Josef Nedoma

The finite element solution of parabolic equations

*Aplikace matematiky*, Vol. 23 (1978), No. 6, 408–438

Persistent URL: http://dml.cz/dmlcz/103769

# THE FINITE ELEMENT SOLUTION OF PARABOLIC EQUATIONS

JOSEF NEDOMA

(Received February 24, 1977)

In papers on solution of parabolic differential equations by the finite element method, error bounds are derived either in the case that the union of finite elements (straight or curved) matches exactly the given domain (e.g. in Zlámals papers) or in the case of curved elements which do not cover, in general, the given domain (e.g. in Ciarlet-Raviarts papers). In the former case the error bounds are given for fully (i.e. both in space and time) discretized approximate solutions. In the latter case the numerical integration is taken into account; however, the error bounds are given only for semidiscrete (not discretized in time) approximate solutions. Error bounds introduced in this paper are given for fully discretized approximate solutions and for arbitrary curved domains. Simplicial curved elements in $n$-dimensional space are applied. Degrees of accuracy of quadrature formulas are determined so that numerical integration does not worsen the optimal order of convergence in $L_2$-norm of the method.

## 1. NOTATION. THE CONSTRUCTION OF FINITE ELEMENT SPACE. ISOPARAMETRIC INTEGRATION

The norm and the scalar product in the space $L^2(A)$ is denoted by $\|\cdot\|_{0,A}$ and $(\cdot,\cdot)_{0,A}$ respectively.

$H^m(A) \equiv W_2^{(m)}(A)$, $m = 0, 1, \ldots$. Here $W_2^{(m)}(A)$ is a Sobolev space with the norm

$$\|v\|_{W_2{}^{(m)}(A)} \equiv \sqrt{\Big(\sum_{|\alpha|\leqq m} \|D^\alpha v\|_{0,A}^2\Big)}, \text{ where } D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}\,.$$

We denote

$$|v|_{W_2{}^{(m)}(A)} = \sqrt{\Big(\sum_{|\alpha|=m} \|D^\alpha v\|_{0,A}^2\Big)}\,, \quad \|v\|_{m,A} = \|v\|_{W_2{}^{(m)}(A)}\,, \quad |v|_{m,A} = |v|_{W_2{}^{(m)}(A)}\,.$$

$H_0^1(A)$ is the closure of the set $C_0^\infty(A)$ (i.e. the set of infinitely differentiable functions with compact support in $A$) in the norm $\|\cdot\|_{1,A}$. $H^{-1}(A)$ is the space dual to $H_0^1(A)$ (with dual norm).

$L^\infty(H^{(m)}(A))$ is the space of all functions $v(x, t)$, $x = (x_1, \ldots, x_n) \in A$, $t \in [0, T]$ such that $v(x, t) \in H^m(A)$, $\forall t \in [0, T]$ and the function $\|v(x, t)\|_{m,A}$ is bounded for almost all $t \in [0, T]$. We denote

$$\|v\|_{L^\infty(H^m)} = \operatorname*{vrai\,sup}_{t\in[0,T]} \|v\|_{m,A} \, .$$

In the same way as in [1] we define the *k-regular family* $\{e\}_k$ of simplicial isoparametric finite elements $e$ in the following manner:
We are given

(a) A set $\hat{\Sigma} = \bigcup_{i=1}^{N} \{\hat{a}_i\}$ of $N$ distinct points from $R^n$ such that its closed convex hull $\hat{T}$ is a unit $n$-simplex.

(b) A finite dimensional space $\hat{P}$ of functions defined on $\hat{T}$ with $\dim \hat{P} = N$ such that $\hat{\Sigma}$ is *$\hat{P}$-unisolvent*, i.e. the Lagrange interpolation problem[1]): "Find $\hat{p} \in \hat{P}$ such that $\hat{p}(\hat{a}_i) = \alpha_i$, $1 \leqq i \leqq N$" has a unique solution for any real numbers $\alpha_i$. We suppose $\hat{P} \in C^{k+1}(\hat{T})$, $\hat{P} \supset \hat{P}(1)$. Here for any integer $r \geqq 0$, $\hat{P}(r)$ is the space or restrictions to $\hat{T}$ of all polynomials of degree $\leqq r$ in $n$ variables $\hat{x}_1, \ldots, \hat{x}_n$.

(c) A set $\Sigma = \bigcup_{i=1}^{N} \{a_i\}$ of $N$ distinct points from $R^n$.

