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ON EVOLUTION INEQUALITIES OF A MODIFIED NAVIER-STOKES TYPE, III

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In the present last part of our paper we apply the abstract results obtained in [7], [8] to a unilateral boundary value problem for a system of modified Navier-Stokes equations which has been studied in [3], [6] (under zero boundary conditions). The unilateral boundary conditions we are going to consider, arise from the problem of the motion of a fluid through a tube: we only prescribe the direction of velocity (completed by certain natural boundary conditions) at the orifices at which the fluid runs into or leaves the tube.

Section 1 presents the statement of our boundary-initial value problem. We then introduce in the following section the function spaces needed and the concept of weak solution to the boundary-initial value problem stated. In Section 3 we collect the existence, uniqueness and regularity results for the problem under consideration.

1. STATEMENT OF THE PROBLEM

Let Ω be a bounded domain in $\mathbb{R}^3$. The boundary Γ of Ω is assumed to be Lipschitzian (cf. [9] for details). Let $x = \{x_1, x_2, x_3\}$ denote the generic point in $\mathbb{R}^3$.

We then consider in $\Omega \times [0, T]$ the following system of partial differential equations for the unknown functions $u = \{u_1, u_2, u_3\}$ and p:

$$(1.1)\quad \begin{cases} \dfrac{\partial u_i}{\partial t} - \dfrac{\partial}{\partial x_j}\left[\left(\mu_0 + \mu_1 |\nabla u|^{r-2}\right) \dfrac{\partial u_i}{\partial x_j}\right] + u_j \dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial p}{\partial x_i} = f_i, \quad {}^{1)}\ (i = 1, 2, 3)\,; \\ \operatorname{div} u = 0\,. \end{cases}$$

Here $f = \{f_1, f_2, f_3\}$ is a given function, μ_0 and μ_1 are positive constants, while r is a real number >2 (it will be specified in the following section). Further,

$$|\nabla u| = \left[\sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_k}\right)^2\right]^{1/2}.$$

[1]) We use the convention that a repeated subscript means summation over 1, 2, 3.

The function u represents the velocity of the motion of a viscous, incompressible fluid which runs through Ω, whereas the function p describes the pressure existing in the fluid. The term

$$-\mu_1 \frac{\partial}{\partial x_j}\left(|\nabla u|^{r-2} \frac{\partial u_i}{\partial x_j}\right)$$

arises from the concept of a motion with big gradient of velocity (cf. [3], [4], [5; Appendix]). We also refer to the paper [1] where a related motivation for introducing this term may be found. An "axiomatic" approach which yields similar nonlinearities (of polynomial type with respect to certain tensor invariants), is presented in [2].

The system (1.1) thus represents a modification of the usual system of Navier-Stokes equations, and it formally turns into the latter when neglecting the nonlinear term $-\mu_1 \frac{\partial}{\partial x_j}\left(|\nabla u|^{r-2} \frac{\partial u_i}{\partial x_j}\right)$.

The system (1.1) (under zero boundary conditions upon u) has been extensively studied in [3] where basic existence, uniqueness and regularity results may be found. An existence theorem for (1.1) under relatively mild conditions upon the data has been proved in [6; Chap. 2.5].

The boundary conditions upon u and p considered in the present paper, arise from the problem of the flow of a fluid (whose motion in Ω is governed by (1.1)) through a tube: the fluid runs into Ω along a certain part of Γ, while it leaves Ω along another one.

In order to give a precise formulation of this situation we suppose that the boundary Γ is decomposed into three mutually disjoint parts Γ_k such that mes $(\Gamma_k) > 0$ ($k = 1, 2, 3$). Let $v = v(x)$ denote the unit outer normal at a point $x \in \Gamma$.[2]) The boundary conditions imposed upon u and p are then as follows:

$$(1.2a) \quad \begin{cases} u_\tau = 0\,, \quad u \cdot v \leqq 0\,, \;{}^{3)} \\ (\mu_0 + \mu_1|\nabla u|^{r-2})\dfrac{\partial u}{\partial v} \cdot v - p \leqq 0 \quad on \quad \Gamma_1 \times [0, T]\,, \\ (\mu_0 + \mu_1|\nabla u|^{r-2})\dfrac{\partial u}{\partial v} \cdot u - pu \,.\, v = 0\,; \end{cases}$$

---

[2]) Note that v exists a. e. (with respect to the surface measure) on Γ (see [9] for details).

