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ON THE APPROXIMATE SOLUTION OF THE MULTI-GROUP TIME-DEPENDENT TRANSPORT EQUATION BY P_L-METHOD

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The P_L-method, sometimes called the spherical-harmonics method, is one of the most powerful tools available for solving the neutron transport equation especially for the steady-state one-velocity equation. This paper deals with a study of the P_L-method for an approximation of solution of the multi-group time-dependent neutron transport mixed problem with three-dimensional geometry.

1. INTRODUCTION

Denote by $N(t, \boldsymbol{x}, \boldsymbol{\omega}, c)$ the neutron density function, which represents the flux of neutrons at the time t at the position $\boldsymbol{x} = (x_1, x_2, x_3)$. The velocity of the moving neutron is denoted by c (it is sometimes interpreted as an energy of neutron) and the direction of the motion of the neutron is denoted by the unit vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$. We consider the following integro-differential equation (see [1], [2])

$$(1.1) \qquad \frac{\partial N}{\partial t} + c\boldsymbol{\omega} \,.\, \operatorname{grad} N + c\sigma N =$$

$$= \int_\Omega \int_0^\infty \frac{\sigma_s(\boldsymbol{x}, c')}{4\pi} h(\boldsymbol{x}, c', \boldsymbol{\omega}' \to \boldsymbol{\omega}, c)\, c' N(t, \boldsymbol{x}, \boldsymbol{\omega}', c')\, \mathrm{d}\boldsymbol{\omega}'\, \mathrm{d}c' + F\,,$$

where $\sigma(\boldsymbol{x}, c)$, $\sigma_s(\boldsymbol{x}, c)$ are total and differential cross sections for scattering neutrons (characterizing the medium — $\sigma(\boldsymbol{x}, c)$, $\sigma_s(\boldsymbol{x}, c)$ is the probability per unit time that a neutron in position $\boldsymbol{x}$ with speed c will undergo a collision), $F = F(t, \boldsymbol{x}, \boldsymbol{\omega}, c)$ represents extraneous neutron sources, $h(\boldsymbol{x}, c', \boldsymbol{\omega}' \to \boldsymbol{\omega}, c)$ describes the transfer of neutron energy, $(h(\boldsymbol{x}, c', \boldsymbol{\omega}' \to \boldsymbol{\omega}, c)\, \mathrm{d}\boldsymbol{\omega}\, \mathrm{d}c$ is the probability that a neutron in position $\boldsymbol{x}$, with energy c', moving in the direction $\boldsymbol{\omega}'$ after collision is moving in the range of directions $\langle \boldsymbol{\omega}, \boldsymbol{\omega} + \mathrm{d}\boldsymbol{\omega} \rangle = \langle \omega_1, \omega_1 + \mathrm{d}\omega_1 \rangle \times \langle \omega_2, \omega_2 + \mathrm{d}\omega_2 \rangle \times \times \langle \omega_3, \omega_3 + \mathrm{d}\omega_3 \rangle$ and velocities $\langle c, c + \mathrm{d}c \rangle$.

We shall now assume that in our medium there are only neutrons with discrete distributions of velocities (energies) $c_1 < c_2 < \ldots < c_l$ and that h depends only on the angle of the directions ω, ω' (precisely on $\cos(\widehat{\omega, \omega'}) = \mu_0 = \omega_1\omega'_1 + \omega_2\omega'_2 + \omega_3\omega'_3$).

After a rearrangement of some terms in the equation (1.1) we obtain the usual multi-group transport equation (see [6], [16]) ($j = 1, 2, \ldots, l$):

$$(1.2) \qquad \frac{1}{c_j}\frac{\partial u_j}{\partial t} + \omega \,.\, \operatorname{grad} u_j + \sigma_j(x)\, u_j = \sum_{k=1}^{l} \frac{1}{4\pi} \int_\Omega \sigma_k^r(x)\, h_{jk}(\mu_0)\, u_k(t, x, \omega')\, \mathrm{d}\omega' + f_j \,.$$

Here $u_j = u_j(t, x, \omega) = c_j n_j$, where n_j is the neutron density of the j-th velocity group of neutrons with a speed $c_j > 0$, $f_j = f_j(t, x, \omega)$ is the source function, $\sigma_j(x)$, $\sigma_j^r(x)$ are the total and differential cross sections, respectively, related with the velocity group j, h_{kj} represents a probability that after a collision the neutrons pass from the k-th velocity group to the j-th velocity group. For example, if in our medium two nuclear reactions are taking place – scattering and fission – then instead of $\sigma_k^r(x)\, h_{jk}$ we have $\sigma_k^s(x)\, h_{jk}^s(\mu_0) + \sigma_k^f \nu_k^f(x)\, h_{jk}^f$ (ν_k is the mean number of secondary neutrons per fission in the group k). From the physical assumptions it follows that $h_{jk}^s = 0$ for $j > k$ and therefore for scattering the 1-st – j-th terms in the sum (1.2) can be left out.

Our approach to the problem is based on some results of [5], [7], [8].

2. STATEMENT OF THE PROBLEM

Let us denote the region of the medium by G and assume that G is a bounded convex domain in the three-dimensional Euclidean space R_3 with boundary ∂G, consisting of a finite number of sufficiently smooth hypersurfaces with the outward unit normal vector $\boldsymbol{n} = \boldsymbol{n}(x) = (n_1, n_2, n_3)$, Ω – the unit sphere with the centre at $x \in G$ is a set of directions ω.

Assuming an l-group formalism, $\mathbf{u}, \boldsymbol{\varphi}, \mathbf{f}$ are vectors of order l with components $u_j(t, x, \omega)$, $\varphi_j(x, \omega)$, $f_j(t, x, \omega)$, we consider the equation (1.2) in the form

$$(2.1) \qquad \mathbf{Du} \equiv \mathbf{Lu} - \mathbf{Hu} = \mathbf{f}\,,$$

where the operator $\mathbf{L}$ is diagonal with elements L_j, where

$$(2.2) \qquad L_j u_j \equiv \frac{1}{c_j}\frac{\partial u_j}{\partial t} + \omega \,.\, \operatorname{grad} u_j + \sigma_j u_j\,, \quad j = 1, 2, \ldots, l\,,$$

and

$$(2.3) \qquad \mathbf{Hu} \equiv \int_\Omega \mathfrak{H}(x, \mu_0)\, \mathbf{u}(t, x, \omega')\, \mathrm{d}\omega'\,,$$

where the j-th component of vector $\mathbf{Hu}$ is

$$\sum_{k=1}^{l} \frac{\sigma_k^r(x)}{4\pi} \int_\Omega h_{jk}(\mu_0)\, u_k(t, x, \omega)\, d\omega .$$

l-dimensional vector-valued function $\mathbf{Hu}$ is given by the sum of integrals in the equations (1.2).

The boundary condition to be imposed in the present paper is that no neutrons enter G from outside through the surface ∂G. Define

$$\Gamma = \Gamma_+ \cup \Gamma_- = \partial G \times \Omega ,$$
$$\Gamma_- = \{(x, \omega) \in \partial G \times \Omega,\ n . \omega < 0\} ,$$
$$\Gamma_+ = \{(x, \omega) \in \partial G \times \Omega,\ n . \omega \geqq 0\} .$$

Then this boundary condition is expressed by

(2.4) $$\mathbf{u}(t, x, \omega) = \mathbf{0} \quad \text{on} \quad \langle 0, T\rangle \times \Gamma_- .$$

The initial condition will be

(2.5) $$\mathbf{u}(0, x, \omega) = \boldsymbol{\varphi}(x, \omega) .$$

We further introduce the abbreviations:

$$(u_j, v_j)_Q = \int_Q u_j(t, x, \omega)\, \bar{v}_j(t, x, \omega)\, dt\, dx\, d\omega ; \quad Q = (0, T) \times G \times \Omega ,$$

$$[\mathbf{u}, \mathbf{v}]_Q = \sum_{j=1}^{l} (u_j, v_j)_Q ; \quad [\mathbf{u}, \mathbf{v}] = \sum_{j=1}^{l} u_j v_j .$$

Denote by $\mathscr{C}_2^k = \mathscr{C}_2^k(\langle 0, T\rangle; L_2(G \times \Omega))$ the cartesian product (taken l-times) of spaces $C_2^k = C^k(\langle 0, T\rangle; L_2(G \times \Omega))$ with the norm

(2.6) $$\|u_j\|_{C_2{}^k} = \sum_{\alpha=0}^{k} \sup_{t \in \langle 0, T\rangle} \left\| \frac{\partial^\alpha u_j}{\partial t^\alpha} \right\|_{L_2(G \times \Omega)} .$$

Then

$$\|\mathbf{u}\|_{\mathscr{C}_2{}^k} = \left(\sum_{j=1}^{l} \|u_j\|^2_{C_2{}^k}\right)^{1/2} .$$

The cartesian product of spaces $C(G \times \Omega)$ or $C(Q)$ will be denoted by $\mathscr{C}(G \times \Omega)$ and $\mathscr{C}(Q)$ respectively.

Analogously $\mathscr{L}_2$ will be the cartesian product of spaces L_2 with the norm

$$\|\boldsymbol{\varphi}\|_{\mathscr{L}_2} = \left(\sum_{j=1}^{l} \|\varphi_j\|_{L_2}\right)^{1/2} .$$

We introduce the following *Hypothesis*:

i) $\sigma_k, \sigma_k^r \in L_\infty(G)$, $k = 1, 2, \ldots, l$,

ii) $\sigma_k^r(x) \geqq 0$, $\sigma_k(x) > 0$ and there exist constants $\sigma_{k0} > 0$ such that $\sigma_k(x) > \sigma_{k0}$, $k = 1, 2, \ldots, l$,

iii) $\int_{-1}^{1} h_{jk}^2(\mu_0)\,d\mu_0 < \infty$, $h_{jk}(\mu_0) \geqq 0$, $j, k = 1, 2, \ldots, l$.

Lemma 2.1. *Under the Hypotheses* i), ii), iii), *suppose that* $\mathbf{u} \in \mathscr{L}_2(G \times \Omega)$ *for all* $t \in \langle 0, T\rangle$. *Then for all* $t \in \langle 0, T\rangle$

$$\mathbf{Hu} \in \mathscr{L}_2(G \times \Omega) \quad \textit{and} \quad \|\mathbf{Hu}\|_{\mathscr{L}_2(G\times\Omega)} \leqq \text{const}\ \|\mathbf{u}\|_{\mathscr{L}_2(G\times\Omega)}\,.$$

Proof. Using the results of [4],

$$\text{(a)} \qquad \int_\Omega |h_{jk}(\mu_0)|^2\,d\omega' = 2\pi \int_{-1}^{1} h_{jk}^2(\mu_0)\,d\mu_0$$

and Hölder's inequality we have

$$\int_{G\times\Omega} \left| \int_\Omega \frac{\sigma_k^r(x)}{4\pi} h_{jk}(\mu_0)\, u_k(t, x, \omega')\,d\omega' \right|^2 dx\,d\omega \leqq$$

$$\leqq \text{const} \int_{G\times\Omega} |u_k|^2\,dx\,d\omega \left[\int_{\Omega\times\Omega} |h_{jk}(\mu_0)|^2\,d\omega\,d\omega' \right].$$

Corollary. *For* $\mathbf{u} \in \mathscr{C}_2^k$ *it is* $\mathbf{Hu} \in \mathscr{C}_2^k$ *and* $\|\mathbf{Hu}\|_{\mathscr{C}_2^k} \leqq \text{const}\ \|\mathbf{u}\|_{\mathscr{C}_2^k}$.