Then the simplicial finite element $e \in \{e\}_h$ is the image (i.e. $e = F_e(\hat{T})$) of the set $\hat{T}$ through the unique mapping $F_e : \hat{T} \to R^n$ which satisfies

$$F_e \in \hat{p}^n \, , \quad F_e(\hat{a}_i) = a_i \, , \quad 1 \leqq i \leqq N \, .$$

We suppose

(d) For all $h$, the mapping $F_e$ is a $C^{k+1}$-diffeomorphism and there exist constants $c_l$, $0 \leqq l \leqq k + 1$, independent of $h$, such that for all $h$:

(1) $$\sup_{\hat{x}\in\hat{T}} \max_{|\alpha|=l} |D^\alpha F_e(\hat{x})| \leqq c_l h^l \, , \quad 1 \leqq l \leqq k + 1$$

and

(2) $$0 < \frac{1}{c_0} h^n \leqq |J_e(\hat{x})| \leqq c_0 h^n \, ,$$

where $J_e(\hat{x})$ is the Jacobian of the mapping $F_e$ at the point $\hat{x} \in \hat{T}$.

With every element $e$ we associate the finite dimensional space $P_e$ (with $\dim = N$) of functions

(3) $$P_e = \{p_e : e \to R; \; p_e = \hat{p}(F_e^{-1}), \; \forall \hat{p} \in \hat{P}\} \, .$$

The *e-interpolate* $\pi_e u$ of a given function $u : e \to R$ is the unique function which satisfies

(4) $$\pi_e u \in P_e \, , \quad \pi_e \, u(a_i) = u(a_i) \, , \quad 1 \leqq i \leqq N \, .$$

---

[1]) The analogous analysis can be given for Hermite interpolation.

For a $k$-regular family $\{e\}_h$ of finite elements the following interpolation theorem is true (see [1], Theorem 2, p. 429):

*Interpolation theorem.*

Let a $k$-regular family $\{e\}_h$ of simplicial finite elements such that $\hat{P}(k) \subset \hat{P}$ be given. Let

$$k > \frac{n}{2} - 1 \tag{5}$$

Then for any integer $m$ such that $0 \leqq m \leqq k + 1$, there exists a constant $c$ independent of $h$ such that for any $e \in \{e\}_h$ and for any function $u \in H^{k+1}(e)$ we have

$$|u - \pi_e u|_{m,e} \leqq c h^{k+1-m} \|u\|_{k+1,e} \tag{6}$$

We define now a *k-regular trtangulatton* $\mathscr{C}_h$ of a set $\Omega$:

Let $\Omega$ be a bounded domain. Let $\Omega_h$ be the union of a finite number of simplicial finite elements $e = F_e(\hat{T})$. Every element $e$ is determined by $N$ points $a_{i,e}$. We suppose that all points $a_{i,e}$ belong to $\bar{\Omega}$. The family of elements constructed in this way is called a triangulation of $\Omega$ (or of $\Omega_h$) and is denoted by $\mathscr{C}_h$. We say that a triangulation $\mathscr{C}_h$ of $\Omega$ is $k$-regular if the family of all elements from which the triangulation is formed is $k$-regular and if for the boundary elements (i.e. for elements $e$ such that $e \not\subset \Omega$) of the triangulation $\mathscr{C}_h$ we have

$$\max_{y' \in \Delta} |\psi_h(y') - \psi(y')| \leqq c h^{k+1} \tag{7}$$

where $c$ is a constant independent of $h$ and the notation is that of Fig. 1.

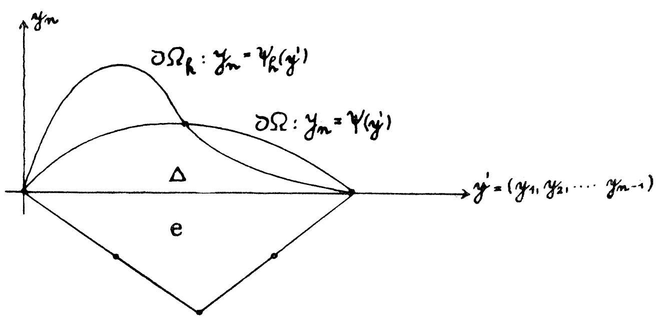


Fig. 1.

With a given $k$-regular triangulation $\mathscr{C}_h$, we associate the finite dimensional space $V_h$ of functions $v$ defined by

$$V_h = \{v \in C^0(\bar{\Omega}_h);\ v_e \in P_e \text{ for all } e \in \mathscr{C}_h,\ v = 0 \text{ on } \partial\Omega_h\}\,, \tag{8}$$

where $v_e$ is the restriction of the function $v$ to the set $e$.

In our paper we suppose that $\hat{P} = \hat{P}(k)$. This restriction is not essential. It enables us to give simpler proofs.

Let $\Phi(x)$ be any function defined on the element $e$. Then the function $\Phi(F_e(\hat{x}))$ is defined on $\hat{T}$. In the sequel we will denote it by $\Phi^*(\hat{x})$.