[3]) $u \cdot v = u_i v_i$, $\frac{\partial u}{\partial v} = \left\{\frac{\partial u_1}{\partial v}, \frac{\partial u_2}{\partial v}, \frac{\partial u_3}{\partial v}\right\}$; $u_\tau = u - (u \cdot v)\, v$ (tangential component of u).

$$
(1.2b)\quad \begin{cases} u_\tau = 0\,, \quad u \cdot \nu \geqq 0\,, \\ (\mu_0 + \mu_1|\nabla u|^{r-2})\dfrac{\partial u}{\partial \nu} \cdot \nu - p \geqq 0 \quad on \quad \Gamma_2 \times [0, T]\,, \\ (\mu_0 + \mu_1|\nabla u|^{r-2})\dfrac{\partial u}{\partial \nu} \cdot u - pu \cdot \nu = 0\,; \end{cases}
$$

$$
(1.2c)\qquad u = 0 \quad on \quad \Gamma_3 \times [0, T]\,.
$$

The first two conditions in (1.2a) express the fact that the fluid runs into Ω along Γ_1, while the first two conditions in (1.2b) mean that it leaves Ω along Γ_2. The remaining conditions in (1.2a, b) may be understood as "natural boundary conditions" (with respect to Green's formula) of the problem under consideration. Condition (1.2c) expresses the fact that no motion of the fluid takes place along Γ_3 (the fluid "adheres" at Γ_3).

We complete the boundary conditions (1.2a-c) by the initial condition

$$
(1.3)\qquad u = u_0 \quad in \quad \Omega\,.
$$

Let us finally refer to [10] where another type of unilateral boundary conditions for the (usual) Navier-Stokes equations is considered.

2. DEFINITION OF THE WEAK SOLUTION

2.1. *Notation. Preliminaries.* Let $W^1_s(\Omega)$ $(1 \leqq s < +\infty)$ denote the usual Sobolev space (cf. e.g. [9]). We then introduce the spaces

$$
\mathscr{V} = \{u \in [C^\infty(\bar{\Omega})]^3 : \operatorname{div} u = 0 \quad in \quad \Omega\,, \quad u = 0 \quad on \quad \Gamma_3\}
$$

and

$$
H = closure\ of\ \mathscr{V}\ in\ [L^2(\Omega)]^3\,,
$$

$$
V = closure\ of\ \mathscr{V}\ in\ [W^1_2(\Omega)]^3\,,
$$

$$
W = closure\ of\ \mathscr{V}\ in\ [W^1_r(\Omega)]^3\,.
$$

H is a Hilbert space with respect to the scalar product

$$
(u, v) = \int_\Omega u_i v_i \,\mathrm{d}x \quad (|u| = (u, u)^{1/2})\,.
$$

Further, observing that $u = 0$ a. e. on Γ_3 (in the sense of traces), we have respectively for any $u \in V$ or $u \in W$ (cf. [9])

$$
m_1\|u\| \leqq \|u\|_{[W^1{}_2(\Omega)]^3} \leqq m_2\|u\| \quad \forall u \in V
$$

$$
n_1|||u||| \leqq \|u\|_{[W^1{}_r(\Omega)]^3} \leqq n_2|||u||| \quad \forall u \in W
$$

where m_k and n_k $(k = 1, 2)$ denote positive constants, and

$$((u, v)) = \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,\mathrm{d}x\,, \quad \|u\| = ((u, u))^{1/2} \quad \textit{for} \quad u, v \in V,$$

$$|||u||| = \left\{\int_\Omega \left[\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2\right]^{r/2} \mathrm{d}x\right\}^{1/r} \quad \textit{for} \quad u \in W.$$

Thus, V is a Hilbert space with respect to the scalar product $((\,,\,))$, while W is a separable, reflexive Banach space with respect to the norm $|||\;|||$. The imbedding $V \subset H$ is compact (cf. [9]). According to our abstract framework of [7] we denote by (u^*, u) the dual pairing between $u^* \in W^*$ and $u \in W$.

