in $C_{2\pi}$ (in the sense of the topology in $C_{2\pi}$)*,

(iii) *for a function $f \in H$ and an arbitrary real number c, the function g defined by the equality $g(x) = f(x + c)$ satisfies*

$$g \in H \quad \text{and} \quad \|g\|_H = \|f\|_H ,$$

(iv) $\|f\|_{C_{2\pi}} \leqq K\|f\|_H$ *for all* $f \in H$.

The class of all periodic spaces will be denoted by $\mathscr{H}$.

Definition 2.2. *The set of all convergent series $d = \{d_i\}_{i=-\infty}^{\infty}$ with positive elements, i.e. $d_i > 0$ and $\sum_{i=-\infty}^{\infty} d_i < +\infty$ will be called the class $\mathscr{P}$.*

Theorem 2.1. *Let the series $d \in \mathscr{P}$. Let us take a sequence $\{f_k\}_{k=-\infty}^{\infty}$ of complex numbers such that*

$$(1) \qquad \sum_{k=-\infty}^{\infty} |f_k|^2/d_k < +\infty$$

holds.

Then the series

$$(2) \qquad \sum_{k=-\infty}^{\infty} f_k e^{ikx}$$

converges for all x.

The proof follows easily from the following inequalities:

$$(3) \qquad \left|\sum_{k=-\infty}^{\infty} f_k e^{ikx}\right| \leqq \sum_{k=-\infty}^{\infty} |f_k| \cdot \frac{\sqrt{d_k}}{\sqrt{d_k}} \leqq \left[\sum_{k=-\infty}^{\infty} \frac{|f_k|^2}{d_k} \cdot \sum_{k=-\infty}^{\infty} d_k\right]^{1/2} < +\infty .$$

We remark that different functions correspond to different sequences.

Theorem 2.2. *The space of all functions of the form (2) is a Hilbert space if we introduce the scalar product by*

$$(f, g) = \sum_{k=-\infty}^{\infty} \frac{f_k \bar{g}_k}{d_k} ,$$

where $f(x) = \sum f_k e^{ikx}$, $g(x) = \sum g_k e^{ikx}$.

Proof. All properties of the scalar product are obvious (the triangle inequality follows from the Minkowski inequality). It remains to prove the completeness. Thus, let us have a Cauchy sequence of functions $f^{(n)}$, i.e. $\|f^{(n)} - f^{(m)}\| < \varepsilon$ for m, n sufficiently large. This means that for sufficiently large m, n (independently of k) we have the inequality

$$(4) \qquad \sum_{k=-\infty}^{\infty} \frac{|f_k^{(n)} - f_k^{(m)}|^2}{d_k} < \varepsilon .$$

It follows that the sequence of numbers $f_k^{(n)}$ converge as $n \to \infty$ for every k. Thus, the limits f_k exist. We shall show that this sequence satisfies (1). Passing to the limit in (4) we obtain

$$\sum_{k=-\infty}^{\infty} \frac{|f_k - f_k^{(m)}|^2}{d_k} < \varepsilon$$

and

$$\sum \frac{|f_k|^2}{d_k} \leqq 2 \sum \frac{|f_k - f_k^{(m)}|^2 + |f_k^{(m)}|^2}{d_k} < +\infty \, .$$

Definition 2.3. *The Hilbert space generated by the series $d \in \mathscr{P}$ in the way just described is denoted by H_d. The class of all such spaces will be called $\mathscr{H}_{\mathscr{P}}$.*

Theorem 2.3. *The classes $\mathscr{H}$ and $\mathscr{H}_{\mathscr{P}}$ are identical.*

For the proof see [1].

Remark. It follows from the proof that the functions e^{ikx} belong to every space of $\mathscr{H}$ (or $\mathscr{H}_{\mathscr{P}}$) for every k. Further, it holds $\|e^{ikx}\|^2_{H_d} = 1/d_k$.

The following theorem characterizes elements of spaces from the class $\mathscr{H}$.

Theorem 2.4. *A function f is an element of a space $H \in \mathscr{H}$ if and only if its Fourier series is absolutely convergent.*

The proof is in [1].

In what follows, the following subclass of the class $\mathscr{H}$ plays an important role.

Definition 2.4. *We shall denote by $\mathscr{H}_1$ the subclass of the class $\mathscr{H}$ which contains spaces generated by series $d \in \mathscr{P}$ with the properties:*

(i) $d_k = d_{-k}$,

(ii) $d_{k+1} \leqq d_k$ *for* $k \geqq 0$,

(iii) $\displaystyle\sum_{t=-\infty}^{\infty} \frac{d_{k+tn}}{d_k} < D$ *for all n and* $k \leqq \left[\frac{n}{2}\right]$.

Here, D is independent of n and k, but it can depend on d. The class of series with these properties will be called $\mathscr{P}_1$.

The first two properties are more or less natural. In the following we will need the restriction given by the third property, too. It can be shown in a way similar to Theorem 5.1 of [2] that the third property depends neither on the first nor on the second property. Putting $d_k = d_{-k}$ and defining $d_k = 1/2^{2^{2^s}}$ for $k = 2^{2^s}, \ldots, 2^{2^{s+1}} - 1$, $s = 0, \ldots$ and $d_{-1} = d_0 = d_1 = 1$ we obtain a series satisfying (i) and (ii) but not satisfying (iii).

Besides the classes introduced above, classes of Hilbert spaces with smoother elements will be used.