Let us suppose that we have at our disposal a quadrature formula of a degree $d$ over the reference set $\hat{T}$. In other words,

$$\int_{\hat{T}} \Phi^*(\hat{x})\, d\hat{x} \quad \text{is approximated by} \quad \sum_r \hat{\omega}_r\, \Phi^*(\hat{b}_r) \tag{9}$$

for some specified points $\hat{b}_r \in \hat{T}$ and weights $\hat{\omega}_r$ which will be assumed once and for all to satisfy

$$\hat{\omega}_r > 0\ ^{2)}\,. \tag{10}$$

Concerning $\hat{b}_r$ we suppose that for every $r$, $\hat{b}_r$ lies either inside $\hat{T}$ or it coincides with some of the points $\hat{a}_i$.

With the quadrature scheme (9) we associate the error

$$\hat{E}(\Phi^*) = \int_{\hat{T}} \Phi^*(\hat{x})\, d\hat{x} - \sum_r \hat{\omega}_r\, \Phi^*(\hat{b}_r)\,. \tag{11}$$

Using the standard formula for the change of variables in multiple integrals, we find that

$$\int_e \Phi(x)\, dx \quad \text{is approximated by} \quad \sum_r \omega_{r,e}\, \Phi(b_{r,e})\,, \tag{12}$$

where

$$\omega_{r,e} = \hat{\omega}_r\, J_e(\hat{b}_r)\ ^{3)}\,, \quad b_{r,e} = F_e(\hat{b}_r)\,. \tag{13}$$

We see that the quadrature scheme (9) over the reference set $\hat{T}$ induces the quadrature scheme (12) over the finite element $e$, a circumstance which we call "*isoparametric*" numerical integration. With the scheme (12), we associate the error

$$E_e(\Phi) = \int_e \Phi(x)\, dx - \sum_r \omega_{r,e}\, \Phi(b_{r,e}) \tag{14}$$

so that we have

$$E_e(\Phi) = \hat{E}(\Phi^* J_e) \quad \text{and} \quad \hat{E}(\Phi^*) = E(\Phi J_e^{-1})\,. \tag{15}$$

---

[2]) This assumption is by no means necessary but it yields simpler proofs.

[3]) We may, and will, assume that the Jacobian $J_e(\hat{x}) > 0$ for all $\hat{x} \in \hat{T}$.

## 2. FORMULATION OF THE PROBLEM

Let $x = (x_1, \ldots, x_n) \in R^n$. Let $\Omega$ be a bounded domain in $R^n$. Let functions $g(x)$, $g_{ij}(x)$, $i, j = 1, \ldots, n$ defined on $\bar{\Omega}$ and a function $f(x, t)$ defined on $\Omega \times [0, T]$ be smooth enough. Let

$$(16) \qquad g_{ij}(x) = g_{ji}(x)\,, \quad g(x) \geqq g_0\ (= \text{const}) > 0\,, \quad \forall x \in \bar{\Omega}$$

and let the differential operator

$$(17) \qquad L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left( g_{ij}(x) \frac{\partial}{\partial x_i} \right)$$

be strongly elliptic, i.e. there exists a constant $g_1 > 0$ such that

$$(18) \qquad \sum_{i,j=1} g_{ij}(x)\, \xi_i \xi_j \geqq g_1 \sum_{i=1} \xi_i^2 \quad \text{for all} \quad x \in \bar{\Omega} \quad \text{and for all } (\xi_1 \ldots \xi_1) \in R^n\,.$$

Let $a(u, v)$ be the bilinear form corresponding to the operator $L$, i.e.

$$(19) \qquad a(u, v) = \int_\Omega \sum_{i,j=1} g_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, \mathrm{d}x\,.$$

We study the following problem:

*Find a functton $u(x, t)$ such that*

$$(20) \qquad u \in L^\infty(H_0^1(\Omega))\,, \quad \frac{\partial u}{\partial t} \in L^\infty(H^{-1}(\Omega))\,,$$

$$\left( g \frac{\partial u}{\partial t}, v \right)_{0,\Omega} + a(u, v) = (f, v)_{0,\Omega_h}\,, \quad \forall v \in H_0^1(\Omega) \quad \text{and} \quad t \in (0, T]\,,$$

$$u(x, 0) = u_0(x) \in L^2(\Omega)\,.$$

First, we discretize this problem by the finite element method with respect to $x$. Let $\mathscr{C}_h$ be a $k$-regular triangulation of the set $\Omega$ and let $V_h$ be the corresponding finite element space. The union of the elements $e$ from $\mathscr{C}_h$ forms a set $\Omega_h$ which, in general, differs from $\Omega$. We extend the functions $g(x)$, $g_{ij}(x)$, $u_0(x)$ to a greater set $\tilde{\Omega} \supset \bar{\Omega}$ so that the conditions (16) and (18) are satisfied (with positive constants $\tilde{g}_0$, $\tilde{g}_1$). In this way we obtain functions $\tilde{g}(x)$, $\tilde{g}_{ij}(x)$, $\tilde{u}_0(x)$. Obviously, for sufficiently small $h$

$$(21) \qquad \Omega_h \subset \tilde{\Omega}\,.$$

The solution $u$ of the problem (20) is supposed to satisfy

$$(22) \qquad u, \frac{\partial u}{\partial t} \in L^\infty(H^{k+3}(\Omega))\,.$$