Let us finally introduce the set

$$\begin{aligned} K = \{u \in V\colon u \,.\, v &\leqq 0 \quad \textit{a. e. on} \quad \Gamma_1\,,\\ u \,.\, v &\geqq 0 \quad \textit{a. e. on} \quad \Gamma_2\,,\\ u_\tau &= 0 \quad \textit{a. e. on} \quad \Gamma_1 \cup \Gamma_2\}\,. \end{aligned}$$

It is readily verified that K is a closed, convex subset of V. Further, setting

$$\begin{aligned} \mathscr{V}_0 &= \{u \in [C_c^\infty(\Omega)]^3\colon \operatorname{div} u = 0 \;\; \textit{in} \;\; \Omega\}\,,\\ W_0 &= \textit{closure of}\, \mathscr{V}_0 \;\textit{in}\; [W_r^1(\Omega)]^3 \end{aligned}$$

we have $u = 0$ a. e. on Γ for any $u \in W_0$, and thus $W_0 \subset K$.

Let us define, for sufficiently small $\eta > 0$,

$$\begin{aligned} \Omega_3^\eta &= \{x \in \bar{\Omega}\colon \operatorname{dist}(x, \Gamma_3) < \eta\}\,,\\ \Gamma_k^\eta &= \Gamma_k \setminus (\Gamma_k \cap \Omega_3^\eta) \quad (k = 1, 2)\,. \end{aligned}$$

The following result yields a further information about K.

Lemma. *Suppose*:

(i) *Ω is star-shaped with respect to the origin*;

(ii) *the surface Γ_k^δ $(k = 1, 2)$ belongs to the class C^2 (for a sufficiently small $\delta > 0$).*

Then there exists a function $w \in W$ such that

$$w \in K\,, \quad w \notin W_0\,.$$

Proof. 1° Let S denote a (closed) surface of class C^2 such that:

a) $S \subset (\Gamma_1 \cup \Gamma_2 \cup \Omega_3^{3\delta})$;

b) $S \cap (\Gamma_1 \cup \Gamma_2) = \Gamma_1^{2\delta} \cup \Gamma_2^{2\delta}$;

c) $0 < \frac{1}{2}\delta \leqq \operatorname{dist}(x, \Gamma_3) \leqq 2\delta$ *for any* $x \in S \cap \Omega_0^{3\delta}$

(note that such a surface exists by virtue of the fact that Γ is Lipschitzian[4])). Let $\tilde{\Omega}$ denote the bounded domain whose boundary is S.

2° Let $\alpha \in [W_r^{2-1/r}(\Gamma)]^3$ be a vector field on Γ having the following properties:

$$\alpha = 0 \quad a.\,e.\ on \quad (\Gamma_1 \cap \Omega_3^{3\delta}) \cup (\Gamma_2 \cap \Omega_3^{3\delta}) \cup \Gamma_3\,;$$

$$\alpha \neq 0 \quad on\ a\ subset\ of \quad \Gamma_1^{3\delta} \cup \Gamma_2^{3\delta} \quad with\ positive\ surface\ measure\,;$$

$$\alpha\,.\,\nu \leqq 0 \quad a.\,e.\ on \quad \Gamma_1^{3\delta}\,;$$

$$\alpha\,.\,\nu \geqq 0 \quad a.\,e.\ on \quad \Gamma_2^{3\delta}\,;$$

$$\alpha_\tau = 0 \quad a.\,e.\ on \quad \Gamma_1^{3\delta} \cup \Gamma_2^{3\delta}\,;$$

$$\int_{\Gamma^{3\delta}{}_1 \cup \Gamma^{3\delta}{}_2} \alpha\,.\,\nu \,\mathrm{d}\Gamma = 0\,.$$