We denote by $C_{2\pi}^{(n)}$ the space of complex 2π-periodic functions with continuous derivatives up to the order n with the usual norm.

Definition 2.5. *We shall say that a Hilbert space is an n-periodic space if it has the following properties:*

(i) $H \subset C_{2\pi}^{(n)}$,

(ii) *H is dense in $C_{2\pi}^{(n)}$ (in the sense of the topology in $C_{2\pi}^{(n)}$)*,

(iii) *for a function $f \in H$ and an arbitrary real number c, the function g defined by the equality $g(x) = f(x + c)$ satisfies*

$$g \in H \quad \text{and} \quad \|g\|_H = \|f\|_H ,$$

(iv) $\|f\|_{C_{2\pi}^{(n)}} \leqq K\|f\|_H$.

The class of all n-periodic spaces will be denoted by $\mathscr{H}^{(n)}$.

Remark. It is $\mathscr{H}^{(n)} \subset \mathscr{H}^{(p)}$ for $p \leqq n$ and $\mathscr{H}^{(0)} = \mathscr{H}$.

Theorem 2.5. *For every space $H \in \mathscr{H}^{(n)}$ there exists a series d such that*

$$\sum_{k=-\infty}^{\infty} k^{2n} \,.\, d_k < +\infty \tag{5}$$

holds and $H = H_d$ (in the sense of Def. 2.3). Conversely, every space H_d with d satisfying (5) is an element of $\mathscr{H}^{(n)}$.

The proof is in [2].

Definition 2.6. *We shall denote by $\mathscr{H}_1^{(n)}$ the class of Hilbert spaces which is the intersection of the classes $\mathscr{H}_1$ and $\mathscr{H}^{(n)}$.*

3. OPTIMAL APPROXIMATION OF A FUNCTIONAL

Let a continuous functional Φ in the space $C_{2\pi}$ be given. Therefore, Φ is a continuous functional over every space $H \in \mathscr{H}$. This is a consequence of the property (iv) of periodic spaces. Now, we shall try to approximate this functional by a linear combination of Dirac functionals $\delta^{2\pi k/n}$ on an equidistant mesh. We use the notation $\delta^{2\pi k/n}(f) = f(2\pi k/n)$. It is well known that there exists an optimal approximation of the given form for which the relation

$$\left\|\Phi - \sum_{k=1}^{n} a_k \delta^{2\pi k/n}\right\|_{H_d} = \min_{p_i} \left\|\Phi - \sum_{k=1}^{n} p_k \delta^{2\pi k/n}\right\|_{H_d} \tag{6}$$

holds. We shall denote by $\chi(\Phi, H_d, n)$ the norm of the error functional corresponding to the optimal approximation (i.e. the left-hand side of (6)).

Now, we shall calculate explicitly the quantity $\chi(\Phi, H_d, n)$. Thus, let $x_k = 2\pi k/n$, $k = 1, \ldots, n$ be the equidistant mesh. Further, let $\varphi = \sum \varphi_k e^{ikx}$ be the element of H_d representing the functional Φ according to the Riesz-Fischer theorem, i.e.

$$(f, \varphi)_{H_d} = \Phi(f) \quad \text{for all} \quad f \in H_d \,.$$

From this we obtain successively

$$\sum_{s=-\infty}^{\infty} \frac{f_s \bar{\varphi}_s}{d_s} = \sum_{s=-\infty}^{\infty} f_s \, \Phi(e^{isx}) \,,$$

$$\bar{\varphi}_s = \Phi(e^{isx}) \,.\, d_s \,, \quad \varphi_s = \overline{\Phi(e^{isx})} \,.\, d_s \quad (d_s \text{ is real}) \,.$$

With the notation $b_s = \overline{\Phi(e^{isx})}$ we have $\varphi_s = b_s \,.\, d_s$. We note that the coefficients b_s are bounded as they are Fourier coefficients of a distribution of the first order. The representing function $\psi^{2\pi k/n}$ of the Dirac functional $\delta^{2\pi k/n}$ is therefore

$$\psi^{2\pi k/n}(x) = \sum_{s=-\infty}^{\infty} d_s e^{is(x-2\pi k/n)}$$

and as a special case

$$\psi^0(x) = \psi^{2\pi}(x) = \sum_{s=-\infty}^{\infty} d_s e^{isx} .$$

The functional Φ_{opt} of the optimal approximation in H_d is given by its representing function

$$\varphi_{\text{opt}}(x) = \sum_{s=1}^{n} \bar{a}_s \, \psi^{2\pi s/n}(x) .$$

Thus, it is

$$\Phi_{\text{opt}}(f) = (f, \varphi_{\text{opt}})_{H_d} = \sum_{s=1}^{n} a_s (f, \psi^{2\pi s/n})_{H_d} = \sum_{s=1}^{n} a_s f\left(\frac{2\pi s}{n}\right) .$$

In other words: the coefficients a_s are coefficients of the linear combination of values at the mesh points.

The coefficients a_s are determined uniquely from the system

$$\sum_{s=1}^{n} \bar{a}_s (\psi^{2\pi s/n}, \psi^{2\pi k/n})_{H_d} = (\varphi, \psi^{2\pi k/n})_{H_d} , \quad k = 1, \ldots, n .$$

This is a consequence of the projection property of the optimal approximation in a Hilbert space. In our case it means that the difference of the functional and its optimal approximation is orthogonal to all Dirac functionals.