The validity of the assumption (22) may be assured by a sufficient smoothness of the given functions and the boundary $\partial\Omega$ of the set $\Omega$. By the Calderon theorem, for every $t \in (0, T]$ there exist extensions $\tilde{u}(x, t)$, $\tilde{u}_t$ [4]) of the functions $u$, $\partial u/\partial t$ such that

(23) $$\|\tilde{u}\|_{k+3,\tilde{\Omega}} \leqq c\|u\|_{k+3,\Omega}\,, \quad \|\tilde{u}_t\|_{k+3,\tilde{\Omega}} \leqq c\left\|\frac{\partial u}{\partial t}\right\|_{k+3,\Omega}\,,$$

where $c$ is a constant independent of $h$ and of $t$ (it depends on $\Omega$ only).

Let us denote

(24) $$\tilde{f}(x, t) = \tilde{g}(x)\widetilde{\frac{\partial u}{\partial t}} - \tilde{L}\tilde{u}\,,$$

where

(25) $$\tilde{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(\tilde{g}_{ij}(x)\frac{\partial}{\partial x_i}\right).$$

According to (20) we define now the following semidiscrete problem (see [2]): *Find a functton $u_s(x, t)$ such that*

(26) $$u_s,\ \frac{\partial u_s}{\partial t} \in L^\infty(V_h(\Omega_h))\,,$$

$$\left(\tilde{g}(x)\frac{\partial u_s}{\partial t}\,,\ v\right)_{0,\Omega_h} + \tilde{a}(u_s, v) = (\tilde{f}, v)_{0,\Omega_h}\,, \quad \forall v \in V_h\,, \quad t \in (0, T]\,,$$

$$u_s(x, 0) = u_0 \in V_h$$

where $u_0$ is an approximate of $\tilde{u}_0(x)$ and $\tilde{a}(u, v)$ is the bilinear form

(27) $$\tilde{a}(u, v) = \int_{\Omega_h} \sum_{i,j=1}^{n} \tilde{g}_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,\mathrm{d}x\,.$$

We named the problem (26) semidiscrete because it is discretized with respect to $x$ only. It is obvious that (26) is a system of ordinary differential equations with an unknown vector function of parameter $t$. This suggests the way how to discretize the problem with respect to $t$. We solve the system of ordinary differential equations by $\nu$-step A-stabil method (for $\nu = 1, 2$) of order $q$. We divide the time interval $[0, T]$ into a finite number of mutually equal parts $\Delta t$. We introduce the notation

(28) $$\Phi^m = \Phi^m(x) = \Phi(x, m\,\Delta t)\,, \quad m = 0, 1, \ldots$$

for any function $\Phi(x, t)$.

[4]) The identity $\tilde{u}_t = \partial\tilde{u}/\partial t$ is not supposed to be true.

According to (26) and to the described way of the time discretization we define the following discrete problem:

*Find a function $u_d(x, t)$ such that*

(29)
$$u_d = V_h \quad \text{for any} \quad t = 0\,, \quad \Delta t, 2\,\Delta t, \ldots, T\,,$$

$$(\tilde{g}(x) \sum_{j=0}^{\nu} \alpha_j u_d^{m+j}, v)_{0,\Omega_h} + \Delta t\; \tilde{a}(\sum_{j=0}^{\nu} \beta_j u_d^{m+j}, v) =$$

$$= \Delta t(\sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}, v)_{0,\Omega_h}\,, \quad \forall v \in V_h\,, \quad m = 0, 1, \ldots\,,$$

$$u_d^0 = u_0 \in V_h\,.$$

It is easy to prove that the problem (29) has one and only one solution. This solution can be considered an approximate of the function $\tilde{u}(x, t)$.

Since it is either too costly or simply impossible to evaluate exactly the integrals $(\cdot, \cdot)_{0,\Omega_h}$, $\tilde{a}(\cdot, \cdot)$, we must now take into account the fact that approximate integration is used for their computation. For this purpose we use the isoparametric numerical integration, i.e. with agreement with (12) we replace

(30)
$$(w, z)_{0,\Omega_h} \approx (w, z)_h\,, \quad \tilde{a}(w, z) \approx a_h(w, z)$$

where

(31)
$$(w, z)_h = \sum_{e \in \mathscr{C}_h} \sum_r \omega_{r,e}\, w(b_{r,e})\, z(b_{r,e})\,,$$

(32)
$$a_h(w, z) = \sum_{e \in \mathscr{C}_h} \sum \omega_{r,e} \left[ \sum_{i,j=1}^{n} g_{ij}(b_{r,e}) \frac{\partial w(b_{r,e})}{\partial x_i} \frac{\partial z(b_{r,e})}{\partial x_j} \right]$$

(in (32) $g_{ij}$ is written instead of $\tilde{g}_{ij}$ because $b_{r,e} \in \bar{\Omega}$ for sufficiently small $h$).