We then define

$$\beta = \begin{cases} \alpha & a.\,e.\ on \quad \Gamma_1^{2\delta} \cup \Gamma_2^{2\delta}\,, \\ 0 & a.\,e.\ on \quad S \setminus (\Gamma_1^{2\delta} \cup \Gamma_2^{2\delta})\,. \end{cases}$$

It is readily verified that

$$\beta \in [W_r^{2-1/r}(\Gamma)]^3\,, \quad \int_S \beta\,.\,\nu \,\mathrm{d}S = 0\,.$$

3° From [5; Theorem 3, p. 102] we conclude the existence of a function $\tilde{w} \in [W_r^2(\tilde{\Omega})]^3$ such that

$$\operatorname{div} \tilde{w} = 0 \quad a.\,e.\ in \quad \tilde{\Omega}\,, \quad \tilde{w} = \beta \quad a.\,e.\ on \quad S\,.$$

Let us now define

$$w = \begin{cases} \tilde{w} & a.\,e.\ in \quad \tilde{\Omega}\,, \\ 0 & a.\,e.\ in \quad \Omega \setminus \tilde{\Omega}\,. \end{cases}$$

Observing that $\tilde{w} = \beta = 0$ a. e. on $S \setminus (\Gamma_1^{2\delta} \cup \Gamma_2^{2\delta})$ we readily obtain $w \in [W_r^1(\Omega)]^3$. Further, it is easy to see that

$$w\,.\,\nu \leqq 0 \quad a.\,e.\ on \quad \Gamma_1\,;$$

$$w\,.\,\nu \geqq 0 \quad a.\,e.\ on \quad \Gamma_2\,;$$

$$w_\tau = 0 \quad a.\,e.\ on \quad \Gamma_1 \cup \Gamma_2\,;$$

$$w \neq 0 \quad on\ a\ subset\ of \quad \Gamma_1 \cup \Gamma_2 \quad with\ positive\ surface\ measure\,;$$

$$w = 0 \quad a.\,e.\ on \quad \Gamma_3\,.$$

---

[4]) Without any further reference, δ is assumed to be so small that no overlapping of the subsets of Γ which are considered in the course of the proof occurs.

4° It remains to show that there exists a sequence of functions $\{w_n\} \subset \mathscr{V}$ such that $w_n \to w$ in $[W_r^1(\Omega)]^3$ as $n \to \infty$. But this can be achieved by using two standard techniques: firstly, carrying out the transformation $x \mapsto \lambda x$ $(0 < \lambda < 1$; cf. hypothesis (i)) (cf. [9; Theorem 3.2, p. 67]), and secondly, using mollifiers for the transformed function (cf. [11; p. 22]). We may therefore drop further details.

Remark. The assertion of the above lemma continues to hold in the case of two dimensions. The argument in the third step of our above proof can then be simplified (cf. [5; p. 41]).

2.2. *Definition of the weak solution.* Let $r \geqq 12/5$. Further, let $f \in L^{s'}(0, T; W^*)$ $(s' = s/(s-1), s \geqq 2)$ and $u_0 \in H$.

Definition. *The function $u \in L^s(0, T; W)$ is called a weak solution to (1.1)–(1.3) if the following conditions are satisfied:*

$$u(t) \in K \quad \text{for a. e.} \quad t \in [0, T]\,; \tag{2.1}$$

$$u' \in L^2(0, T; W^*)\,; \tag{2.2}$$

$$\left\{\begin{aligned} &\int_0^T (u', v - u)\, dt + \\ &+ \int_0^T \int_\Omega (\mu_0 + \mu_1 |\nabla u|^{r-2}) \frac{\partial u_i}{\partial x_j} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\right) dx\, dt + \\ &+ \int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j} (v_i - u_i)\, dx\, dt \geqq \int_0^T (f, v - u)\, dt \\ &\forall v \in L^s(0, T; W) \quad \text{with} \quad v(t) \in K \quad \text{for a. a.} \quad t \in [0, T]\,; \end{aligned}\right. \tag{2.3}$$

$$u(0) = u_0\,. \tag{2.4}$$

Let us note that the third term on the left hand side in (2.3) is well-defined.[5]) Indeed, since $r \geqq 12/5$ one may find a number q such that

$$\frac{1}{r} + \frac{1}{q} = \frac{1}{2}, \quad 1 < q \leqq \frac{3r}{3-r}\,.$$ [6])