Let $u_j, v_j \in W_2^1(G)$ (for fixed $(t, \omega) \in \langle 0, T\rangle \times \Omega$), then Green's formula (generally for complex-valued functions) holds

$$\text{(2.7)} \qquad \int_G \omega v_j \,.\, \text{grad}\, u_j\,dx = -\int_G \omega u_j \,.\, \text{grad}\, v_j\,dx + \int_{\partial G} \boldsymbol{n} \,.\, \omega u_j v_j\,ds\,,$$

where the derivatives should be taken in the sense of Sobolev and the surface integral for traces. If u_j, v_j and ∂G are sufficiently smooth, then (2.7) is obvious via the integration by parts. Hence it is valid also in $W_2^1(G)$.

The formula (2.7) will play an important role hereafter.

We define a diagonal matrix-operator Λ with elements Λ_j, where

$$\Lambda_j u_j \equiv \omega \,.\, \text{grad}\, u_j + \sigma_j u_j\,, \quad j = 1, 2, \ldots, l\,,$$

with the domain $\mathscr{D}(\Lambda)$ given by

$$\mathscr{D}(\Lambda) = \{\mathbf{u} \in \mathscr{L}_2(G \times \Omega);\ \Lambda\mathbf{u} \in \mathscr{L}_2(G \times \Omega),\ \forall t \in \langle 0, T\rangle;\ u_j \in C^1(Q)$$
$$\text{for } j = 1, 2, \ldots, l \text{ and satisfies the boundary condition (2.4)}\}.$$

Obviously, the range of $\Lambda \subset \mathscr{C}$. The closure of $\mathscr{D}(\Lambda)$ in $\mathscr{C}_2^1$ will be denoted again by $\mathscr{D}(\Lambda)$. Λ is a densely defined closable operator in this space. We denote its closure again by Λ.

Lemma 2.2. *Under the assumption* ii) $\mathbf{\Lambda}$ *is dissipative on* $\mathscr{L}(\mathbf{\Lambda})$, *i.e.*

$$\mathrm{Re}\,[\mathbf{\Lambda u}, \mathbf{u}]_{G\times\Omega} \geqq 0\,, \quad \mathbf{u} \in \mathscr{L}(\mathbf{\Lambda})\,. \tag{2.8}$$

Proof. According to the identities

$$\mathrm{Re}\,(\bar{u}_j \,\mathrm{grad}\, u_j) = \mathrm{Re}\,(u_j \,\mathrm{grad}\, \bar{u}_j) = \mathrm{Re}\, u_j \,\mathrm{grad}\,(\mathrm{Re}\, u_j) + \\ + \mathrm{Im}\, u_j \,\mathrm{grad}\,(\mathrm{Im}\, u_j)$$

and (2.7) we have

$$\mathrm{Re} \int_{G\times\Omega} \bar{u}_j \omega \,.\, \mathrm{grad}\, u_j \,\mathrm{d}x \,\mathrm{d}\omega = \frac{1}{2} \int_{G\times\Omega} \omega \,.\, \mathrm{grad}\,(u_j \bar{u}_j) \,\mathrm{d}x \,\mathrm{d}\omega = \\ = \frac{1}{2} \int_{\partial G\times\Omega} \boldsymbol{n} \,.\, \omega u_j \bar{u}_j \,\mathrm{d}s \,\mathrm{d}\omega\,.$$

It follows that

$$\mathrm{Re}\,(\Lambda_j u_j, u_j)_{G\times\Omega} = \mathrm{Re} \int_{G\times\Omega} \bar{u}_j \omega \,.\, \mathrm{grad}\, u_j \,\mathrm{d}x \,\mathrm{d}\omega + \mathrm{Re} \int_{G\times\Omega} \sigma_j u_j \bar{u}_j \,\mathrm{d}x \,\mathrm{d}\omega = \\ = \frac{1}{2} \int_{\partial G\times\Omega} \boldsymbol{n} \,.\, \omega u_j \bar{u}_j \,\mathrm{d}s \,\mathrm{d}\omega + \int_{G\times\Omega} \sigma_j u_j \bar{u}_j \,\mathrm{d}x \,\mathrm{d}\omega\,.$$

Using the hypothesis ii) and the boundary condition (2.4) we have

$$\int_{\partial G\times\Omega} \boldsymbol{n} \,.\, \omega u_j \bar{u}_j \,\mathrm{d}s \,\mathrm{d}\omega = \int_{\Gamma_+} \boldsymbol{n} \,.\, \omega u_j \bar{u}_j \,\mathrm{d}s \,\mathrm{d}\omega \geqq 0 \quad \text{because} \quad \boldsymbol{n} \,.\, \omega \geqq 0 \quad \text{on} \quad \Gamma_+\,;$$

$$\int_{G\times\Omega} \sigma_j u_j \bar{u}_j \,\mathrm{d}x \,\mathrm{d}\omega \geqq \sigma_{j0} \int_{G\times\Omega} u_j \bar{u}_j \,\mathrm{d}x \,\mathrm{d}\omega = \sigma_{j0} \int_{G\times\Omega} [(\mathrm{Re}\, u_j)^2 + \\ + (\mathrm{Im}\, u_j)^2] \,\mathrm{d}x \,\mathrm{d}\omega \geqq 0\,,$$

which was to be proved.

Remark 2.1. *In the course of proving Lemma* 2.2 *we obtained*

$$\mathrm{Re}\,(\Lambda_j u_j, u_j)_{G\times\Omega} \geqq \sigma_{j0} \|u_j\|^2_{L_2(G\times\Omega)}\,. \tag{2.9}$$

Remark 2.2. *If* (2.8) *holds we will say that the boundary condition* (2.4) *is dissipative.*

Denote

$$\mathbf{D}^* = \mathbf{L}^* - \mathbf{H}^*\,, \tag{2.10}$$

where $\mathbf{L}^*$ is formally adjoint to $\mathbf{L}$, therefore $\mathbf{L}^*$ is also a diagonally matrix-operator with elements L_j^*, where $L_j^* v_j = -(1/c_j) \,.\, (\partial v_j/\partial t) - \omega \,.\, \mathrm{grad}\, v_j + \sigma_j v_j$. Similarly to $\mathbf{H}$, the operator $\mathbf{H}^*$ is a matrix-integral operator

$$\mathbf{H}^* \mathbf{v} = \int_\Omega \mathfrak{H}^*(x, \mu_0)\, \mathbf{v}(t, x, \omega)\, d\omega\,, \tag{2.11}$$

where the j-th component of the vector $\mathbf{H}^*\mathbf{v}$ is

$$\sum_{k=1}^{l} \frac{1}{4\pi} \sigma_j^r(x) \int_\Omega h_{kj}(\mu_0)\, v_k(t, x, \omega)\, d\omega\,.$$

3. SOLUTION OF THE PROBLEM. A PRIORI BOUND

In order to study the solution of Problem (2.1), (2.4), (2.5) we use the following notation

$\mathscr{R}(\mathbf{D}) \equiv \{\mathbf{u} \in \mathscr{C}_2^1;\ \mathbf{u}(0, x, \omega) = \boldsymbol{\varphi}(x, \omega),\ \boldsymbol{\varphi} \in \mathscr{L}_2(G \times \Omega);\ \mathbf{u}(t, x, \omega) = 0$ *on* $\langle 0, T\rangle \times \times \Gamma_-$ (*in the sense of traces*); $\boldsymbol{\omega}\,.\,\mathrm{grad}\, u_j \in C_2,\ t \in \langle 0, T\rangle\}$;

$\mathscr{R}(\mathbf{D}^*) \equiv \{\mathbf{v} \in \mathscr{C}(\bar{Q});\ \mathbf{v}_t \in \mathscr{C}(Q),\ \boldsymbol{\omega}\,.\,\mathrm{grad}\, v_j \in C(\bar{Q});\ \mathbf{v}(T, x, \omega) = \mathbf{0};\ \mathbf{v}(t, x, \omega) = \mathbf{0}$ *on* $\langle 0, T\rangle \times \Gamma_+\}$.

The problem (2.1), (2.4), (2.5) can be formulated as follows: To find $\mathbf{u} \in \mathscr{R}(\mathbf{D})$ such that

$$[\mathbf{u}, \mathbf{D}^*\mathbf{v}]_Q - [\mathbf{c}^{-1}\boldsymbol{\varphi}, \mathbf{v}(0, x, \omega)]_{G\times\Omega} = [\mathbf{f}, \mathbf{v}]_Q\,, \quad \forall \mathbf{v} \in \mathscr{R}(\mathbf{D}^*)\,. \tag{3.1}$$

If, moreover, $\mathbf{u}$ is a sufficiently smooth function on $\bar{Q}$ (for details see [16]), then it is a solution in the classical sense ($\mathbf{c}^{-1}$ is the diagonal matrix with the elements $1/c_j$).

In [13] conditions are given for the existence and uniqueness of the solution of general time-dependent multi-velocity transport equation in the space $\mathscr{L}_2(Q)$ and a construction of the solution is given by a successive approximations. Analogous results by methods of integral equations are obtained in [17], [18]. Our considerations are based on similar ideas which were used for mono-velocity time-dependent transport equation in [5]. For the solution of Problem (2.1), (2.4), (2.5) we shall obtain an a priori estimate for $\mathbf{u}$, which is based on an energy inequality.