Utilizing the developments of the functions $\psi^{2\pi k/n}$ we obtain further

$$\sum_{s=1}^{n} \bar{a}_s \sum_{j=-\infty}^{\infty} d_j e^{-2\pi i s j/n} e^{ij2\pi k/n} = \sum_{j=-\infty}^{\infty} b_j d_j e^{2\pi i k j/n} , \quad k = 1, \ldots, n$$

and

$$\sum_{j=-\infty}^{\infty} d_j \sum_{s=1}^{n} \bar{a}_s e^{2\pi i j(k-s)/n} = \sum_{j=-\infty}^{\infty} b_j d_j e^{2\pi i k j/n} , \quad k = 1, \ldots, n .$$

Changing the order of summation and introducing the subscript l we obtain

$$\sum_{j=-\infty}^{\infty} \sum_{l=1}^{n} d_{jn+l} \sum_{s=1}^{n} \bar{a}_s e^{2\pi i(k-s)(jn+l)/n} = \sum_{j=-\infty}^{\infty} \sum_{l=1}^{n} b_{jn+l} d_{jn+l} e^{2\pi i k(jn+l)/n} ,$$

$$\sum_{j=-\infty}^{\infty} \sum_{l=1}^{n} d_{jn+l} \sum_{s=1}^{n} \bar{a}_s e^{2\pi i(k-s)(j+l/n)} = \sum_{j=-\infty}^{\infty} \sum_{l=1}^{n} b_{jn+l} d_{jn+l} e^{2\pi i k(j+l/n)} ,$$

$$\sum_{j=-\infty}^{\infty} \sum_{l=1}^{n} d_{jn+l} \sum_{s=1}^{n} \bar{a}_s e^{2\pi i(k-s)l/n} = \sum_{j=-\infty}^{\infty} \sum_{l=1}^{n} b_{jn+l} d_{jn+l} e^{2\pi i k l/n} .$$

Using the short notation $\sum_{j=-\infty}^{\infty} d_{jn+l} = D_l$, $\sum_{j=-\infty}^{\infty} b_{jn+l} d_{jn+l} = B_l$ we have

$$\sum_{l=1}^{n} \sum_{s=1}^{n} \bar{a}_s e^{2\pi i(k-s)l/n} D_l = \sum_{l=1}^{n} B_l e^{2\pi i k l/n} , \quad k = 1, \ldots, n .$$

Multiplying by $e^{-2\pi ipk/n}$, $p = 1, \ldots, n$ and summing with respect to k gives

$$\sum_{s=1}^{n} \bar{a}_s \sum_{l=1}^{n} \sum_{k=1}^{n} e^{2\pi i(l-p)k/n} e^{-2\pi i s l n} D_l = \sum_{l=1}^{n} B_l \sum_{k=1}^{n} e^{2\pi i(l-p)k/n} .$$

Since

$$\sum_{k=1}^{n} e^{2\pi irk/n} = n \quad \text{for} \quad r = 0 \pmod n)$$
$$= 0 \quad \text{otherwise}$$

we obtain

$$\sum_{s=1}^{n} \bar{a}_s e^{-2\pi isp/n} = \frac{B_p}{D_p}$$

whence in a similar way we finally obtain

$$a_l = \frac{1}{n} \sum_{p=1}^{n} \frac{\bar{B}_p}{D_p} e^{-2\pi ilp/n} , \quad l = 1, \ldots, n .$$

It is evident that the optimal coefficients depend on the space H_d.

The norm of the error functional $\Phi - \Phi_{\text{opt}}$ is given as the norm of the corresponding representing function $\varphi - \varphi_{\text{opt}}$ in H_d. Since the projection property yields $(\varphi - \varphi_{\text{opt}}, \varphi_{\text{opt}}) = 0$, we have $\|\varphi - \varphi_{\text{opt}}\|^2 = \|\varphi\|^2 - \|\varphi_{\text{opt}}\|^2$. We know that $\|\varphi\|^2 = \sum_{k=-\infty}^{\infty} |b_k|^2 d_k$. The quantity $\|\varphi_{\text{opt}}\|^2$ can be calculated directly from the explicit form of the coefficients a_s. After some manipulations one has

$$\|\varphi_{\text{opt}}\|^2 = \|\sum_{s=1}^{n} \bar{a}_s \psi^{2\pi s/n}\|^2 = \sum_{j=-\infty}^{\infty} \frac{|B_j|^2}{D_j^2} d_j .$$

The last sum may be written in the form

$$\sum_{j=-\infty}^{\infty} \sum_{l=1}^{n} \frac{|B_{jn+l}|^2}{(D_{jn+l})^2} d_{jn+l} .$$

However, it is easily seen that $D_{jn+l} = D_l$ and $B_{jn+l} = B_l$ for all j so that we finally have

$$\chi^2(\Phi, H_d, n) = \sum_{j=-\infty}^{\infty} |b_j|^2 d_j - \sum_{l=1}^{n} \frac{|B_l|^2}{D_l} .$$

We repeat that this norm depends on the given functional (coefficients b_j), on the given space (coefficients d_j) and on the number of mesh points n.

Now, a natural question of the behaviour of the norm $\|\varphi - \varphi_{\text{opt}}\|$ with n tending to infinity arises. We shall prove that the norm tends to zero. That will prove that our choice of approximating functionals is appropriate.