Observing the imbedding $W_r^1(\Omega) \subset L^q(\Omega)$ we then obtain by Hölder's inequality

$$\left|\int_\Omega u \frac{\partial v}{\partial x_j} w dx\right| \leqq \|u\|_{L^2(\Omega)} \|v\|_{W_r^1(\Omega)} \|w\|_{L^q(\Omega)} \leqq \text{const} \|u\|_{L^2(\Omega)} \|v\|_{W_r^1(\Omega)} \|w\|_{W_r^1(\Omega)}$$

[5]) The second term will be considered in the next section.

[6]) We assume that $12/5 \leqq r < 3$. In case $r \geqq 3$ our conclusions obviously continue to hold, they even get simplified and may be strengthened (cf. the following section).

for any $u, v, w \in W_r^1(\Omega)$ $(j = 1, 2, 3)$. Hence

$$\left|\int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \,\mathrm{d}x\right| \leqq \text{const}\, |u|\, |||v|||\, |||w||| \tag{2.5}$$

for any $u, v, w \in W$. Thus, the function

$$t \mapsto \int_\Omega u_j(t) \frac{\partial u_i(t)}{\partial x_j} (v_i(t) - u_i(t))\,\mathrm{d}x$$

belongs to $L^1(0, T)$, where u is a weak solution to (1.1)–(1.3), $v \in L^s(0, T; W)$ being arbitrary.

Let $f \in L^2(0, T; H)$, and let $\{u, p\}$ be a sufficiently regular solution to (1.1)–(1.3) (i.e. both u and p are sufficiently smooth, their derivatives are integrable to appropriate powers in $\Omega \times [0, T]$, (1.1), (1.2a-c) are satisfied a. e.). We show that u is a weak solution to (1.1)–(1.3).

To this end, let $v \in L^s(0, T; W)$ with $v(t) \in K$ for a. a. $t \in [0, T]$. We then multiply the i-th equation in (1.1) by $v_i - u_i$, integrate over Ω and sum on $i = 1, 2, 3$. Integration by parts of the second and fourth terms of the integral identity obtained yields

$$\int_\Omega u(v_i - u_i)\,\mathrm{d}x + \int_\Omega u_j \frac{\partial u_i}{\partial x_j}(v_i - u_i)\,\mathrm{d}x +$$
$$+ \int_\Omega (\mu_0 + \mu_1|\nabla u|^{r-2}) \frac{\partial u_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\right)\mathrm{d}x =$$
$$= \int_\Omega f_i(v_i - u_i)\,\mathrm{d}x + \int_{\Gamma_1 \cup \Gamma_2} \left[(\mu_0 + \mu_1|\nabla u|^{r-2})\frac{\partial u}{\partial \nu} \cdot (v - u) - p(v - u) \cdot \nu\right] \mathrm{d}S\,.$$

Since $v(t) \in K$ for a. a. $t \in [0, T]$ (i.e. in particular $v_\tau(t) = 0$ a. e. on $\Gamma_1 \cup \Gamma_2$ where $v_\tau(t) = v(t) - (v(t) \cdot \nu)\,\nu$) there exists a real non-negative function $\lambda_k = \lambda_k(t)$ on Γ_k $(k = 1, 2)$ (depending on v) such that

$$v(t) = -\lambda_1(t)\,\nu \quad \textit{for a. a.} \quad t \in [0, T]\,, \quad \textit{a. e. on} \quad \Gamma_1\,,$$
$$v(t) = \lambda_2(t)\,\nu \quad \textit{for a. a.} \quad t \in [0, T]\,, \quad \textit{a. e. on} \quad \Gamma_2\,.$$