Theorem 3.1. *Let* $\mathbf{f} \in \mathscr{C}_2^1$, $\boldsymbol{\varphi} \in \mathscr{L}_2(G \times \Omega)$, $\boldsymbol{\varphi}(x, \omega) = \mathbf{0}$ *on* Γ_-, $h_{jk}(\mu_0) \in L_2(-1, 1)$ *and let* $\mathbf{u}$ *be a real solution of Problem* (2.1), (2.4), (2.5) *in the sense of* (3.1); *then*

$$\|\mathbf{u}\|_{\mathscr{C}_2^1} \leqq \chi_1(\|\boldsymbol{\varphi}\|_{\mathscr{L}_2} + \|\tilde{\boldsymbol{\varphi}}\|_{\mathscr{L}_2}) + \chi_2 \|\mathbf{f}\|_{\mathscr{C}_2^1}\,; \tag{3.2}$$

$$[\boldsymbol{n}\,.\,\boldsymbol{\omega}\mathbf{u}, \mathbf{u}]_{\partial G\times\Omega} \leqq 2\|\mathbf{u}\|_{\mathscr{L}_2} \left\{\frac{1}{c_{\min}} \|\mathbf{u}_t\|_{\mathscr{L}_2} + \sigma_0^r \|\mathbf{u}\|_{\mathscr{L}_2} + \|\mathbf{f}\|_{\mathscr{L}_2}\right\}, \tag{3.3}$$

$$\forall t \in \langle 0, T\rangle\,.$$

The constants χ_1, χ_2 *depend only on* $\sup_{k,x} \sigma_k^r(x)$, l, T, $c_{\max}$, $\inf_k \sigma_{k0} > 0$.

Proof. We multiply Eq. (2.1) by the function $2\mathbf{u}$ and integrate over $G \times \Omega$ (assuming t fixed). We estimate the form $[\mathbf{Du}, \mathbf{u}]_{G\times\Omega} = [\mathbf{Lu}, \mathbf{u}]_{G\times\Omega} - [\mathbf{Hu}, \mathbf{u}]_{G\times\Omega}$. We have (using (2.3))

$$[\mathbf{Hu}, \mathbf{u}]_{G\times\Omega} = [\mathbf{u}, \mathbf{Hu}]_{G\times\Omega} = \left[\mathbf{u}, \int_\Omega \dot{\mathfrak{H}}(x, \mu_0)\, \mathbf{u}(t, x, \omega')\, \mathrm{d}\omega'\right]_{G\times\Omega} =$$

$$= \int_{G\times\Omega} \sum_{j=1}^{l} u_j(t, x, \omega) \sum_{k=1}^{l} \int_\Omega \frac{\sigma_k^r(x)}{4\pi}\, h_{jk}(\mu_0)\, u_k(t, x, \omega')\, \mathrm{d}\omega'\, \mathrm{d}x\, \mathrm{d}\omega.$$

By Schwarz's inequality and the result (a) used in the proof of Lemma 2.1 it follows

$$\left|\int_{G\times\Omega} \left(\frac{\sigma_k^r(x)}{4\pi}\, u_j(t, x, \omega) \int_\Omega h_{jk}(\mu_0)\, u_k(t, x, \omega')\, \mathrm{d}\omega'\right) \mathrm{d}x\, \mathrm{d}\omega\right| \leqq$$

$$\leqq \sup_{x\in G} \sigma_k^r(x)\, \tilde{h}_{jk} \|u_j\|_{L_2(G\times\Omega)} \cdot \|u_k\|_{L_2(G\times\Omega)},$$

where

$$\tilde{h}_{jk} = \left(\frac{1}{2}\int_{-1}^{1} h_{jk}^2(\mu_0)\, \mathrm{d}\mu_0\right)^{1/2}.$$

From here and from the assumptions i) – iii) it follows that

$$[\mathbf{Hu}, \mathbf{u}]_{G\times\Omega} \leqq \sup_{x\in G} \sigma_k^r(x) \sum_{j=1}^{l} \|u_j\|_{L_2(G\times\Omega)} \sum_{k=1}^{l} \|u_k\|_{L_2(G\times\Omega)} \leqq$$

$$= l \max_{j,k} \left(\sup_{x\in G} \sigma_k^r(x)\, \tilde{h}_{jk}\right) \|\mathbf{u}\|^2_{\mathscr{L}_2(G\times\Omega)} = l\sigma_0^r \|\mathbf{u}\|^2_{\mathscr{L}_2(G\times\Omega)}.$$

By the obvious inequality $2|a|\,|b| \leqq (1/\varepsilon)\, a^2 + \varepsilon b^2$ ($\varepsilon > 0$, a, b real), we further obtain

$$2[\mathbf{f}, \mathbf{u}]_{G\times\Omega} \leqq \frac{1}{\varepsilon} \|\mathbf{f}\|^2_{\mathscr{L}_2(G\times\Omega)} + \varepsilon \|\mathbf{u}\|^2_{\mathscr{L}_2(G\times\Omega)}.$$

By (2.9) we can write

$$[\Lambda\mathbf{u}, \mathbf{u}]_{G\times\Omega} \geqq \min_j \sigma_{j0} \|\mathbf{u}\|^2_{\mathscr{L}_2(G\times\Omega)} = \sigma_0 \|\mathbf{u}\|^2_{\mathscr{L}_2}.$$

Then

$$2[\mathbf{Lu}, \mathbf{u}]_{G\times\Omega} \geqq \frac{1}{c_{\max}} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{u}\|^2_{\mathscr{L}_2} + 2\sigma_0 \|\mathbf{u}\|^2_{\mathscr{L}_2}$$

(it is easily shown that $\|\mathbf{u}\|^2_{\mathscr{L}_2}$ is differentiable with respect to t).

By combining these results we obtain (for all $t \in \langle 0, T\rangle$)

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{u}\|^2_{\mathscr{L}_2} \leqq \sigma^* \|\mathbf{u}\|^2_{\mathscr{L}_2} + \frac{c_{\max}}{\varepsilon} \|\mathbf{f}\|^2_{\mathscr{L}_2}, \quad \sigma^* = c_{\max}(2\sigma_0^r l - 2\sigma_0 + \varepsilon)$$

(we take such an ε to guarantee $\sigma^* > 0$).

The integration and $\|\mathbf{u}(0, x, \omega)\|^2_{\mathscr{L}_2} = \|\boldsymbol{\varphi}\|^2_{\mathscr{L}_2}$ leads to

$$\|\mathbf{u}\|^2_{\mathscr{C}_2} \leqq \chi_1 \|\boldsymbol{\varphi}\|^2_{\mathscr{L}_2} + \chi_2 \|\mathbf{f}\|_{\mathscr{C}_2}, \tag{3.4}$$

where

$$\chi_1 = \sqrt{e^{\sigma^* T}}, \quad \chi_2 = \sqrt{\left(\frac{c_{\max}}{\varepsilon \sigma^*}\left(e^{\sigma^* T} - 1\right)\right)} \quad \text{for } \sigma^* > 0 .$$

Applying this procedure to the equation $\mathbf{D}\mathbf{u}_t = \mathbf{f}_t$ we get

$$\|\mathbf{u}_t\|_{\mathscr{C}_2} \leqq \chi_1 \|\tilde{\boldsymbol{\varphi}}\|_{\mathscr{L}_2} + \chi_2 \|\mathbf{f}_t\|_{\mathscr{C}_2}, \tag{3.5}$$

where $\tilde{\boldsymbol{\varphi}}$ is defined by

$$\frac{1}{c_j} \tilde{\varphi}_j(x, \omega) = -\omega \, . \, \operatorname{grad} u_j(0, x, \omega) - \sigma_j u_j(0, x, \omega) + $$
$$+ \sum_{k=1}^{l} \frac{\sigma^r_k(x)}{4\pi} \int_{\Omega} h_{jk}(\mu_0) \, u_k(0, x, \omega') \, d\omega' + f_j(0, x, \omega) .$$

From (3.2) and from the equation (2.1) in the form

$$\mathbf{c}^{-1} \frac{\partial \mathbf{u}}{\partial t} + \Lambda \mathbf{u} = \mathbf{H}\mathbf{u} + \mathbf{f}, \tag{3.6}$$

we get by an analogous procedure (3.3) ($\mathbf{c}^{-1}$ is the diagonal matrix with the elements $1/c_j$, $j = 1, 2, \ldots, l$).

Theorem 3.2. *Let* $\mathbf{f}$, $\boldsymbol{\varphi}$, h_{jk} *satisfy the hypotheses of Theorem* 3.1 *and let the assumptions* i) – iii) *hold. Then the solution* $\mathbf{u}$ *of Problem* (2.1), (2.4), (2.5) *is uniquely determined and depends continuously upon the data* $\mathbf{f}$, $\boldsymbol{\varphi}$, σ_k, σ^r_k.

Proof. See [16].

4. CONSTRUCTION OF AN APPROXIMATE PROBLEM BY P_L-METHOD

In this section we shall construct an approximate problem in the following form

$$\mathbf{D}^{(n)} \mathbf{u}^{(n)} = \mathbf{f}^{(n)} \quad \text{on} \quad Q, \tag{4.1}$$

$$\mathbf{u}^{(n)}(0, x, \omega) = \boldsymbol{\varphi}^{(n)}(x, \omega) \quad \text{on} \quad \bar{G} \times \Omega, \tag{4.2}$$

$$\mathbf{u}^{(n)} \in N^-(\partial G), \tag{4.3}$$

where $\mathbf{f}^{(n)}$, $\boldsymbol{\varphi}^{(n)}$ are approximations of $\mathbf{f}$ and $\boldsymbol{\varphi}$ respectively. Condition (4.3) is an approximation of the boundary condition (2.4) in the *Marschak-Vladimirov* sense. The solution of Problem (4.1) – (4.3) will be an approximate solution of Problem (2.1), (2.4), (2.5). The convergence $\mathbf{u}^{(n)}$ to $\mathbf{u}$ depends on the *boundary space* $N^-(\partial G)$ as well as on the convergence of $\mathbf{f}^{(n)}$, $\boldsymbol{\varphi}^{(n)}$ and $\mathbf{D}^{(n)}$ to $\mathbf{f}$, $\boldsymbol{\varphi}$ and $\mathbf{D}$ respectively.

As Ω is a unit sphere, we shall characterize $\omega \in \Omega$ by a couple of angle coordinates ϑ, ψ in the sense of the sperical coordinate system. Then equation (2.1) or (3.6) can be written in the form

$$(4.4) \qquad \mathbf{c}^{-1} \frac{\partial \mathbf{u}}{\partial t} + \sqrt{(1-\mu^2)} \cos \psi \frac{\partial \mathbf{u}}{\partial x_1} + \sqrt{(1-\mu^2)} \sin \psi \frac{\partial \mathbf{u}}{\partial x_2} + \mu \frac{\partial \mathbf{u}}{\partial x_3} + \\ + \boldsymbol{\sigma} \mathbf{u} = \int_{-1}^{1} \int_{0}^{2\pi} \mathfrak{H}(x, \mu_0)\, \mathbf{u}(t, x, \mu', \psi')\, \mathrm{d}\mu'\, \mathrm{d}\psi' + \mathbf{f}(t, x, \mu, \psi)\,,$$

where $\mathbf{c}^{-1}$ and $\boldsymbol{\sigma}$ are diagonal matrices with the elements $1/c_j$, $\sigma_j(x)$ respectively, $\mu = \cos \vartheta$, $\mathbf{u} = \mathbf{u}(t, x, \mu, \psi)$, $\mu_0 = \omega \cdot \omega' = \mu\mu' + \sqrt{(1-\mu^2)}\sqrt{(1-\mu'^2)}\cos(\psi - \psi')$.