Theorem 3.1. *For an arbitrary functional Φ and an arbitrary space H_d it holds*

$$\lim_{n \to \infty} \|\varphi - \varphi_{\text{opt}}\| = 0$$

Proof. From the expression $\sum_{j=-\infty}^{\infty} |b_j|^2 d_j - \sum_{l=1}^{n} \frac{|B_l|^2}{D_l}$ we obtain

$$(7) \qquad \sum_{l=-[(n-1)/2]}^{[n/2]} \left(\sum_{t=-\infty}^{\infty} |b_{l+nt}|^2 \, d_{l+nt} - \frac{|B_l|^2}{D_l} \right)$$

after easy manipulations.

Now, we fix the subscript l and introduce the notation

$$\sum_{t=-\infty, t \neq 0}^{\infty} |b_{l+nt}|^2 \, d_{l+nt} = K'_l \,, \quad B'_l = B_l - b_l d_l \,, \quad D'_l = D_l - d_l \,.$$

The expression in parentheses in (7) can be written as

$$|b_l|^2 \, d_l + K'_l - \frac{|b_l d_l + B'_l|^2}{d_l + D'_l} = \frac{1}{d_l + D'_l} \left[|b_l|^2 \, d_l^2 + |b_l|^2 \, d_l D'_l + d_l K'_l + D'_l K'_l - \right.$$
$$\left. - |b_l d_l|^2 - b_l d_l \bar{B}'_l - \bar{b}_l d_l B'_l - |B'_l|^2 \right] \leqq |b_l|^2 \, D'_l + K'_l + 2|\bar{b}_l B'_l|$$

and therefore

$$\|\varphi - \varphi_{\text{opt}}\|^2 \leqq \sum_{l=-[(n-1)/2]}^{[n/2]} \left(|b_l|^2 \, D'_l + K'_l + 2|\bar{b}_l B'_l| \right).$$

It is not difficult to see that the right-hand term tends to zero for n tending to infinity.

4. UNIVERSAL APPROXIMATION OF A FUNCTIONAL

The results of the preceding section imply that the optimal approximation and its convergence behaviour depend on the choice of the space H_d. But, as shown in [1], the results obtained by using different optimal formulae in different spaces may differ significantly. Therefore, we are led to the approach introduced in [1], where we reduced our demands on the optimality of the approximation but required that the approximation be in a sense reasonable for a whole class of periodic spaces.

We shall now deal with some further formulae, but we still retain the equidistant mesh. By a formula we will understand a triangular matrix of coefficients

$$P = \begin{bmatrix} p_1^{(1)} & & & & \\ p_1^{(2)} & p_2^{(2)} & & & \\ p_1^{(3)} & p_2^{(3)} & p_3^{(3)} & & \\ \cdots & \cdots & \cdots & & \\ p_1^{(n)} & p_2^{(n)} & p_3^{(n)} & \cdots & p_n^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

The rows of the matrix will be denoted by p_n.

For a given functional Φ we denote by the symbol

$$\varrho(\Phi, H_d, p_n) = \left\| \varphi - \sum_{k=1}^{n} p_k^{(n)} \psi^{2\pi k/n} \right\|_{H_d}$$

the norm of the error functional of the approximation by the formula P and with n mesh points.

We now introduce basic definitions.

Definition 4.1. *We say that the formula P is asymptotically optimal with respect to a functional Φ and a space H_d when*

$$\lim_{n\to\infty} \frac{\varrho(\Phi, H_d, p_n)}{\chi(\Phi, H_d, n)} = 1$$

holds.

Definition 4.2. *We say that the formula P is optimal in order with respect to a functional Φ and a space H_d when the estimate*

$$\frac{\varrho(\Phi, H_d, p_n)}{\chi(\Phi, H_d, n)} < K(\Phi, H_d)$$

holds for all n and the constant K does not depend on n, but may depend on Φ and H_d.

The first definition is based on the asymptotic equality, the second expresses a weaker requirement, namely, it admits that the formula optimal in order is by a multiplicative constant worse than the optimal one.

Further, we shall say that a formula is universally asymptotically optimal or universally optimal in order with respect to a class of periodic spaces if it is asymptotically optimal or optimal in order for all spaces of that class. Our main task is to construct universally asymptotically optimal formulae in as large classes as possible. By that the risk of an inadequate choice of the space for optimization will be reduced to a minimum. Unfortunately, this cannot be achieved in many cases. Detailed results are in [1]. We recall here only those that are connected with our results. We shall say that a functional is of integral type if it is of the form

$$\Phi(f) = \int_0^{2\pi} g(t) f(t) \, \mathrm{d}t \,, \quad g \in L_2(0, 2\pi) \,.$$

In [1] no more general functionals are considered. It was proved that there exists a universally asymptotically optimal formula with respect to the whole class $\mathscr{H}$ if and only if the function g is a trigonometric polynomial. For functionals of integral type a universally optimal in order formula with respect to a subclass of $\mathscr{H}$ was constructed in [1].

Our result contained in the following theorem deals with general functionals.

Theorem 4.1. *Let a continuous functional Φ over $C_{2\pi}$ be given. Let us denote, as previously, $b_k = \overline{\Phi(e^{ikx})}$. Then the formula P with*

$$p_k^{(n)} = \frac{1}{n} \sum_{s=-[(n-1)/2]}^{[n/2]} \overline{b_s^{(n)}} e^{-2\pi i s k/n} \,, \tag{8}$$

where

$$b_s^{(n)} = b_s \quad \text{for} \quad |s| < \frac{n}{2}$$

and (in the case of an even n)

$$b_{n/2}^{(n)} = \tfrac{1}{2}(b_{n/2} + b_{-n/2}) \,,$$

is a universally optimal in order formula for the functional Φ with respect to the class $\mathscr{H}_1$.