Taking into account the third and fourth boundary conditions in (1.2a), (1.2b), we get

$$\int_{\Gamma_1 \cup \Gamma_2} \left[(\mu_0 + \mu_1|\nabla u|^{r-2})\frac{\partial u}{\partial \nu} \cdot (v - u) - p(v - u) \cdot \nu\right] \mathrm{d}S =$$
$$= -\int_{\Gamma_1} \lambda_1 \left[(\mu_0 + \mu_1|\nabla u|^{r-2})\frac{\partial u}{\partial \nu} \cdot \nu - p\right] \mathrm{d}S +$$

$$+\int_{\Gamma_2} \lambda_2 \left[(\mu_0 + \mu_1 |\nabla u|^{r-2}) \frac{\partial u}{\partial \nu} \,.\, \nu - p \right] \mathrm{d}S \geqq 0\,.$$

The inequality in (2.3) is now immediate.

Remark. Let $f \in L^2(0, T; H)$. Suppose that u is a weak solution to (1.1)$-_\tau$1.3) possessing appropriate regularity properties (e.g. $u \in L^s(0, T; W \cap [W_q^2(\Omega)]^3)$ (for a suitable $q > 1$), $u' \in L^2(0, T; H)$). Then it can be shown that there exists a function $p \in L^2_{\mathrm{loc}}(\Omega)$ with $\partial p/\partial x_i \in L^2(\Omega)$ $(i = 1, 2, 3)$ such that $\{u, p\}$ satisfies the equations (1.1) a. e. in $\Omega \times [0, T]$. If in addition the conditions of the lemma in 2.1 are satisfied then it can be proved that $\{u, p\}$ fulfils the second and third boundary conditions in (1.2a) and (1.2b) a. e. on $\Gamma_1 \times [0, T]$ and a. e. on $\Gamma_2 \times [0, T]$, respectively.

3. RESULTS

Let us introduce mappings A_k: $W \to W^*$ $(k = 0, 1)$ by

$$(A_0 u, v) = \mu_0 \int_\Omega \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x\,, \quad u, v \in W\,;$$

$$(A_1 u, v) = \mu_1 \int_\Omega |\nabla u|^{r-2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x\,, \quad u, v \in W\,.$$

A simple calculation shows that

$$(A_1 u, u) = \mu_1 |||u|||^r\,, \quad |||A_1 u|||_* = \mu_1 |||u|||^{r-1} \quad \forall u \in W$$

i.e. A_1 is the duality mapping from W into W^* with respect to the gauge function $\psi(\sigma) = \mu_1 \sigma^{r-1}$. Further, it can be easily verified that A_1 is the gradient of the functional $u \mapsto (1/r)\, \mu_1 |||u|||^r$ $(u \in W)$. Thus, setting

$$A = A_0 + A_1\,, \quad F(u) = \frac{1}{2}(A_0 u, u) + \frac{1}{r} \mu_1 |||u|||^r$$

the operator A and the functional F satisfy the conditions (1.1)$-$(1.5) in [8] (suppose $r > 3$; cf. the remark at the end of [7]).

Further, the estimate (2.5) implies that for each pair u, $v \in W$ there exists a (uniquely determined) element $B(u, v) \in W^*$ such that

$$(B(u, v), w) = \int_\Omega u_j \frac{\partial v_i}{\partial x_j} w_i \, \mathrm{d}x \quad \forall w \in W$$

(note that the estimates (1.4) in [7] are obviously satisfied).