We shall consider the system of $(n+1)^2$ base functions (*spherical harmonics*)

$$(4.5) \qquad \{C_0^0, C_1^0, C_2^0, \ldots, C_n^0; C_1^1, C_2^1, \ldots, C_n^1; S_1^1, S_2^1, \ldots, S_n^1; \ldots \\ \ldots; C_{n-1}^{n-1}; C_n^{n-1}; S_{n-1}^{n-1}, S_n^{n-1}; C_n^n; S_n^n\}\,.$$

$$C_p^m = C_p^m(\mu, \psi) = P_p^{(m)}(\mu) \cos m\psi\,; \quad p = 0, 1, 2, \ldots, n\,; \quad m = 0, 1, 2, \ldots, p\,;$$

$$S_p^m = S_p^m(\mu, \psi) = P_p^{(m)}(\mu) \sin m\psi\,; \quad p = 1, 2, \ldots, n\,; \quad m = 1, 2, \ldots, p\,;$$

$$P_p^{(m)}(\mu) = (1-\mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_p(\mu)\,, \quad p \geqq 0\,, \quad m \leqq p\,;$$

$P_p(\mu)$ are Legendre polynomials.

Applying the *Galerkin procedure* to the velocity variables in Eq. (4.4), i.e. multiplying each term of the j-th equation of system (4.4) by base function (4.5) and integrating over $\langle -1, 1\rangle \times \langle 0, 2\pi\rangle$ (see [1], [2], [16]) we obtain, after some rearrangement, a first order system of partial differential equations

$$(4.6) \qquad \frac{1}{c_j} B_j \frac{\partial U_j}{\partial t} + \sum_{i=1}^{3} A_{ji} \frac{\partial U_j}{\partial x_i} + \sigma_j B_j U_j = \\ = \sum_{k=1}^{l} H_{jk} U_k + B_j F_j\,; \quad j = 1, 2, \ldots, l\,.$$

Here $U_j = U_j(t, x)$ are vector-valued functions with $(n+1)^2$ components (ordered by (4.5))

$$\mathrm{U}_{j,p}^{c,m} = \mathrm{U}_{j,p}^{c,m}(t, x) = \int_{-1}^{1} \int_{0}^{2\pi} u_j(t, x, \mu, \psi)\, C_p^m(\mu, \psi)\, \mathrm{d}\mu\, \mathrm{d}\psi\,,$$

$$\mathrm{U}_{j,p}^{s,m} = \mathrm{U}_{j,p}^{s,m}(t, x) = \int_{-1}^{1} \int_{0}^{2\pi} u_j(t, x, \mu, \psi)\, S_p^m(\mu, \psi)\, \mathrm{d}\mu\, \mathrm{d}\psi$$

(analogously for $F_j = F_j(t, x)$). It can be easily shown that B_j (for all j) is a diagonal and positive matrix and A_{ji} are symmetric (for details see [16]). $H_{jk} = \frac{1}{2}\sigma_k^r(x)\, B_k \widetilde{H}_{jk}$,

$\widetilde{H}_{jk}$ is a diagonal matrix with $(n+1)^2$ elements (ordered again by (4.5))

$$h_{jk}^0, h_{jk}^1, h_{jk}^2, \ldots, h_{jk}^n; h_{jk}^1, h_{jk}^2, \ldots, h_{jk}^n; h_{jk}^1, h_{jk}^2, \ldots, h_{jk}^n; \ldots$$
$$\ldots; h_{jk}^{n-1}, h_{jk}^n; h_{jk}^{n-1}, h_{jk}^n; h_{jk}^n; h_{jk}^n,$$

where

$$h_{jk}^s = \int_{-1}^{1} h_{jk}(\mu_0)\, P_s(\mu_0)\, d\mu_0$$

and we denote

$$h_{jk}^{(n)} = \sum_{s=0}^{n} \frac{2s+1}{2} h_{jk}^s P_s(\mu_0).$$

B_j, A_{ji} are constant matrices, too.

If

$$\boldsymbol{U} = (U_1, U_2, \ldots, U_l); \quad \boldsymbol{F} = (F_1, F_2, \ldots, F_l),$$
$$\boldsymbol{B} = \sum_{j=1}^{l} \oplus B_j, \quad \boldsymbol{B}_c = \sum_{j=1}^{l} \oplus \frac{1}{c_j} B_j,$$
$$\boldsymbol{B}_\sigma = \sum_{j=1}^{l} \oplus \sigma_j B_j, \quad \boldsymbol{A}_i = \sum_{j=1}^{l} \oplus A_{ji}, \quad i = 1, 2, 3$$

(direct sum of matrices), then we can write (4.6) in the form

$$\boldsymbol{B}_c \frac{\partial \boldsymbol{U}}{\partial t} + \sum_{i=1}^{3} \boldsymbol{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} + \boldsymbol{RU} = \boldsymbol{BF}, \tag{4.7}$$

where $\boldsymbol{R} = \boldsymbol{B}_\sigma - \boldsymbol{E}$. The matrices $\boldsymbol{B}, \boldsymbol{A}_i, \boldsymbol{E}$ are square matrices of order $\alpha = l(n+1)^2$ and B_j, A_{ji}, H_{jk} are their submatrices.

We shall seek the solution $\boldsymbol{U} = \boldsymbol{U}(t, x)$ of (4.7) in the cylinder $(0, T) \times G$ satisfying the initial condition

$$\boldsymbol{U}(0, x) = \boldsymbol{\Phi}(x), \quad x \in G \tag{4.8}$$

and the boundary condition

$$\boldsymbol{U} \in \mathcal{N}^-(\partial G), \tag{4.8'}$$

where the boundary space will be prescribed by a boundary matrix (see (4.12)). Function $\boldsymbol{\Phi}$ is determined by $\boldsymbol{\varphi}(x, \omega)$ as a vector-valued function with the components (ordered by (4.5))

$$\Phi_{j,p}^{c,m}(x) = \int_{-1}^{1} \int_{0}^{2\pi} \varphi_j(x, \mu, \psi)\, C_p^m(\mu, \psi)\, d\mu\, d\psi,$$
$$\Phi_{j,p}^{s,m}(x) = \int_{-1}^{1} \int_{0}^{2\pi} \varphi_j(x, \mu, \psi)\, S_p^m(\mu, \psi)\, d\mu\, d\psi.$$

The equation (4.7) forms a *symmetric hyperbolic system* (see [11], [14]).

We shall now describe the construction of the boundary space $\mathcal{N}^-(\partial G)$ or $\boldsymbol{N}^-(\partial G)$.

As is well-known, the solution $\mathbf{u} = (u_1, u_2, \ldots, u_l) \in \mathscr{R}(\mathbf{D})$ of Problem (2.1), (2.4), (2.5) may be represented in the form

$$u_j(t, x, \mu, \psi) = \sum_{p=0}^{\infty} \sum_{m=0}^{p} \frac{2p+1}{2\pi(1+\delta_{m0})} \frac{(p-m)!}{(p+m)!} \mathrm{U}_{j,p}^{c,m}(t, x)\, C_p^m(\mu, \psi) +$$

$$+ \sum_{p=1}^{\infty} \sum_{m=1}^{p} \frac{2p+1}{2\pi} \frac{(p-m)!}{(p+m)!} \mathrm{U}_{j,p}^{s,m}(t, x)\, S_p^m(\mu, \psi)\,,$$

or formally

$$u_j(t, x, \mu, \psi) = \sum_{\beta}^{\infty} \varepsilon_\beta\, \mathrm{U}_{j\beta}(t, x)\, Y_\beta(\mu, \psi)\,,$$

where Y_β, $\beta = 0, 1, 2, \ldots$, represent spherical harmonics base functions (4.5), $\mathrm{U}_{j\beta}$ are Fourier coefficients of u_j, ε_β are numerical coefficients dependent on p, m.

As an *approximate solution* of the problem (2.1), (2.4), (2.5) we shall take

$$\text{(4.9)} \qquad \mathbf{u}^{(n)}(t, x, \mu, \psi) = (u_1^{(n)}, u_2^{(n)}, \ldots, u_l^{(n)})\,;$$

$$u_j^{(n)}(t, x, \mu, \psi) = \sum_{\beta}^{n} \varepsilon_\beta\, \mathrm{U}_{j\beta} Y_\beta(\mu, \psi)\,,$$

(sum of $(n + 1)^2$ members).

In this expression the approximations $\mathbf{f}^{(n)}$, $\boldsymbol{\varphi}^{(n)}$ of $\mathbf{f}$, $\boldsymbol{\varphi}$ in (4.1), (4.2) will be represented by

$$\text{(4.10)} \qquad f_j^{(n)}(t, x, \mu, \psi) = \sum_{\beta}^{n} \varepsilon_\beta\, \mathrm{F}_{j\beta} Y_\beta\,; \quad \varphi_j^{(n)}(x, \mu, \psi) = \sum_{\beta}^{n} \varepsilon_\beta\, \Phi_{j\beta} Y_\beta\,.$$

To be able to formulate the boundary condition (4.8′) for equation (4.7) we must take the weak Marschak-Vladimirov condition in the form

$$\text{(4.11)} \qquad \int_{\Omega^-} (\boldsymbol{n} \cdot \boldsymbol{\omega})^{1+q}\, u_j^{(n)}(t, x, \omega)\, C_{2(p-q)}^m(\omega)\, \mathrm{d}\omega = 0\,,$$

$$\int_{\Omega^-} (\boldsymbol{n} \cdot \boldsymbol{\omega})^{1+q} u_j^{(n)}(t, x, \omega)\, S_{2(p-q)}^m(\omega)\, \mathrm{d}\omega = 0\,,$$

$(t, x, \omega) \in \langle 0, T\rangle \times \Gamma_-$; $j = 1, 2, \ldots, l$; $m = 0, 1, 2, \ldots, 2p - 3q$; $p = 2q, 2q + 1$ $2q + 2, \ldots, [n/2] + 2[(n + 1)/2] - (n + 1)$: $q = 0$ for n odd, $q = 1$ for n even We integrate over those directions $\boldsymbol{\omega} \in \Omega$ for which $\boldsymbol{n} \cdot \boldsymbol{\omega} < 0$ holds ($\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ is the outward unit normal vector at the point $\boldsymbol{x} \in \partial G$).

After substituting from (4.8′) into (4.11) and integrating we obtain the matrix form of the boundary conditions

$$\text{(4.12)} \qquad M_j^- U_j = 0\,, \quad t \in \langle 0, T\rangle\,, \quad \boldsymbol{x} \in \partial G\,.$$

The elements of the matrix M_j^- are independent of j and are calculated by means of integration formulas for spherical harmonics. This procedure is described also in [7] and others.