Proof. We know the norm of the optimal approximation. We have to calculate the norm of the formula given by (8). The representing function for this formula in the space H_d is

$$\varphi_{P,n}(x) = \sum_{s=1}^{n} \overline{p_s^{(n)}} \sum_{k=-\infty}^{\infty} d_k e^{-2\pi iks/n} e^{ikx} = \sum_{k=-\infty}^{\infty} d_k \Big(\sum_{s=1}^{n} \overline{p_s^{(n)}} e^{-2\pi iks/n} \Big) e^{ikx} . \tag{9}$$

For the norm of the corresponding error functional we have successively

$$\varrho^2(\Phi, H_d, P) = \Big\| \sum_{k=-\infty}^{\infty} b_k d_k e^{ikx} -$$

$$\sum_{k=-\infty}^{\infty} d_k \Big(\sum_{s=1}^{n} \frac{1}{n} \sum_{p=-[(n-1)/2]}^{[n/2]} b_p^{(n)} e^{2\pi ips/n} \Big) e^{-2\pi iks/n} e^{ikx} \Big\|^2 =$$

$$= \Big\| \sum_{k=-\infty}^{\infty} b_k d_k e^{ikx} - \sum_{k=-\infty}^{\infty} \sum_{l=-[(n-1)/2]}^{[n/2]} d_{kn+l} \cdot$$

$$\cdot \Big(\sum_{p=-[(n-1)/2]}^{[n/2]} b_p^{(n)} \sum_{s=1}^{n} \frac{1}{n} e^{2\pi i(p-l)s/n} \Big) e^{i(kn+l)x} \Big\|^2 =$$

$$= \Big\| \sum_{k=-\infty}^{\infty} \sum_{l=-[(n-1)/2]}^{[n/2]} d_{kn+l} \Big(b_{kn+l} - \sum_{p=-[(n-1)/2]}^{[n/2]} b_p^{(n)} \cdot \delta_{pl} \Big) e^{i(kn+l)x} \Big\|^2 =$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-[(n-1)/2]}^{[n/2]} \big| b_{kn+l} - b_l^{(n)} \big|^2 d_{kn+l} =$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-[(n-1)/2]}^{[n/2]} \Big(\big|b_{kn+l}\big|^2 - b_{kn+l} \overline{b_l^{(n)}} - \overline{b_{kn+l}} b_l^{(n)} + \big|b_l^{(n)}\big|^2 \Big) d_{kn+l} =$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-[(n-1)/2]}^{[n/2]} \big|b_{kn+l}\big|^2 d_{kn+l} - \sum_{l=-[(n-1)/2]}^{[n/2]} \Big(\overline{b_l^{(n)}} B_l + b_l^{(n)} \overline{B_l} - \big|b_l^{(n)}\big|^2 D_l \Big) .$$

Let us now consider the difference

$$\varrho^2 - \chi^2 = \sum_{l=-[(n-1)/2]}^{[n/2]} \Big(\frac{|B_l|^2}{D_l} - b_l^{(n)} \overline{B_l} - \overline{b_l^{(n)}} B_l + \big|b_l^{(n)}\big|^2 D_l \Big) =$$

$$= \sum_{l=-[(n-1)/2]}^{[n/2]} \frac{\big|b_l^{(n)} D_l - B_l\big|^2}{D_l} .$$

With the notation D_l' as before and $B_l'' = B_l - d_l b_l^{(n)}$ we obtain

$$\varrho^2 - \chi^2 = \sum_{l=-[(n-1)/2]}^{[n/2]} \frac{\big|b_l^{(n)} D_l' - B_l''\big|^2}{D_l'} \cdot \frac{D_l'}{D_l} .$$

For a fixed l, the Schwarz inequality gives in the case $l \neq n/2$ (i.e. for all l with n odd)

$$\frac{\big|b_l^{(n)} D_l' - B_l''\big|^2}{D_l'} = \frac{\Big| \sum\limits_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \big(b_l^{(n)} d_{kn+l} - b_{kn+l} d_{kn+l} \big) \Big|^2}{\sum\limits_{\substack{k=-\infty \\ k \neq 0}}^{\infty} d_{kn+l}} \leqq$$

$$\leqq \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left|b_l^{(n)} - b_{kn+l}\right|^2 d_{kn+l} = \sum_{k=-\infty}^{\infty} \left|b_l^{(n)} - b_{kn+l}\right|^2 d_{kn+l} .$$

The subscript $k = 0$ in the last sum gives a zero contribution and may be therefore included in the summation.

The case $l = n/2$ for an even n is treated analogously and for the sum we have the estimate

$$\varrho^2 - \chi^2 \leqq \sum_{l=-[(n-1)/2]}^{[n/2]} \sum_{k=-\infty}^{\infty} \left|b_l^{(n)} - b_{kn+l}\right|^2 d_{kn+l} \cdot \frac{D_l'}{D_l} \leqq \left(1 - \frac{1}{D}\right) \varrho^2$$

whence finally

$$\varrho/\chi \leqq \sqrt{D}$$

which completes the proof.

5. FURTHER PROPERTIES OF OPTIMAL AND UNIVERSAL APPROXIMATIONS

In this section we shall state further properties of our types of approximation. In the first place we shall show the connection between the optimal approximation and the approximation (8) and the spline and trigonometric interpolations, respectively. We shall briefly call the concept of the abstract spline interpolation in a Hilbert space. Let Ω be a finite set of functionals ω_i, $i = 1, \ldots, n$ over a Hilbert space H. Let k_i, $i = 1, \ldots, n$ be the corresponding representing elements. Denote by K the linear hull of the elements k_i, $i = 1, \ldots, n$ and by S_n the projector onto K.