Under the assumption $r > 3$ the estimate (2.5) may be sharpened as follows. Firstly, we obtain by virtue of the imbedding $W_2^1(\Omega) \subset L^6(\Omega)$ the estimates

$$\left| \int_\Omega u \frac{\partial v}{\partial x_j} w \, dx \right| \leqq \|u\|_{L^6(\Omega)} \|v\|_{W_r^1(\Omega)} \|w\|_{L^2(\Omega)} \leqq$$
$$\leqq \text{const} \|u\|_{W_2^1(\Omega)} \|v\|_{W_r^1(\Omega)} \|w\|_{L^2(\Omega)}$$

for arbitrary $u, v, w \in W_r^1(\Omega)$ $(j = 1, 2, 3)$. Secondly, using the imbedding $W_r^1(\Omega) \subset \subset C(\bar{\Omega})$ we obtain

$$\left| \int_\Omega u \frac{\partial v}{\partial x_j} w \, dx \right| \leqq \max_{\bar{\Omega}} |u| \int_\Omega \left| \frac{\partial v}{\partial x_j} w \right| dx \leqq$$
$$\leqq \text{const} \|u\|_{W_r^1(\Omega)} \|v\|_{W_2^1(\Omega)} \|w\|_{L^2(\Omega)}$$

and

$$\left| \int_\Omega u \frac{\partial v}{\partial x_j} w \, dx \right| \leqq \text{const} \|u\|_{L^2(\Omega)} \|v\|_{W_2^1(\Omega)} \|w\|_{W_r^1(\Omega)} .$$

Thus

$$|(B(u, v), w)| \leqq \text{const} \|u\| \; |||v||| \; |w| ,$$
$$|(B(u, v), w)| \leqq \text{const} \; |||u||| \; \|v\| \; |w| ,$$
$$|(B(u, v), w)| \leqq \text{const} \; |u| \; \|v\| \; |||w|||$$

for all $u, v, w \in W$, i.e. the bilinear mapping B satisfies (1.6) in [8].

Finally, set $\varphi = I_K$ where I_K denotes the indicator function of K, i.e.

$$I_K(u) = \begin{cases} 0 & \textit{if} \quad u \in K , \\ +\infty & \textit{if} \quad u \in V \setminus K . \end{cases}$$

The functional I_K is proper, convex and semi-continuous on V. Condition (1.5) in [7] is immediate.

Thus, taking into account the definition of the mappings A and B it is easily seen that the evolution problem (2.1)–(2.4) is a special case of our abstract theory developed in the preceding two parts of our paper.

Applying the results of [7], [8] to the present case we obtain: Let $r > 3$. Then it holds:

1° *Let the data satisfy the conditions*

$$f = f_1 + f_2 : f_1 \in L^2(0, T; H) , \quad f_2, f_2' \in L^{r'}(0, T; W^*) ;$$
$$u_0 \in W \cap K .$$

Then there exists exactly one function $u \in L^\infty(0, T; W) \cap C([0, T]; H)$ *such that*

$$u(t) \in K \quad \textit{for a. a.} \quad t \in [0, T]\,; \tag{3.1}$$

$$u' \in L^2(0, T; H)\,; \tag{3.2}$$

$$\left\{\begin{aligned} &\int_0^T (u', v - u)\,\mathrm{d}t + \\ &+ \int_0^T \int_\Omega (\mu_0 + \mu_1 |\nabla u|^{r-2}) \frac{\partial u_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\right) \mathrm{d}x\,\mathrm{d}t + \\ &+ \int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j}(v_i - u_i)\,\mathrm{d}x\,\mathrm{d}t \geqq \int_0^T (f, v - u)\,\mathrm{d}t \\ &\forall v \in L^r(0, T; W) \quad \textit{with} \quad v(t) \in K \quad \textit{for a. a.} \quad t \in [0, T]\,; \end{aligned}\right. \tag{3.3}$$

$$u(0) = u_0\,. \tag{3.4}$$

2° *Suppose that the data fulfil the following conditions*:

$$f = f_1 + f_2 : f_1 \in L^2(0, T; V^*)\,, \quad f_2, f_2' \in L^{r'}(0, T; W^*)\,;$$

$$u_0 \in \overline{W \cap K}^H\,.$$

Then there exists exactly one function $u \in L^r(0, T; W) \cap C([0, T]; H)$ *which satisfies* (3.1), (3.4) *and the inequality*