Let us denote by $\boldsymbol{M}^- = \sum_{j=1}^{l} \oplus M_j^-$ a quasidiagonal matrix with blocks M_j^- on the diagonal. Then

$$\mathcal{N}^-(\partial G) \equiv \{\boldsymbol{U} = \boldsymbol{U}(t, x);\ \boldsymbol{M}^-\boldsymbol{U} = 0 \quad \text{on} \quad \langle 0, T\rangle \times \partial G\}\,.$$

We further introduce the adjoint boundary condition to (4.11)

$$(4.13) \qquad \int_{\Omega^+} (\boldsymbol{n}\,.\,\boldsymbol{\omega})^{1+q}\, v_j^{(n)}(t, \boldsymbol{x}, \boldsymbol{\omega})\, C_{2(p-q)}^m(\boldsymbol{\omega})\, \mathrm{d}\boldsymbol{\omega} = 0\,,$$

$$\int_{\Omega^+} (\boldsymbol{n}\,.\,\boldsymbol{\omega})^{1+q}\, v_j^{(n)}(t, \boldsymbol{x}, \boldsymbol{\omega})\, S_{2(p-q)}^m(\boldsymbol{\omega})\, \mathrm{d}\boldsymbol{\omega} = 0\,,$$

$(t, \boldsymbol{x}, \boldsymbol{\omega}) \in \langle 0, T\rangle \times \Gamma_+$ (we integrate over those directions $\boldsymbol{\omega} \in \Omega$ for which $\boldsymbol{n}\,.\,\boldsymbol{\omega} \geqq 0$ holds). The other conditions are the same as in (4.11).

In (4.13) we assume

$$v_j^{(n)}(t, \boldsymbol{x}, \mu, \psi) = \sum_{\beta}^{n} \varepsilon_\beta\, \mathrm{V}_{j\beta} Y_\beta(\mu, \psi)\,.$$

The conditions (4.13) can again be written in the matrix form as

$$(4.14) \qquad M_j^+ V_j = 0\,, \quad t \in \langle 0, T\rangle\,, \quad \boldsymbol{x} \in \partial G\,.$$

Denoting $\boldsymbol{M}^+ = \sum_{j=1}^{l} \oplus M_+^j$ we define

$$\mathcal{N}^+(\partial G) \equiv \{\boldsymbol{V} = \boldsymbol{V}(t, x);\ \boldsymbol{M}^+\boldsymbol{V} = \boldsymbol{0} \quad \text{on} \quad \langle 0, T\rangle \times \partial G\}\,.$$

Let $\langle \boldsymbol{U}, \boldsymbol{V}\rangle = \sum_{j=1}^{l} \langle U_j, V_j\rangle$ be the usual inner product of α-dimensional vectors ($\alpha = l(n+1)^2$). Using elementary rearrangements (see [16]) we have (Y is a vector with $(n+1)^2$ components Y_β, i.e. (4.5)):

$$u_j^{(n)} = \langle B_j U_j, Y\rangle\,; \quad \sigma_j u_j^{(n)} = \langle \sigma_j B_j U_j, Y\rangle\,;$$

$$\frac{1}{4\pi} \sum_{k=1}^{l} \int_{\Omega} \sigma_k^r(x)\, h_{jk}^{(n)}(\mu_0)\, u_k^{(n)}(t, \boldsymbol{x}, \boldsymbol{\omega}')\, \mathrm{d}\boldsymbol{\omega}' = \langle \sum_{k=1}^{l} H_{jk} U_k, Y\rangle\,;$$

$$\boldsymbol{\omega}\,.\,\mathrm{grad}\, u_j^{(n)} + \sigma_j u_j^{(n)} + r_j^{(n)} = \left\langle \sum_{i=1}^{3} A_{ji} \frac{\partial U_j}{\partial x_i} + \sigma_j B_j U_j, Y \right\rangle,$$

where

$$r_j^{(n)} = \frac{1}{2\pi} \left\{ \sum_{m=0}^{n} \frac{1}{2} \left[-\frac{(n-m+2)!}{(n+m)!}\, C_{n+1}^{m-1} + \frac{(n-m)!}{(n+m)!}\, C_{n+1}^{m+1} \right] \frac{\partial \mathrm{U}_{j,n}^{c,m}}{\partial x_1} + \right.$$

$$+\sum_{m=1}^{n}\frac{1}{2}\left[-\frac{(n-m+2)!}{(n+m)!}S_{n+1}^{m-1}+\frac{(n-m)!}{(n+m)!}S_{n+1}^{m+1}\right]\frac{\partial U_{j,n}^{s,m}}{\partial x_1}+$$

$$+\frac{1}{2\pi}\left\{\sum_{m=0}^{n}\frac{1}{2}\left[\frac{(n-m+2)!}{(n+m)!}S_{n+1}^{m-1}+\frac{(n-m)!}{(n+m)!}S_{n+1}^{m+1}\right]\frac{\partial U_{j,n}^{c,m}}{\partial x_2}+\right.$$

$$+\sum_{m=1}^{n}\left(-\frac{1}{2}\right)\left[\frac{(n-m+2)!}{(n+m)!}C_{n+1}^{m-1}+\frac{(n-m)!}{(n+m)!}C_{n+1}^{m+1}\right]\frac{\partial U_{j,n}^{s,m}}{\partial x_2}+$$

$$+\frac{1}{2\pi}\left\{\sum_{m=0}^{n}\frac{(n-m+1)!}{(n+m)!}C_{n+1}^{m}\frac{\partial U_{j,n}^{c,m}}{\partial x_3}+\sum_{m=1}^{n}\frac{(n-m+1)!}{(n+m)!}S_{n+1}^{m}\frac{\partial U_{j,n}^{s,m}}{\partial x_3}\right\}.$$

By these identities, after multiplying every equation of the system (4.6) by Y, we get $(j = 1, 2, \ldots, l)$

(4.15)
$$\frac{1}{c_j}\frac{\partial u_j^{(n)}}{\partial t}+\boldsymbol{\omega}\,.\,\mathrm{grad}\,u_j^{(n)}+\sigma_j u_j^{(n)}+r_j^{(n)}=$$
$$=\frac{1}{4\pi}\sum_{k=1}^{l}\int_{\Omega}\sigma_k^r(\boldsymbol{x})\,h_{jk}^{(n)}(\mu_0)\,u_k^{(n)}\,\mathrm{d}\boldsymbol{\omega}'+f_j^{(n)}\,,$$

whose operator form is (4.1), where

(4.16)
$$\mathbf{D}^{(n)}\mathbf{u}^{(n)}\equiv\mathbf{c}^{-1}\frac{\partial\mathbf{u}^{(n)}}{\partial t}+\boldsymbol{\Lambda}\mathbf{u}^{(n)}+\mathbf{r}^{(n)}-\mathbf{H}^{(n)}\mathbf{u}^{(n)}\,,$$
$$\mathbf{r}^{(n)}=\left(r_1^{(n)},r_2^{(n)},\ldots,r_l^{(n)}\right).$$

For the integral operator $\mathbf{H}^{(n)}$ Lemma 2.1 holds under the same hypotheses on the kernel $\sigma_k^r(\boldsymbol{x})\,h_{jk}^{(n)}(\mu_0)$ instead of $\sigma_k^r(\boldsymbol{x})\,h_{jk}(\mu_0)$.

On the other hand, it is not possible to extend the validity of Lemma 2.2 to $\mathbf{u}^{(n)}$, as $\mathbf{u}^{(n)}\notin\mathscr{L}(\boldsymbol{\Lambda})$ (the boundary condition is not fulfilled).

We say that $\mathbf{u}^{(n)}\in N^-(\partial G)$ if the corresponding $\boldsymbol{U}\in\mathcal{N}^-(\partial G)$ and vice versa. Similarly we define the boundary space $N^+(\partial G)$.

$\mathbf{D}^{(n)*}$ is defined analogously as $\mathbf{D}^*$:

(4.17)
$$\mathbf{D}_j^{(n)*}v_j^{(n)}=-\frac{1}{c_j}\frac{\partial v_j^{(n)}}{\partial t}-\boldsymbol{\omega}\,.\,\mathrm{grad}\,v_j^{(n)}+\sigma_j v_j^{(n)}-$$
$$-\frac{1}{4\pi}\sum_{k=1}^{l}\sigma_j^r(\boldsymbol{x})\int_{\Omega}h_{kj}^{(n)}(\mu_0)\,v_k^{(n)}(t,\boldsymbol{x},\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}\,,$$

that is

$$\mathbf{H}^{(n)*}\mathbf{v}^{(n)}=\int_{\Omega}\mathfrak{H}^{(n)*}(\boldsymbol{x},\mu_0)\,\mathbf{v}^{(n)}(t,\boldsymbol{x},\boldsymbol{\omega})\,\mathrm{d}\boldsymbol{\omega}\,.$$

For $\forall\mathbf{u}^{(n)},\mathbf{v}^{(n)}\in\mathscr{L}_2(Q)$ we have

$$[\mathbf{H}^{(n)}\mathbf{u}^{(n)},\mathbf{v}^{(n)}]_Q=[\mathbf{u}^{(n)},\mathbf{H}^{(n)*}\mathbf{v}^{(n)}]_Q\,.$$

Let the operator $\boldsymbol{K}$ be defined by

$$\boldsymbol{KU} = \boldsymbol{B}_c \frac{\partial \boldsymbol{U}}{\partial t} + \sum_{i=1}^{3} \boldsymbol{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} + \boldsymbol{RU}, \quad (t, x) \in \langle 0, T \rangle \times G, \tag{5.1}$$

and let $\boldsymbol{K}^*$ be the adjoint operator of $\boldsymbol{K}$:

$$\boldsymbol{K}^*\boldsymbol{V} = -\boldsymbol{B}_c \frac{\partial \boldsymbol{V}}{\partial t} - \sum_{i=1}^{3} \boldsymbol{A}_i \frac{\partial \boldsymbol{U}}{\partial x_i} + \boldsymbol{R}^{\mathrm{T}}\boldsymbol{V}, \tag{5.2}$$

where $\boldsymbol{R}^{\mathrm{T}}$ denotes the transpose matrix to $\boldsymbol{R}$ (in the case of complex valued coefficients this is to be replaced by conjugate transpose).

Let $\mathscr{C}_{\alpha,2} = \mathscr{C}(\langle 0, T \rangle; L_2(G))$ be the cartesian product of $\alpha = l(n+1)^2$ spaces $C(\langle 0, T \rangle; L_2(G))$ and $\mathscr{C}_\alpha^1 \equiv \mathscr{C}_\alpha^1(\langle 0, T \rangle \times G)$ the cartesian product of the spaces $C^1(\langle 0, T \rangle \times G)$.