Definition 5.1. *Let an arbitrary element $f \in H$ be given. The element $S_n f$ will be called the spline interpolating element of f with respect to the set Ω of functionals.*

It is evident that the element $S_n f$ has the characteristic properties of a spline interpolant, namely

$$\omega_i(S_n f) = \omega_i(f) , \quad i = 1, \ldots, n ,$$

$\|S_n f\| \leqq \|g\|$ for all $g \in H$ satisfying $\omega_i(g) = \omega_i(f)$, $i = 1, \ldots, n$. In what follows we choose $\omega_k = \delta^{2\pi k/n}$ in all spaces $H_d \in \mathscr{H}$.

Let us further denote by T_n the operator of the usual trigonometric interpolation with an equidistant mesh. We suppose the polynomial $T_n f$ to be in the form

$$T_n f(x) = \sum_{l=-[n/2]}^{[n/2]} F_l e^{ilx}$$

and for an even n we demand in addition $F_{-n/2} = F_{n/2}$.

Let $f(x) = \sum_{k=-\infty}^{\infty} f_k e^{ikx}$. The equalities

$$T_n f\left(\frac{2\pi s}{n}\right) = f\left(\frac{2\pi s}{n}\right), \quad s = 1, \ldots, n ,$$

yield

$$\sum_{l=-[n/2]}^{[n/2]} F_l e^{2\pi i l s/n} = \sum_{k=-\infty}^{\infty} f_k e^{2\pi i k s/n} , \quad s = 1, \ldots, n .$$

For an odd n, we can transform the right-hand side into

$$\sum_{k=-\infty}^{\infty} \sum_{l=-[n/2]}^{[n/2]} f_{kn+l} e^{2\pi i l s/n} = \sum_{l=-[n/2]}^{[n/2]} \sum_{k=-\infty}^{\infty} f_{kn+l} e^{2\pi i l s/n},$$

whence we have

$$F_l = \sum_{k=-\infty}^{\infty} f_{kn+l} \quad \text{for} \quad |l| < \frac{n}{2}.$$

For an even n we obtain in a similar way that the right-hand sum is equal to

$$\sum_{l=-[n/2]+1}^{[n/2]-1} e^{2\pi i s l/n} \sum_{k=-\infty}^{\infty} f_{kn+l} + \tfrac{1}{2} \sum_{k-=\infty}^{\infty} f_{kn+n/2} \left(e^{(2\pi i s/n)n/2} + e^{(-2\pi i s/n)n/2}\right),$$

whence we have the same expression as for an odd n for F_l, $|l| < n/2$, and in addition we have

$$F_{n/2} = \tfrac{1}{2} \sum_{k=-\infty}^{\infty} f_{kn+n/2}.$$

Let the functional of the universal approximation given by the formula (8) be denoted by Φ_{univ}. The relation of the approximation and the interpolation can be now formulated.

Theorem 5.1. *For the optimal approximation Φ_{opt} and the approximation Φ_{univ} of a given functional Φ over $C_{2\pi}$ the following equalities hold:*

$$\Phi_{\text{opt}}(f) = \Phi(S_n f),$$
$$\Phi_{\text{univ}}(f) = \Phi(T_n f),$$

where S_n and T_n are the operators of spline and trigonometric interpolations, respectively.

Proof. Let φ and φ_{opt} be the representing functions of Φ and Φ_{opt}, respectively. The first equality can be rewritten as

$$(f, \varphi_{\text{opt}}) = (S_n f, \varphi).$$

The validity of this equation is obvious because of the symmetry of the projector S_n.

The second equality will be proved only for the case of an odd n. For an even n the proof is similar. Thus, let $n = 2m + 1$. With the use of the representing function of Φ_{univ} in H_d given by (9) we obtain

$$\Phi_{\text{univ}}(f) = (f, \varphi_{P,n}) = \sum_{j=-\infty}^{\infty} f_j \Big(\sum_{s=1}^{n} p_s^{(n)} e^{2\pi i j s/n} \Big) =$$

$$= \sum_{j=-\infty}^{\infty} f_j \sum_{s=1}^{n} \frac{1}{n} \sum_{k=-m}^{m} \overline{b_k^{(n)}} e^{-2\pi i s k/n} e^{2\pi i j s/n} \Big) =$$

$$= \sum_{j=-m}^{m} \sum_{t=-\infty}^{\infty} f_{tn+j} \sum_{s=1}^{n} \frac{1}{n} \sum_{k=-m}^{m} \overline{b_k^{(n)}} e^{2\pi i s(j-k)/n} =$$

$$= \sum_{j=-m}^{m} \sum_{t=-\infty}^{\infty} f_{tn+j} \sum_{k=-m}^{m} \overline{b_k^{(n)}} \cdot \delta_{kj} = \sum_{j=-m}^{m} \sum_{t=-\infty}^{\infty} f_{tn+j} \overline{b_j^{(n)}}.$$

On the other hand we have

$$\Phi(T_n f) = (T_n f, \varphi) = \sum_{l=-m}^{m} \frac{F_l \cdot \bar{\varphi}_l}{d_l} = \sum_{l=-m}^{m} \sum_{t=-\infty}^{\infty} f_{tn+l} \cdot \bar{b}_l \,.$$

The equality holds.