$$\begin{aligned} &\int_0^T (v', v - u)\,\mathrm{d}t + \\ &+ \int_0^T \int_\Omega (\mu_0 + \mu_1 |\nabla u|^{r-2}) \frac{\partial u_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial u_i}{\partial x_j}\right) \mathrm{d}x\,\mathrm{d}t + \\ &+ \int_0^T \int_\Omega u_j \frac{\partial u_i}{\partial x_j}(v_i - u_i)\,\mathrm{d}x\,\mathrm{d}t \geqq \int_0^T (f, v - u)\,\mathrm{d}t - \tfrac{1}{2}|v(0) - u_0|^2 \end{aligned}$$

for all $v \in L^r(0, T; W)$ *with* $v' \in L^{r'}(0, T; W^*)$.

3° (i) *Let*

$$f \in L^2(0, T; V^*)\,, \quad f' \in L^{r'}(0, T; W^*)\,, \quad t^\alpha f' \in L^2(0, T; V^*)\,;$$

$$u_0 \in W \cap K$$

where $\alpha \geqq \frac{1}{2}$. *Then there exists exactly one function* $u \in L^\infty(0, T; W) \cap C([0, T]; H)$ *which satisfies* (3.1)–(3.4). *Furthermore, it holds*

$$u \in C([0, T]; V)\,, \quad t^\alpha u' \in L^\infty(0, T; H) \cap L^2(0, T; V)\,.$$

(ii) *If the data satisfy the conditions*

$$f, f' \in L^2(0, T; V^*), \quad u_0 \in W \cap K;$$

$$\left|(f(0), v) + \int_\Omega (\mu_0 + \mu_1 |\nabla u_0|^{r-2}) \frac{\partial u_{0i}}{\partial x_j} \frac{\partial v_i}{\partial x_j} dx + \right.$$
$$\left. + \int_\Omega u_{0j} \frac{\partial u_{0i}}{\partial x_j} v_i \, dx \right| \leqq \text{const} |v|$$

for all $v \in W$, *then the function* u *from* (i) *additionally satisfies*

$$u \in C([0, T]; V), \quad u' \in L^\infty(0, T; H) \cap L^2(0, T; V).$$

For proving the results stated we only note that the uniqueness of the solution to (3.1)–(3.4) follows by passing from (3.3) to the pointwise inequality and using a standard device (cf. the proof of Theorem 2 in [8]). Finally, the functional I_K is subdifferentiable at each point of K, and

$$\partial I_K(u_0) = \{w \in V: ((w, v - u_0)) \leqq 0 \quad \forall v \in K\}.$$

Hence, Theorem 2, (ii) in [8] applies.

Remark. It is easy to see that the theorem in [7] also yields the existence of a weak solution to (1.1)–(1.3) when $\mu_0 = 0$ (cf. [6; Chap. 2.5]).

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Souhrn

O EVOLUČNÍCH NEROVNOSTECH MODIFIKOVANÉHO NAVIEROVA-STOKESOVA TYPU, III

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V článku se aplikují teoretické výsledky z předchozích dvou částí na problém jednostranných okrajových podmínek pro modifikovanou Navier-Stokesovu rovnici (1.1). Uvažované jednostranné okrajové podmínky odpovídají úloze o proudění kapaliny trubicí, při níž je předepsán směr rychlosti v místě vtékání kapaliny do trubice a u jejího ústí. Tyto podmínky jsou popsány vztahy (1.2a) na části hranice Γ_1 (odpovídající vtoku kapaliny) a (1.2b) na části hranice Γ_2 (odpovídající výtoku), přičemž třetí rovnice v obou případech znamená jisté dodatečné přirozené podmínky (související s Greenovou formulí); (1.2c) představuje podmínku nulové rychlosti na plášti trubice Γ_3.

V prvním odstavci je formulován problém, v druhém jsou zavedeny potřebné prostory a pojem slabého řešení a ve třetím jsou shromážděny výsledky o existenci, jednoznačnosti a regularitě pro daný problém.

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