For (real) vector-valued functions $\boldsymbol{U}(t, x)$, $\boldsymbol{V}(t, x)$ with $\alpha = l(n+1)^2$ components, ordered by (4.5), we define

$$\langle \boldsymbol{U}, \boldsymbol{V} \rangle_G = \int_G \langle \boldsymbol{U}, \boldsymbol{V} \rangle \, \mathrm{d}x \,; \quad \langle \boldsymbol{U}, \boldsymbol{V} \rangle_{\partial G} = \int_{\partial G} \langle \boldsymbol{U}, \boldsymbol{V} \rangle \, \mathrm{d}s \,, \tag{5.3}$$

where $\langle \boldsymbol{U}, \boldsymbol{V} \rangle$ is the usual scalar product of α-dimensional vectors.

We will make use of the following lemmas by *Friedrichs*.

Lemma 5.1. *For any functions $\boldsymbol{U}, \boldsymbol{V} \in \mathscr{C}_\alpha^1(\langle 0, T \rangle \times \bar{G})$, where G has a smooth boundary ∂G, we have:*

$$\begin{aligned} \langle \boldsymbol{KU}, \boldsymbol{V} \rangle_{\langle 0,T \rangle \times G} - \langle \boldsymbol{U}, \boldsymbol{K}^*\boldsymbol{V} \rangle_{\langle 0,T \rangle \times G} &= \langle \boldsymbol{B}_c \, \boldsymbol{U}(T, x), \boldsymbol{V}(T, x) \rangle_G - \\ &- \langle \boldsymbol{B}_c \, \boldsymbol{U}(0, x), \boldsymbol{V}(0, x) \rangle_G + \langle \mathscr{A}\boldsymbol{U}, \boldsymbol{V} \rangle_{\langle 0,T \rangle \times \partial G} \,; \end{aligned} \tag{5.4}$$

here $\mathscr{A} = n_1 \boldsymbol{A}_1 + n_2 \boldsymbol{A}_2 + n_3 \boldsymbol{A}_3$, $\boldsymbol{n} = (n_1, n_2, n_3)$ being the unit outward normal. The matrix $\mathscr{A}$ is called a boundary matrix.

To prove (5.4) it is enough to use Green's formula – the integration by-parts for the functions $\boldsymbol{U}$, $\boldsymbol{V}$. It is clear that (5.4) can be proved for the function from W_2^1.

Lemma 5.2. *For any function $\boldsymbol{U} \in \mathscr{C}_2^1(\langle 0, T \rangle \times \bar{G})$ we have*

$$\begin{aligned} 2\langle \boldsymbol{KU}, \boldsymbol{U} \rangle_{\langle 0,T \rangle \times G} &= \langle (\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) \, \boldsymbol{U}, \boldsymbol{U} \rangle_{\langle 0,T \rangle \times G} + \\ &+ \langle \boldsymbol{B}_c \, \boldsymbol{U}(T, x), \boldsymbol{U}(t, x) \rangle_G - \langle \boldsymbol{B}_c \, \boldsymbol{U}(0, x), \boldsymbol{U}(0, x) \rangle_G + \langle \mathscr{A}\boldsymbol{U}, \boldsymbol{U} \rangle_{\langle 0,T \rangle \times \partial G} \,. \end{aligned} \tag{5.5}$$

Proof. By Lemma 5.1.

Lemma 5.3. *The boundary spaces $\mathscr{N}^-(\partial G)$, $\mathscr{N}^+(\partial G)$ are $\mathscr{A}$-orthogonal, i.e.*

$$\langle \mathscr{A}\boldsymbol{U}, \boldsymbol{V} \rangle_{\langle 0,T \rangle \times \partial G} = 0 \,, \quad \textit{for} \quad \boldsymbol{U} \in \mathscr{N}^-(\partial G), \ \boldsymbol{V} \in \mathscr{N}^+(\partial G) \,. \tag{5.6}$$

Proof. It is sufficient to prove that the spaces $N^-(\partial G)$, $N^+(\partial G)$ are $\mathscr{A}$-orthogonal. If $\mathbf{u}^{(n)}$, $\mathbf{v}^{(n)}$ are given by $\boldsymbol{U}$, $\boldsymbol{V}$ by means of (4.9) (4.14) then for all $t \in \langle 0, T \rangle$ (for details see [16])

$$\langle \mathscr{A}\boldsymbol{U}, \boldsymbol{V} \rangle_{\partial G} = [\boldsymbol{n} \, . \, \omega \mathbf{u}^{(n)}, \mathbf{v}^{(n)}]_{\partial G \times \Omega} = 0 \, . \tag{5.7}$$

The following result is based on Lemma 5.4 concerning the polynomials.

Lemma 5.4. *Let $Q_n(\mu)$, $\tilde{Q}_n(\mu)$ be arbitrary polynomials of degree $\leqq n$ satisfying the relations $(m < -1)$*

$$\int_{-1}^{0} \mu(1 - \mu^2)^m \, T_l(\mu^2) \, Q_n(\mu) \, \mathrm{d}\mu = 0 \, , \tag{5.8}$$

$$\int_{0}^{1} \mu(1 - \mu^2)^m \, T_l(\mu^2) \, \tilde{Q}_n(\mu) \, \mathrm{d}\mu = 0 \, , \quad l = 0, 1, 2, \ldots, r; \; n = 2r + 1 \, ,$$

$$\int_{-1}^{0} \mu^2(1 - \mu^2)^m \, T_l(\mu^2) \, Q_n(\mu) \, \mathrm{d}\mu = 0 \, , \tag{5.9}$$

$$\int_{0}^{1} \mu^2(1 - \mu^2)^m \, T_l(\mu^2) \, \tilde{Q}_n(\mu) \, \mathrm{d}\mu = 0 \, , \quad l = 0, 1, 2, \ldots, r - 1; \; n = 2r \, ,$$

where $T_l(\mu^2)$ are arbitrary polynomials of argument μ^2 of degree $\leqq r$. Then

$$\int_{-1}^{1} \mu(1 - \mu^2)^m \, Q_n(\mu) \, \tilde{Q}_n(\mu) \, \mathrm{d}\mu = 0 \, , \tag{5.10}$$

$$\int_{-1}^{1} \mu(1 - \mu^2)^m \, Q_n^2(\mu) \, \mathrm{d}\mu \qquad \geqq 0 \, . \tag{5.11}$$

Proof. If we consider the functions $\mathbf{u}^{(n)}(t, x, \omega)$, $\mathbf{v}^{(n)}(t, x, \omega)$ as functions of the arguments $\omega = (\xi, \tau, \mu)$, $\xi = \cos \psi \sin \vartheta$, $\tau = \sin \psi \sin \vartheta$, $\mu = \cos \vartheta$, where $\xi^2 + \tau^2 + \mu^2 = 1$, we can express $u_j^{(n)}(t, x, \omega)$ as a linear combination of the harmonic polynomials $Y_p(\vartheta, \psi)$:

$$u_j^{(n)}(t, x, \omega) = K_n(\xi, \tau, \mu) = \sum_{p=0}^{n} \alpha_p \, Y_p(\vartheta, \psi) \, .$$

Then (4.11) can be written (for n odd) as

$$\int_{\Omega^-} \mu \, K_n(\xi, \tau, \mu) \, L_{2s}(\xi, \tau, \mu) \, \mathrm{d}\omega = 0 \, , \tag{5.12}$$

where $L_{2s}(\xi, \tau, \mu)$ is a polynomial on the unit sphere of an even degree satisfying

$$L_{2s}(\xi, \tau, \mu) = L_{2s}(-\xi, -\tau, -\mu) \, .$$

The integral (5.12) can be expressed as a linear combination of integrals of the types (5.8), (5.9). Hence (5.10) implies (5.7). The details of the proof of this lemma can be found in [7], [8], [16].

Lemma 5.5. *The boundary space* $\mathscr{N}^-(\partial G)$ *(or* $N^-(\partial G)$*) is dissipative, i.e.*

$$\langle \mathscr{A}\mathbf{U}, \mathbf{U}\rangle_{\partial G} = [\mathbf{n} \,.\, \omega \mathbf{u}^{(n)}, \mathbf{u}^{(n)}]_{\partial G \times \Omega} \geqq 0\,, \quad \forall t \in \langle 0, T\rangle\,, \tag{5.13}$$

for $\mathbf{U} \in \mathscr{N}^-(\partial G)$ *(or* $\mathbf{u}^{(n)} \in N^-(\partial G)$*).*

Proof is based on (5.11) since

$$\langle \mathscr{A}\mathbf{U}, \mathbf{U})_{\partial G} = \sum_{j=1}^{l} \langle A_j U_j, U_j\rangle_{\partial G}\,, \quad A_j = \sum_{i=1}^{l} n_i A_{ji}\,.$$

The boundary conditions (4.3) or (4.9) are called dissipative (non-negative for $\mathbf{K}$ or $\mathbf{D}^{(n)}$) if at every point of the boundary, the matrix $\mathscr{A}$ is non-negative over the boundary space $\mathscr{N}^-(\partial G)$, i.e. if the inequality (5.13) holds. Under this assumption the space $\mathscr{N}^-(\partial G)$ is the maximal one on which the matrix $\mathscr{A}$ is non-negative.

According to the results of [8] we can easily prove that the matrix $\mathscr{A}$ does not change its rank on G.

The domains of the operators $\mathbf{K}$, $\mathbf{K}^*$ are as follows:

$$W(\mathbf{K}) \equiv \{\mathbf{U} \in \mathscr{C}_{\alpha,2} \cap \mathscr{N}^-(\partial G);\ \mathbf{U}(0, x) = \Phi(x)\}\,,$$
$$W(\mathbf{K}^*) \equiv \{\mathbf{V} \in \mathscr{C}_\alpha^1 \cap \mathscr{N}^+(\partial G);\quad \mathbf{V}(T, x) = 0\}\,.$$

We say that $\mathbf{U} \in W(\mathbf{K})$ is a *weak solution* of the problem (4.7)–(4.9) if

$$\langle \mathbf{U}, \mathbf{K}^*\mathbf{V}\rangle_{\langle 0,T\rangle \times G} - \langle \mathbf{B}_c\Phi, \mathbf{V}(0, x)\rangle_G = \langle \mathbf{BF}, \mathbf{V}\rangle_{\langle 0,T\rangle \times G} \tag{5.14}$$

for all $\mathbf{V} \in W(\mathbf{K}^*)$.

We say that $\mathbf{U} \in W(\mathbf{K})$ is a *strong solution* of the problem (4.7)–(4.9) if there exists a sequence $\mathbf{U}^N \in \mathscr{C}_\alpha^1$ of functions satisfying the boundary conditions $\mathbf{M}^-\mathbf{U}^N =$ $= 0$ at every point $x \in \partial G$, such that

$$\|\mathbf{U}^N - \mathbf{U}\|_{\mathscr{C}_{\alpha,2}} \to 0\,; \quad \|\mathbf{U}^N(0, x) - \Phi(x)\|_{\mathscr{L}_2(G)} \to 0\,;$$
$$\|\mathbf{KU}^N - \mathbf{BF}\|_{\mathscr{C}_{\alpha,2}} \to 0 \quad \text{as } N \to \infty\,.$$

Friedrichs [11] proved the existence of a weak solution. He also proved the equivalence of the strong and weak solutios for the mixed problem for the symmetric hyperbolic system under the following assumptions:

i) the boundary ∂G is sufficiently smooth,

ii) the boundary condition is maximally dissipative,

iii) the rank of the boundary matrix $\mathscr{A}$ is constant on ∂G.