Now, we will pay our attention to another fact. The spaces from the class $\mathscr{H}_1$ do not contain all continuous functions. Nevertheless, we may ask whether our universal formula gives a convergent approximation of the given functional for an arbitrary function of $C_{2\pi}$. A partial result is given by the following theorem.

Theorem 5.2. *Let a functional Φ be given by*

$$\Phi(f) = \frac{1}{2\pi} \int_0^{2\pi} g(t) f(t) \, \mathrm{d}t$$

with $g \in L_2(0, 2\pi)$.

Then the formula given by (8) is convergent for all functions from $C_{2\pi}$.

Proof. We have, in fact, to prove weak convergence of a sequence of functionals. The proof will be based on the Banach-Steinhaus principle of uniform boundedness. We know that the convergence takes place for a dense set of functions in $C_{2\pi}$, e.g., for an arbitrary space $H_d \in \mathscr{H}_1$. The only thing we must prove ist he uniform boundedness of norms of the approximating functionals given by the quantities

$$\sum_{k=1}^{n} \left|p_k^{(n)}\right| .$$

We have

$$\left(\sum_{k=1}^{n} \left|p_k^{(n)}\right|\right)^2 \leqq n \cdot \sum_{k=1}^{n} \left|p_k^{(n)}\right|^2 =$$

$$= \frac{1}{n} \sum_{k=1}^{n} \sum_{s=-[(n-1)/2]}^{[n/2]} \sum_{j=-[(n-1)/2]}^{[n/2]} \overline{b_s^{(n)}} b_j^{(n)} e^{2\pi i (j-s)k/n} = \sum_{s=-[(n-1)/2]}^{[n/2]} \left|b_s^{(n)}\right|^2 \leqq \left\|g\right\|_{L_2}^2 .$$

Because of the independence of the norm $\left\|g\right\|_{L_2}$ of n we have the result needed. The theorem is proved.

Although the creation of the theory of universal optimality was motivated by the possibility of avoiding the classical error estimates which include a certain power of the mesh step and the maximum of the corresponding derivative of the function, the relation of the universal formula to the trigonometric interpolation enables us to establish such a classical estimate. Let the given function f be from $C_{2\pi}^{(p)}$ and let $f^{(p+1)}$ exist and be bounded by M_{p+1}. Further, let g be the trigonometric polynomial of degree at most n of the best approximation of f in $C_{2\pi}$. Then, the equality

$$f - T_n f = f - g - T_n(f - g)$$

yields the estimate

$$\left|\Phi(f) - \Phi_{\mathrm{univ}}(f)\right| = \left|\Phi(f - T_n f)\right| \leqq \left\|\Phi\right\|_{C_{2\pi}'} \left\|f - T_n f\right\|_{C_{2\pi}} \leqq$$
$$\leqq \left\|\Phi\right\|_{C_{2\pi}'} \left(\left\|f - g\right\|_{C_{2\pi}} + \left\|T_n(f - g)\right\|_{C_{2\pi}}\right) .$$

The Jackson theorems (see [4]) give us

$$|\Phi(f) - \Phi_{\text{univ}}(f)| \leqq \|\Phi\|_{C_{2\pi}'} \cdot \frac{12^{p+1} M_{p+1}}{n^{p+1}} (1 + \lambda_n),$$

where λ_n is the norm of the interpolation operator T_n. It is known (see again [4]) that $\lambda_n < A + B \ln n$ with some constants A and B. Summarizing, we have an estimate

$$|\Phi(f) - \Phi_{\text{univ}}(f)| \leqq \max_{\langle 0, 2\pi \rangle} |f^{(p+1)}(x)| \cdot h^{p+1}(A' + B' \ln h),$$

where A' and B' are constants independent of f.

It is easily seen that Theorem 4.1 remains true if we replace the space $C_{2\pi}$ by the space $C_{2\pi}^{(n)}$ and the class $\mathscr{H}_1$ by the class $\mathscr{H}_1^{(n)}$. The proof is entirely identical. The only thing we must realize is that the function $\varphi = \sum b_k d_k e^{ikx}$ representing the functional Φ ($b_k = \Phi(e^{ikx})$ as before) is an element of the space $H_d \in \mathscr{H}_1^{(n)}$. The coefficients b_k as coefficients of a distribution of order n are not bounded any more, we have only the estimate $|b_k| \leqq B|k|^n$. On the other hand the sequence defining the space H_d satisfies the condition $\sum k^{2n} d_k < +\infty$. So, we obtain $\|\varphi\|_{H_d}^2 = \sum |b_k|^2 \cdot d_k \leqq \sum B^2 |k|^{2n} d_k < < +\infty$.

Thus, the formula (8) gives a universal approximation in the space $C_{2\pi}^{(n)}$ employing the function values only. In such spaces the values of derivatives of various orders may also be used for approximation, but we do not deal here with such problems. This knowledge about the approximation in the space $C_{2\pi}^{(n)}$ enables us to approximate (using the function values only) distributions of finite order. In this way the set of functionals that are approximated by the formula (8) is essentially extended in comparison with the set of functionals over only the space $C_{2\pi}$.

6. APPLICATIONS

As an application we shall approximate by the formula (8) the functional of principal value which, in fact, is no functional over $C_{2\pi}$.