According to (29) and (30) we define the following problem:

*Find a function $u_h(x, t)$ such that*

(33)
$$u_h \in V_h \quad \text{for} \quad t = 0\,, \quad \Delta t, \ldots, T\,,$$

$$(g(x) \sum_{j=0}^{\nu} \alpha_j u_h^{m+j}, v)_h + \Delta t\; a_h(\sum_{j=0}^{\nu} \beta_j u_h^{m+j}, v) =$$

$$= \Delta t(\sum_{j=0}^{\nu} \beta_j f^{m+j}, v)_h\,, \quad \forall v \in V_h\,, \quad m = 0, 1, \ldots\,,$$

$$u_h^0 = u_0 \in V_h\,.$$

The aim of the paper is to derive error bounds for

(34)
$$\lambda = u(x, t) - u_h(x, t) \quad \text{for} \quad t = 0, \Delta t, \ldots, T$$

in the norm $\|\cdot\|_{0,\Omega \cap \Omega_h}$.

## 3. THE THEOREM ON RITZ APPROXIMATION

A function $\eta(x, t) \in V_h(V_h \subset H_0^1(\Omega_h))$, $\forall t \in (0, T]$ such that

$$(35) \qquad (\tilde{g}(x)\, \tilde{u}_t, v)_{0,\Omega_h} + \tilde{a}(\eta(x, t), v) = (\tilde{f}(x, t), v)_{0,\Omega_h}, \quad \forall v \in V_h$$

is called the *Ritz approximation* of the function $\tilde{u}(x, t)$.

Let us prove that the function $\eta(x, t)$ is an orthogonal projection onto $V_h$ of the function $\tilde{u}(x, t)$ in the energetic norm given by the bilinear form $\tilde{a}(\cdot, \cdot)$, i.e. it satisfies

$$(36) \qquad \tilde{a}(\tilde{u} - \eta, v) = 0, \quad \forall v \in V_h.$$

Really, (35) implies

$$\tilde{a}(\eta, v) = (\tilde{f} - \tilde{g}\tilde{u}_t, v)_{0,\Omega_h}.$$

The Green theorem together with (24) yield

$$\tilde{a}(\tilde{u}, v) = -(\tilde{L}\tilde{u}, v)_{0,\Omega_h} = (\tilde{f} - \tilde{g}\tilde{u}_t, v)_{0,\Omega_h}.$$

The aim of this chapter is to estimate the norm

$$(37) \qquad \|\tilde{u}(x, t) - \eta(x, t)\|_{i,\Omega_h}, \quad i = 0, 1.$$

In the sequel the constants independent of $h$ and $t$ will be denoted by $c$. The notation is generic, i.e., $c$ will not denote the same constant in any two places.

From (27) and (18) it follows that

$$(38) \qquad \tilde{a}(v, v) \geqq \tilde{g}_1 |v|_{1,\Omega_h}^2, \quad \forall v \in H^1(\Omega_h).$$

The continuity assumption and the Cauchy-Schwarz inequality imply

$$(39) \qquad |\tilde{a}(z, v)| \leqq c|z|_{1,\Omega_h} |v|_{1,\Omega_h}, \quad \forall z, \quad v \in H^1(\Omega_h).$$

Let $v \in H^{k+1}(\Omega_h)$ and let $\mathscr{C}_h$ be a $k$-regular triangulation of $\Omega_h$. Let $\pi_h v$ be the function which is equal to $\pi_e v$ on every element $e \in \mathscr{C}_h$. Here $\pi_e v$ is an $e$-interpolate of $v$ (see (4)). The inerpolation theorem (see (6)) implies

$$|v - \pi_e v|_{m,e}^2 \leqq c h^{2(k+1-m)} \|v\|_{k+1,e}^2, \quad 0 \leqq m \leqq k + 1.$$

Hence

$$(40) \qquad \|v - \pi_h v\|_{1,\Omega_h} \leqq c h^k \|v\|_{k+1,\Omega_h}.$$

Obviously $\pi_h \tilde{u} = \pi_h u = 0$ on $\partial\Omega_h$. Hence

$$(41) \qquad \pi_h \tilde{u} \in V_h \quad \text{for any} \quad t \in (0, T].$$

From (38), (36), (41) and (39) we obtain

$$|\tilde{u} - \eta|_{1,\Omega_h}^2 \leqq c\tilde{a}(\tilde{u} - \eta,\ \tilde{u} - \eta) \leqq c\tilde{a}(\tilde{u} - \pi_h\tilde{u},\ \tilde{u} - \pi_h\tilde{u}) \leqq c|\tilde{u} - \pi_h\tilde{u}|_{1,\Omega_h}^2.$$