If there exists a constant $c_0 > 0$ such that $\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}} \geqq c_0 \boldsymbol{I}$ on G, where $\boldsymbol{I}$ is the identity matrix, we shall show, using (5.5) and (5.13) that

$$\|\boldsymbol{U}\|_{\mathscr{C}_{\alpha,2}} \leqq \gamma_1 \|\boldsymbol{\Phi}\|_{\mathscr{L}_2} + \gamma_2 \|\mathbf{F}\|_{\mathscr{C}_{\alpha,2}}, \tag{5.15}$$

and the uniqueness follows.

However, for our purposes it would be more important to have an analog of (5.15) with $\mathbf{u}^{(n)}$, $\boldsymbol{\varphi}^{(n)}$, $\mathbf{f}^{(n)}$ instead of $\boldsymbol{U}$, $\boldsymbol{\Phi}$, $\mathbf{F}$. Applying the same procedure to the equation $\mathbf{D}^{(n)}\mathbf{u}^{(n)} = \mathbf{f}^{(n)}$ as was used in the proof of Lemma 3.1, we obtain the inequalities

$$\|\mathbf{u}^{(n)}\|_{\mathscr{C}_2{}^1} \leqq \kappa_1 (\|\boldsymbol{\varphi}^{(n)}\|_{\mathscr{L}_2} + \|\tilde{\boldsymbol{\varphi}}^{(n)}\|_{\mathscr{L}_2}) + \kappa_2 \|\mathbf{f}^{(n)}\|_{\mathscr{C}_2{}^1}, \tag{5.16}$$

$$[\boldsymbol{n} \,.\, \omega \mathbf{u}^{(n)}, \mathbf{u}^{(n)}]_{\partial G \times \Omega} \leqq 2 \|\mathbf{u}^{(n)}\|_{\mathscr{L}_2} \left\{ \frac{1}{c_{\min}} \|\mathbf{u}_t^{(n)}\|_{\mathscr{L}_2} + \right. \tag{5.17}$$

$$\left. + \sigma_0^r \|\mathbf{u}^{(n)}\|_{\mathscr{L}_2} + \|\mathbf{f}^{(n)}\|_{\mathscr{L}_2} \right\}, \quad \forall t \in \langle 0, T \rangle .$$

The function $\tilde{\boldsymbol{\varphi}}$ is obtained by substituting $t = 0$ into (4.15).

Lemma 5.6. *If* $\boldsymbol{U} \in W(\mathbf{K})$ *is a weak solution of the problem* (4.7)–(4.9) *in the sense* (5.14) *and* $\mathbf{u}^{(n)}$ *is defined by* (4.9), *then*

$$[\mathbf{u}^{(n)}, \mathbf{D}^{(n)*}\mathbf{v}^{(n)}]_Q - [\mathbf{c}^{-1}\boldsymbol{\varphi}^{(n)}, \mathbf{v}^{(n)}(0, x, \omega)]_{G \times \Omega} = \tag{5.18}$$
$$[\mathbf{f}^{(n)}, \mathbf{v}^{(n)}]_Q, \quad \textit{for all} \quad \mathbf{v}^{(n)} \in W^{(n)*},$$

where $\mathbf{D}^{(n)*}$ *is given by* (4.16) *and*

$$W^{(n)*} \equiv \{\mathbf{v}^{(n)} \in \mathscr{C}^1(\langle 0, T \rangle \times \bar{G} \times \Omega) \cap N^+(\partial G);\ \mathbf{v}^{(n)}(T, x, \omega) = \mathbf{0}\} .$$

Proof. It can be proved by the following identities (for details see [16])

$$\langle \boldsymbol{U}, \boldsymbol{K}^*\boldsymbol{V} \rangle = [\mathbf{u}^{(n)}, \mathbf{D}^{(n)*}\mathbf{v}^{(n)}]_\Omega;\ \langle \boldsymbol{B}_c \boldsymbol{\Phi}, \boldsymbol{V}(0, x) \rangle =$$
$$= [\mathbf{c}^{-1}\boldsymbol{\varphi}^{(n)}, \mathbf{v}^{(n)}(0, x, \omega)]_\Omega .$$

Remark 5.1. We say that $\mathbf{u}^{(n)} \in W^{(n)}$, if and only if $\boldsymbol{U} \in W(\mathbf{K})$.

6. CONVERGENCE OF THE P_L-METHOD

Theorem 6.1. *Let us assume that* $\mathbf{f} \in \mathscr{C}_2^1$, $\boldsymbol{\varphi} \in \mathscr{L}_2(G \times \Omega)$ *and the hypotheses of* §2 *hold. Let* $\mathbf{u} \in \mathscr{R}(\mathbf{D})$ *be the solution of the problem* (3.1) *and* $\mathbf{u}^{(n)}$ *the solution of the approximate problem* (4.1)–(4.3) *by the* P_L*-method. Then* $\mathbf{u}^{(n)}$ *converges weakly to* $\mathbf{u}$ *in the sense*

$$\lim_{n \to \infty} [\mathbf{u}^{(n)} - \mathbf{u}, \mathbf{w}]_Q, \quad \forall \mathbf{w} \in \mathscr{C}_0^\infty(Q) . \tag{6.1}$$

Proof. Let us denote by index ε the regularized function [3] with radius of regularization ε; for example $\sigma_{k\varepsilon}(x)$, $\sigma^r_{k\varepsilon}(x)$ are the regularized coefficients of the equation (2.1). The transport operator $\mathbf{D}$ with these coefficients is denoted by $\mathbf{D}_\varepsilon$. The same notation is also used for $\mathbf{D}^*_\varepsilon$.

Let us formulate the following problem:

$$\mathbf{D}^*_\varepsilon \mathbf{v} = \mathbf{w}\,, \quad \mathbf{w} \in \mathscr{C}^\infty_0(Q)\,, \tag{6.2}$$

$$\mathbf{v}(T, x, \omega) = \mathbf{0}\,, \quad (x, \omega) \ni G \times \Omega\,, \tag{6.3}$$

$$\mathbf{v}(t, x, \omega) = \mathbf{0} \quad on \quad \langle 0, T \rangle \times \Gamma_+\,. \tag{6.4}$$

From the regularity conditions which are proved to be valid for the monoenergetic boundary-value transport problem in [5], we conclude that $\mathbf{v} \in \mathscr{R}(\mathbf{D}^*)$ (see §3 $-$ $\mathbf{v}$ is a solution of the problem (6.2)$-$(6.4)). If $\mathbf{v}(T, x, \omega) = 0$ then $\boldsymbol{V}(T, x) = 0$, where $\boldsymbol{V}$ is the α-dimensional vector-valued function representing the partial sum of the expansion of the function $\mathbf{v}$ into a Fourier series using the spherical harmonics. Hence $\mathbf{v}^{(n)}(T, x, \omega) = 0$. Furthermore to guarantee $\mathbf{v}^{(n)} \in N^+(\partial G)$ we have to put restrictions (4.13) upon the Fourier coefficients of $\mathbf{v}$.

From (4.15) and (4.1) we get

$$[\mathbf{D}^{(n)}\mathbf{u}^{(n)}, \mathbf{v}]_Q = [\mathbf{f}^{(n)}, \mathbf{v}]_Q\,. \tag{6.5}$$

According to (4.16), (4.17) and by Green's formula (2.7) we have

$$\begin{aligned}[\mathbf{D}^{(n)}\mathbf{u}^{(n)}, \mathbf{v}]_Q = [\mathbf{u}^{(n)}, \mathbf{D}^{(n)*}\mathbf{v}_Q - [\mathbf{c}^{-1}\boldsymbol{\varphi}^{(n)}, \mathbf{v}(0, x, \omega)]_{G\times\Omega} +\\ + [\boldsymbol{n}\,.\,\omega\mathbf{u}^{(n)}, \mathbf{v}]_{\langle 0,T\rangle\times\partial G\times\Omega} + [\mathbf{r}^{(n)}, \mathbf{v}]_Q\,.\end{aligned} \tag{6.6}$$

If we put (6.6) into (6.5) and use the following identities

$$\mathbf{D}^{(n)*}\mathbf{v} = \mathbf{D}^*\mathbf{v} - (\mathbf{H}^{(n)*} - \mathbf{H}^*)\,\mathbf{v}\,; \quad \mathbf{f}^{(n)} = \mathbf{f} + (\mathbf{f}^{(n)} - \mathbf{f})\,;$$
$$\boldsymbol{\varphi}^{(n)} = \boldsymbol{\varphi} + (\boldsymbol{\varphi}^{(n)} - \boldsymbol{\varphi})\,; \quad \mathbf{v} = \mathbf{v}^{(n)} + (\mathbf{v} - \mathbf{v}^{(n)})\,,$$

it will be

$$\begin{aligned}[\mathbf{u}^{(n)}, \mathbf{D}^*\mathbf{v}]_Q - [\mathbf{c}^{-1}\boldsymbol{\varphi}, \mathbf{v}(0, x, \omega)]_{G\times\Omega} =\\ = [\mathbf{f}, \mathbf{v}]_Q + [\mathbf{f}^{(n)} - \mathbf{f}, \mathbf{v}]_Q + [\mathbf{c}^{-1}(\boldsymbol{\varphi}^{(n)} - \boldsymbol{\varphi}), \mathbf{v}(0, x, \omega)]_{G\times\Omega} -\\ - [\boldsymbol{n}\,.\,\omega\mathbf{u}^{(n)}, \mathbf{v}^{(n)}]_{\langle 0,T\rangle\times\partial G\times\Omega} -\\ - [\boldsymbol{n}\,.\,\omega\mathbf{u}^{(n)}, \mathbf{v} - \mathbf{v}^{(n)}]_{\langle 0,T\rangle\times\partial G\times\Omega} +\\ + [\mathbf{u}^{(n)}, (\mathbf{H}^{(n)*} - \mathbf{H}^*)\,\mathbf{v}]_Q + [\mathbf{r}^{(n)}, \mathbf{v}]_Q\,.\end{aligned} \tag{6.7}$$

That is

$$[\mathbf{u}^{(n)}, \mathbf{D}^*\mathbf{v}]_Q - [\mathbf{c}^{-1}\boldsymbol{\varphi}, \mathbf{v}(0, x, \omega)]_{G\times\Omega} = [\mathbf{f}, \mathbf{v}]_Q + \tau_n\,, \tag{6.7'}$$

where τ_n denotes all the members on the right hand side of (6.7) except $[\mathbf{f}, \mathbf{v}]_Q$.