Let us compute v.p. $\int_{-1}^{1} (f(x)/x)\,dx$ for a function defined in the interval $\langle -1, 1 \rangle$. We make the substitution $x = \cos t$ and obtain

$$\text{v.p.} \int_{-1}^{1} \frac{f(x)}{x}\,dx = \lim_{\varepsilon \to 0} \left(\int_0^{(\pi/2)-\varepsilon} + \int_{(\pi/2)+\varepsilon}^{\pi} \right) f(\cos t) \cdot \sin t/\cos t\,dt.$$

Now, we can view the function $f(\cos t)$ as a periodic function defined on the interval $\langle 0, 2\pi \rangle$. Therefore, we consider the expression

$$\lim_{\varepsilon \to 0} \left(\int_0^{(\pi/2)-\varepsilon} + \int_{(\pi/2)+\varepsilon}^{\pi} \right) g(t) \sin t/\cos t\,dt$$

for a function $g \in C_{2\pi}^{(n)}$, where $n \geqq 1$. It is easy to see that this limit exists and that it represents a linear functional over $C_{2\pi}^{(n)}$ which we denote by Φ.

Indeed, we can write

$$\Phi(f) = \lim_{\varepsilon\to 0}\left(\int_0^{(\pi/2)-\varepsilon} + \int_{(\pi/2)+\varepsilon}^{\pi}\right) g(t)\sin t/\cos t \,\mathrm{d}t =$$

$$= \int_0^{\pi} G(t)\,\mathrm{d}t + \lim_{\varepsilon\to 0}\left(\int_0^{(\pi/2)-\varepsilon} + \int_{(\pi/2)+\varepsilon}^{\pi}\right) g(\pi/2)\sin t/\cos t \,\mathrm{d}t\,,$$

where $G(t) = (g(t) - g(\pi/2))\sin t/\cos t$. However, the function G is continuous and the corresponding integral is well-defined. The value of the limit on the right-hand side may be calculated explicitly and is equal to zero. It is evident that the functional Φ is additive and homogeneous. Its continuity follows from the estimate

$$|\Phi(f)| = \left|\int_0^{\pi} \frac{g(t) - g(\pi/2)}{t - \pi/2}\,\frac{t - \pi/2}{\cos t}\sin t\,\mathrm{d}t\right| \leqq$$

$$\leqq \int_0^{\pi} |g'(\xi)| \left|\frac{t - \pi/2}{\cos t}\sin t\right| \mathrm{d}t \leqq \|g\|_{C_{2\pi}^{(n)}} \int_0^{\pi} \left|\frac{t - \pi/2}{\cos t}\sin t\right| \mathrm{d}t\,.$$

In order to apply our formula we calculate first the coefficients b_k. According to the definition we have

$$\bar{b}_k = \Phi(e^{ikx}) = \lim_{\varepsilon\to 0}\left(\int_0^{(\pi/2)-\varepsilon} + \int_{(\pi/2)+\varepsilon}^{\pi}\right) e^{ikt}\sin t/\cos t\,\mathrm{d}t\,.$$

These values can be found explicitly, we have, e.g., $b_0 = 0$, $b_1 = 2$ etc. In this case, the functional is real, i.e. it has a real value for a real function. We can therefore calculate only the real parts of the coefficients b_k. From these values the coefficients $p_k^{(n)}$ of the formula (8) are computed on a computer. We then obtain a linear combination of function values at the points of the whole interval $\langle 0, 2\pi\rangle$. Because of the prolongation of the function $f(\cos t)$ many of the necessary function values repeat. All these properties were taken into consideration when the programme was written. The fast Fourier transform may be also used for the computation.

The following table gives the absolute errors of the approximation and the exact values of the singular integral $\int_{-1}^{1} f(x)/x\,\mathrm{d}x$ computed for five different functions.

m	f_1	f_2	f_3	f_4	f_5
2	5·67 − 2	3·00 − 1	1·32 − 1	5·81 − 1	1·06 − 1
6	2·53 − 4	2·78 − 2	9·65 − 6	4·80 − 3	1·53 − 4
10	1·28 − 6	2·78 − 2	7·60 − 11	2·52 − 6	4·58 − 7
14	6·52 − 9	2·33 − 2	8·80 − 14	6·60 − 8	1·67 − 9
18	3·35 − 11	1·96 − 2		1·65 − 10	6·52 − 12
exact value	−0·54931	0·52325	1·89217	1·24944	1·29584

The computation was performed on the IBM 370/135 computer in double precision for the following functions: $f_1 = 1/(x + 2)$, $f_2 = |x| . \ln(x + 2)$, $f_3 = \sin x$, $f_4 = \ln(x + 2) . \sin e^x$, $f_5 = x . \ln(x + 2)$. The number of points was $n = 2m + 1$. With the exception of the function f_2, the convergence is very good and the error has exponential behaviour. The last approximate values may be influenced by round-off errors. The application of the formula (8) to the function f_2 is, however, illegal, because f_2 does not belong to $C_{2\pi}^{(1)}$. The corresponding singular integral exists.

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Souhrn

UNIVERZÁLNĚ OPTIMÁLNÍ APROXIMACE FUNKCIONÁLŮ

MILAN PRÁGER

V článku je zkonstruována univerzální řádově optimální aproximace obecného funcionálu nad prostorem $C_{2\pi}$ spojitých 2π-periodických funkcí. Tím jsou zobecněny některé výsledky uvedené v [1]. Vyšetřují se některé základní vlastnosti uvedené aproximace. Efektivnost postupu je ilustrována numerickým příkladem výpočtu singulárních integrálů.

Author's address: RNDr. *Milan Práger,* CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.