From here and from (40) it follows

$$|\tilde{u} - \eta|_{1,\Omega_h} \leqq ch^k \|\tilde{u}\|_{k+1,\Omega_h}, \quad \forall t \in (0, T]. \tag{42}$$

Let us denote

$$\psi(x, t) = \begin{cases} \tilde{u}(x, t) - \eta(x, t) & \text{for} \quad x \in \bar{\Omega}_h \\ 0 & \text{for} \quad x \in \tilde{\Omega} - \bar{\Omega}_h . \end{cases} \tag{43}$$

For any $t \in (0, T]$ we solve the homogeneous Dirichlet problem

$$-L\Phi(x, t) = \psi(x, t) \quad \text{in} \quad \Omega, \quad \Phi(x, t) = 0 \quad \text{on} \quad \partial\Omega. \tag{44}$$

If $\partial\Omega$ is smooth enough then

$$\Phi \in H_0^1(\Omega) \cap H^2(\Omega), \tag{45}$$

$$\|\Phi\|_{2,\Omega} \leqq c\|\psi\|_{0,\Omega} \leqq c\|\psi\|_{0,\tilde{\Omega}} = c\|\psi\|_{0,\Omega_h}, \quad \text{i.e.}$$

$$\|\Phi\|_{2,\Omega} \leqq c\|\tilde{u} - \eta\|_{0,\Omega_h}, \quad \forall t \in (0, T].$$

Using the Calderon theorem we extend the function $\Phi$ from $\Omega$ onto $\tilde{\Omega}$. In this way we obtain a function $\tilde{\Phi} \in H^2(\tilde{\Omega})$ such that

$$\|\tilde{\Phi}\|_{2,\tilde{\Omega}} \leqq c\|\Phi\|_{2,\Omega}.$$

From here and from (45) it follows

$$\|\Phi\|_{2,\Omega_h} \leqq c\|\tilde{u} - \eta\|_{0,\Omega_h}. \tag{46}$$

Let $\zeta(x, t)$ be the orthogonal projection of $\Phi(x, t)$ onto the space $V_h(\Omega_h)$ in the energetic norm. Then in the same way as in (42) we get

$$|\tilde{\Phi} - \zeta|_{1,\Omega_h} \leqq ch\|\tilde{\Phi}\|_{2,\Omega_h}.$$

From here and from (46) it follows

$$|\tilde{\Phi} - \zeta|_{1,\Omega_h} \leqq ch\|\tilde{u} - \eta\|_{0,\Omega_h}. \tag{47}$$

It is easy to verify that ($\psi + L\Phi = \psi + \tilde{L}\tilde{\Phi}$ *on* $\Omega$)

$$\|\tilde{u} - \eta\|_{0,\Omega_h}^2 = \int_{\Omega_h - \Omega} (\tilde{u} - \eta)(\psi + \tilde{L}\tilde{\Phi})\, dx - \int_{\Omega_h} (\tilde{u} - \eta)\, \tilde{L}\tilde{\Phi}\, dx. \tag{48}$$

The Green theorem ($\eta = 0$ on $\partial\Omega_h$) yields

$$-\int_{\Omega_h} (\tilde{u} - \eta)\, \tilde{L}\tilde{\Phi}\, dx = \tilde{a}(\tilde{u} - \eta, \tilde{\Phi}) - \int_{\partial\Omega_h} \tilde{u} \frac{\partial\tilde{\Phi}}{\partial\nu}\, dS. \tag{49}$$

Since $\zeta \in V_h$, we get from (36) $\tilde{a}(\tilde{u} - \eta, \zeta) = 0$, i.e. $\tilde{a}(\tilde{u} - \eta, \tilde{\Phi}) = \tilde{a}(\tilde{u} - \eta, \tilde{\Phi} - \zeta)$. From here and from (49) and (48) it follows

$$(50) \qquad \|\tilde{u} - \eta\|^2_{0,\Omega_h} \leqq \left| \int_{\Omega_h - \Omega} (\tilde{u} - \eta)(\psi + \tilde{L}\tilde{\Phi})\,\mathrm{d}x \right| + |\tilde{a}(\tilde{u} - \eta, \tilde{\Phi} - \zeta)| + + \left| \int_{\partial\Omega_h} \tilde{u} \frac{\partial \tilde{\Phi}}{\partial \nu}\,\mathrm{d}S \right| .$$

Before estimating the expressions on the right hand side of (50) we introduce some lemmas and notes.