After subtracting (6.7′) and (3.1) and substituting $\mathbf{D}^*\mathbf{v} = \mathbf{D}^*_\varepsilon\mathbf{v} + (\mathbf{D}^*\mathbf{v} - \mathbf{D}^*_\varepsilon\mathbf{v})$ we obtain

$$(6.8) \qquad [\mathbf{u}^{(n)} - \mathbf{u}, \mathbf{D}^*_\varepsilon\mathbf{v}]_Q = \tau_n + [\mathbf{u}^{(n)} - \mathbf{u}, \mathbf{D}^*_\varepsilon\mathbf{v} - \mathbf{D}^*\mathbf{v}]_Q\,,$$

for $\mathbf{u}^{(n)} \in W^{(n)}$, $\mathbf{u} \in \mathscr{R}(\mathbf{D})$, $\mathbf{v} \in \mathscr{R}(\mathbf{D}^*)$.

Since $\mathbf{D}^*_\varepsilon\mathbf{v} = \mathbf{w} \in \mathscr{C}^\infty_0(Q)$, it is sufficient to show that

$$\lim_{n\to\infty} \{\tau_n + [\mathbf{u}^{(n)} - \mathbf{u}, \mathbf{D}^*_\varepsilon\mathbf{v} - \mathbf{D}^*\mathbf{v}]\}_Q = 0\,.$$

Using the component form of $\mathbf{D}^*\mathbf{v}$ and $\mathbf{D}^*_\varepsilon\mathbf{v}$:

$$\mathrm{D}^*_j v_j = -\frac{1}{c_j}\frac{\partial v_j}{\partial t} - \omega\,.\,\mathrm{grad}\, v_j + \sigma_j v_j - \frac{1}{4\pi}\sum_{k=1}^{l} \sigma^r_j(x)\int_\Omega h_{kj}(\mu_0)\, v_k\, \mathrm{d}\omega'\,,$$

$$\mathrm{D}^*_{j\varepsilon} v_j = -\frac{1}{c_j}\frac{\partial v_j}{\partial t} - \omega\,.\,\mathrm{grad}\, v_j + \sigma_{j\varepsilon} v_j - \frac{1}{4\pi}\sum_{k=1}^{l} \sigma^r_{j\varepsilon}(x)\int_\Omega h_{kj}(\mu_0)\, v_k\, \mathrm{d}\omega'\,,$$

we have

$$[\mathbf{u}^{(n)} - \mathbf{u}, \mathbf{D}^*_\varepsilon\mathbf{v} - \mathbf{D}^*\mathbf{v}]_Q = \sum_{j=1}^{l} (u^{(n)}_j - u_j, \mathrm{D}^*_{j\varepsilon}v_j - \mathrm{D}^*_j v_j)_Q =$$

$$= \sum_{j=1}^{l} \int_Q (u^{(n)}_j - u_j)\left\{(\sigma_{j\varepsilon} - \sigma_j)\, v_j + \frac{1}{4\pi}\sum_{k=1}^{l}(\sigma^r_j - \sigma^r_{j\varepsilon})\int_\Omega h_{kj}(\mu_0)\, v_k(t, x, \omega')\,\mathrm{d}\omega'\right\}\mathrm{d}Q\,.$$

Using the boundedness of $\|\mathbf{u}^{(n)} - \mathbf{u}\|_{\mathscr{C}_2{}^1}$ and Schwarz's inequality we can write

$$|(u^{(n)}_j - u_j, \mathrm{D}^*_{j\varepsilon}v_j - \mathrm{D}^*_j v_j)| \leqq \mathrm{const}\,(\|\sigma_{j\varepsilon} - \sigma_j\|_{L_2(G)} + \\ + \|\sigma^r_j - \sigma^r_{j\varepsilon}\|_{L_2(G)})\,.$$

We choose the radius of regularization $\varepsilon = \mathrm{const}/n^\alpha$, $\alpha > 0$, where the constant depends on the initial condition and on the diameter of the region G. Then $[\mathbf{u}^{(n)} - \mathbf{u}, \mathbf{D}^*_\varepsilon\mathbf{v} - \mathbf{D}^*\mathbf{v}]_Q \to 0$, for $n \to \infty$. Since $\mathbf{f}, \mathbf{f}^{(n)} \in \mathscr{C}^1_2$, $\mathbf{v} \in \mathscr{R}(\mathbf{D}^*) \subset \mathscr{C}(\bar{Q})$, we have

$$[\mathbf{f}^{(n)} - \mathbf{f}, \mathbf{v})_Q \leqq \sum_{j=1}^{l} \int_Q (f^{(n)}_j - f_j)\, v_j\, \mathrm{d}t\, \mathrm{d}x\, \mathrm{d}\omega \leqq 4\pi T \,\mathrm{mes}\, G \sum_{j=1}^{l} \|v_j\|_{C(Q)}\, \|f^{(n)}_j - f_j\|_{C_2{}^1}\,.$$

That is

$$\lim_{n\to\infty} [\mathbf{f}^{(n)} - \mathbf{f}, \mathbf{v}]_Q = 0\,.$$

Similarly

$$[\mathbf{c}^{-1}(\boldsymbol{\varphi}^{(n)} - \boldsymbol{\varphi}), \mathbf{v}(0, x, \omega)]_{G\times\Omega} \leqq$$

$$\leqq \sum_{j=1}^{l} \frac{1}{c_j}\int_{G\times\Omega} (\varphi^{(n)}_j - \varphi_j)\, v_j(0, x, \omega)\, \mathrm{d}x\, \mathrm{d}\omega \leqq$$

$$\leqq \frac{4\pi\, \mathrm{mes}\, G}{\min_j c_j} \sum_{j=1}^{l} \|v_j(0, x, \omega)\|_{C(G\times\Omega)}\, \|\varphi^{(n)}_j - \varphi_j\|_{L_2(G\times\Omega)}\,.$$

Because

$$\lim_{n\to\infty} \|\mathbf{v} - \mathbf{v}^{(n)}\|_{\mathscr{L}_2(\Omega)} = 0\,, \quad \mathbf{v} \in \mathscr{R}(\mathbf{D}^*) \subset \mathscr{C}(\bar{Q})\,,$$

the continuity of the function $\mathbf{v} - \mathbf{v}^{(n)}$ on $\bar{Q}$ and the boundedness of the function $\mathbf{u}^{(n)}$ on ∂G for all $(t, \omega) \in \langle 0, T\rangle \times \Omega$ guarantee that

$$\lim_{n\to\infty} [\boldsymbol{n} \,.\, \omega \mathbf{u}^{(n)}, \mathbf{v} - \mathbf{v}^{(n)}]_{\langle 0,T\rangle \times \partial G \times \Omega} = 0\,.$$

It is clear that

$$[\mathbf{u}^{(n)}, (\mathbf{H}^{(n)*} - \mathbf{H}^*)\,\mathbf{v}]_Q =$$

$$= \sum_{j=1}^{l} \int_Q u_j^{(n)}(t, x, \omega) \frac{1}{4\pi} \sum_{k=1}^{l} \sigma_j^r(x) \int_\Omega (h_{kj}^{(n)}(\mu_0) - h_{kj}(\mu_0))\, v_k(t, x, \omega')\, \mathrm{d}\omega'\, \mathrm{d}Q\,.$$

From the hypotheses i) ii) iii) and from the boundedness of the functions v_k, $u_j^{(n)}$ we obtain

$$[\mathbf{u}^{(n)}, (\mathbf{H}^{(n)*} - \mathbf{H}^*)\,\mathbf{v}]_Q \leqq \text{const} \sum_{k,j=1}^{l} \|h_{kj}^{(n)} - h_{kj}\|^2_{L_2(-1,1)}\,.$$

From (5.10) it follows that

$$[\boldsymbol{n} \,.\, \omega \mathbf{u}^{(n)}, \mathbf{v}^{(n)}]_{\langle 0,T\rangle \times \partial G \times \Omega} = 0 \quad \text{for} \quad \mathbf{v}^{(n)} \in N^+(\partial G),\ \mathbf{u}^{(n)} \in N^-(\partial G)\,.$$

The following identities for the spherical harmonics

$$\lim_{n\to\infty} \sum_{m=0}^{n} \left(\frac{2n+1}{1+\delta_{m0}} \frac{(n-m)!}{(n+m)!}\right)^{1/2} \int_\Omega C_n^m(\omega)\, z(\omega)\, \mathrm{d}\omega = 0\,,$$

$$\lim_{n\to\infty} \sum_{m=0}^{n} \left((2n+1)\frac{(n-m)!}{(n+m)!}\right)^{1/2} \int_\Omega S_n^m(\omega)\, z(\omega)\, \mathrm{d}\omega = 0\,, \quad z \in L_2(\Omega)\,,$$

when used to the components of $\partial \mathbf{U}/\partial x_i$, $i = 1, 2, 3$, instead of $z(\omega)$, give $[\mathbf{r}^{(n)}, \mathbf{v}]_Q \to 0$, for $n \to \infty$ (it is necessary to use the component form $r_j^{(n)}$ of $\mathbf{r}^{(n)}$ (see §4)).

From this consideration it is seen that $\lim \tau_n = 0$ and the proof of Theorem 6.1 is complete.

7. REMARKS

The questions of the strong convergence of the P_L-method for the time-dependent mono-velocity transport equations were studied in [15]. The authors obtained estimates of the rate of convergence for the spherical symmetry and slab geometries.

For the steady state neutron transport equation S. Ukai shows in [19] the order of convergence $0((1/n)^{s+1/2})$ for the transport solution in $W_2^{s+2}(G \times \Omega)$.

For the slab geometry it can be shown that

$$[r_j^{(n)}, v_j]_Q \leqq \text{const} \sqrt{\frac{2}{n}} \ (\text{see } [10]) .$$

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Souhrn

PŘIBLIŽNÉ ŘEŠENÍ *l*-RYCHLOSTNÍ NESTACIONÁRNÍ TRANSPORTNÍ ROVNICE P_L-METODOU

STANISLAV MÍKA

V článku je vyšetřován *l*-rychlostní model obecné lineární nestacionární transportní rovnice. Předpokládá se, že pravděpodobnost reakce (rozptyl, dělení) závisí pouze na úhlu směrů pohybu netronu před a po reakci. Je podána zobecněná formulace problému a jsou odvozeny apriorní odhady. Dále je provedena konstrukce přibližného řešení P_L-metodou. U získaného symetrického hyperbolického systému je ukázána dissipativnost a $\mathscr{A}$-ortogonalita příslušných hraničních prostorů a souvislost s jednorychlostním modelem transportní rovnice vyšetřovaným v [5], [7], [8]. V závěru práce je proveden důkaz slabé konvergence přibližných řešení k přesnému.

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