**Lemma 1.** *Let $\mathscr{C}_h$ be a k-regular triangulation of $\Omega_h$. Let $v(x, t) \in H^1(\tilde{\Omega})$, $\forall t \in (0, T]$ and let*

$$(51) \qquad v(y', y_n, t) = 0 \quad \text{on} \quad \partial\Omega$$

*(for notation see Fig. 1). Then there exist a constant c such that*

$$(52) \qquad \|v\|_{0,\Omega_h - \Omega} \leqq ch^{k+1}|v|_{1,\Omega_h - \Omega} \,.$$

Proof. From (51), from the Schwarz inequality and from (7) it follows

$$(53) \quad |v(y', y_n, t)|^2 = \left| \int_{\psi(y')}^{y_n} \frac{\partial v(y', \tau, t)}{\partial y_n}\,\mathrm{d}\tau \right|^2 \leqq |y_n - \psi(y')| \left| \int_{\psi(y')}^{y_n} \left( \frac{\partial v(y', \tau, t)}{\partial y_n} \right)^2 \mathrm{d}\tau \right| \leqq \leqq ch^{k+1} \left| \int_{\psi(y')}^{y_n} \left( \frac{\partial v(y', \tau, t)}{\partial y_n} \right)^2 \mathrm{d}\tau \right| .$$

By integrating (53) and summing over all the boundary elements $e \in \mathscr{C}_h$ we obtain (52).

Remark 1. The proof implies immediately that the assumption (51) may be replaced by the assumption

$$(51') \qquad v(y', y_n, t) = 0 \quad \text{on} \quad \partial\Omega_h \,.$$

Remark 2. (53) yields immediately the inequality

$$(54) \qquad |v(x, t)| \leqq ch^{k+1}|v|_{1,\infty,\tilde{\Omega}}$$

valid for every $x \in \partial\Omega_h$ (under the assumption $v = 0$ on $\partial\Omega$).

**Lemma 2.**

$$(55) \qquad \|v\|_{0,\delta\Omega_h} \leqq c\|v\|_{1,\Omega_h} , \quad \forall v \in H'(\tilde{\Omega}) \,.$$

Proof. Lemma is a consequence of the proof of Theorem 1.2, p. 15 of [3] and the inclusion $\Omega_h \subset \Omega$ for all $h$.

We estimate now the expressions on the right hand side of (50).

The Schwarz inequality gives

$$(56)\qquad \left|\int_{\Omega_h-\Omega}(\tilde{u}-\eta)(\psi+\tilde{L}\tilde{\Phi})\,\mathrm{d}x\right| \leqq \|\tilde{u}-\eta\|_{0,\Omega_h-\Omega}\,\|\psi+\tilde{L}\tilde{\Phi}\|_{0,\Omega_h-\Omega}\,.$$

From (46) we get

$$\|\psi+\tilde{L}\tilde{\Phi}\|_{0,\Omega_h-\Omega} \leqq \|\psi+\tilde{L}\tilde{\Phi}\|_{0,\Omega_h} \leqq \|\psi\|_{0,\Omega_h} + \|\tilde{L}\tilde{\Phi}\|_{0,\Omega_h} \leqq$$

$$\leqq \|\psi\|_{0,\Omega_h} + c\|\tilde{\Phi}\|_{2,\Omega_h} \leqq \|\psi\|_{0,\Omega_h} + c\|\tilde{u}-\eta\|_{0,\Omega_h}\,.$$

From here and from (43) it follows that

$$(57)\qquad \|\psi+\tilde{L}\tilde{\Phi}\|_{0,\Omega_h-\Omega} \leqq c\|\tilde{u}-\eta\|_{0,\Omega_h}\,.$$

From (38), (36) and (39) we get

$$|\eta|^2_{1,\Omega_h} \leqq c\tilde{a}(\eta,\eta) = c\tilde{a}(\tilde{u},\eta) \leqq c|\tilde{u}|_{1,\Omega_h}\,|\eta|_{1,\Omega_h}\,.$$

Hence

$$|\eta|_{1,\Omega_h} \leqq c|\tilde{u}|_{1,\Omega_h}\,.$$

From here, from (52) and from the Remark 2 (we remember $\tilde{u} = 0$ on $\partial\Omega$ and $\eta = 0$ on $\partial\Omega_h$) we obtain

$$\|\tilde{u}-\eta\|_{0,\Omega_h-\Omega} \leqq \|\tilde{u}\|_{0,\Omega_h-\Omega} + \|\eta\|_{0,\Omega_h-\Omega} \leqq$$

$$\leqq ch^{k+1}\{|\tilde{u}|_{1,\Omega_h-\Omega} + |\eta|_{1,\Omega_h-\Omega}\} \leqq ch^{k+1}|\tilde{u}|_{1,\Omega_h}\,.$$

Substituting from here and from (57) into (56) we get

$$(58)\qquad \left|\int_{\Omega_h-\Omega}(\tilde{u}-\eta)(\psi+\tilde{L}\tilde{\Phi})\,\mathrm{d}x\right| \leqq ch^{k+1}|\tilde{u}|_{1,\Omega_h}\,\|\tilde{u}-\eta\|_{0,\Omega_h}\,.$$

From (39), (42) and (47) we get

$$(59)\qquad |\tilde{a}(\tilde{u}-\eta,\ \tilde{\Phi}-\zeta)| \leqq ch^{k+1}\|\tilde{u}\|_{k+1,\Omega_h}\,\|\tilde{u}-\eta\|_{0,\Omega_h}\,.$$

The Schwarz inequality implies

$$(60)\qquad \left|\int_{\partial\Omega_h}\tilde{u}\,\frac{\partial\tilde{\Phi}}{\partial\nu}\,\mathrm{d}S\right| \leqq \|\tilde{u}\|_{0,\partial\Omega_h}\left\|\frac{\partial\tilde{\Phi}}{\partial\nu}\right\|_{0,\partial\Omega_h}\,.$$

From (55) and (46) it follows that

$$(61)\qquad \left\|\frac{\partial\tilde{\Phi}}{\partial\nu}\right\|_{,\partial\Omega_h 0} \leqq c\sum_{i=1}^{n}\left\|\frac{\partial\tilde{\Phi}}{\partial x_i}\right\|_{0,\partial\Omega_h} \leqq c\sum_{i=1}^{n}\left\|\frac{\partial\tilde{\Phi}}{\partial x_i}\right\|_{1,\Omega_h} \leqq$$

$$\leqq c\|\tilde{\Phi}\|_{2,\Omega_h} \leqq c\|\tilde{u}-\eta\|_{0,\Omega_h}\,.$$

Using (54), (55), (21) and the first Sobolev theorem (see for example [4]) we get (we suppose (5) to be satisfied)

$$\|\tilde{u}\|_{0,\partial\Omega_h} = \sqrt{(\int_{\partial\Omega_h} \tilde{u}^2 \,\mathrm{d}s)} \leqq ch^{k+1}\| \, |\tilde{u}|_{1,\infty,\tilde{\Omega}}\|_{0,\partial\Omega_h} \leqq ch^{k+1}\| \, |\tilde{u}|_{1,\infty,\tilde{\Omega}}\|_{1,\Omega_h} \leqq$$
$$\leqq ch^{k+1} \operatorname{mes} \Omega_h |\tilde{u}|_{1,\infty,\tilde{\Omega}} \leqq ch^{k+1}\|\tilde{u}\|_{k+2,\tilde{\Omega}} \,.$$

Substituting from here and from (61) into (60) we get

$$(62) \qquad \left| \int_{\partial\Omega_h} \tilde{u} \frac{\partial \tilde{\Phi}}{\partial v} \,\mathrm{d}s \right| \leqq ch^{k+1}\|\tilde{u}\|_{k+2,\tilde{\Omega}} \, \|\tilde{u} - \eta\|_{0,\Omega_h} \,.$$

(50), (58), (59) and (60) implies

$$(63) \qquad \|\tilde{u} - \eta\|_{0,\Omega_h} \leqq ch^{k+1}\|\tilde{u}\|_{k+2,\tilde{\Omega}} \,.$$

From here and from (42) it follows that

$$\|\tilde{u} - \eta\|_{1,\Omega_h} \leqq \|\tilde{u} - \eta\|_{0,\Omega_h} + |\tilde{u} - \eta|_{1,\Omega_h} \leqq ch^{k+1}\|\tilde{u}\|_{k+2,\tilde{\Omega}} + ch^{k}\|\tilde{u}\|_{k+1,\tilde{\Omega}} \,.$$

Hence

$$(64) \qquad \|\tilde{u} - \eta\|_{1,\Omega_h} \leqq ch^{k}\|\tilde{u}\|_{k+2,\tilde{\Omega}} \,.$$

Now we can summarize the results into the following

**Theorem 1** (*Theorem on the Ritz approximaton*). *Let $u(x, t)$ be a solutton of the problem (20) such that*

$$(65) \qquad u(x, t) \in H^{k+3}(\Omega) \,, \quad \forall t \in (0, T] \,.$$

*Let a triangulation $\mathscr{C}_h$ of the set $\Omega$ be k-regular. Let $\Omega_h \subset \tilde{\Omega}$ and let*

$$(66) \qquad k > \frac{n}{2} - 1 \,.$$

*Let $\tilde{u}(x, t) \in H^{k+3}(\tilde{\Omega})$, $\forall t \in (0, T]$ be an extension of $u(x, t)$ from $\Omega$ onto $\tilde{\Omega}(\Omega \subset \tilde{\Omega})$ such that*

$$(67) \qquad \|\tilde{u}\|_{k+3,\tilde{\Omega}} \leqq c_1\|u\|_{k+3,\Omega}$$

*where $c_1$ is a constant (independent of $h$ and $t$).*

*Let $\eta(x, t) \in V_h(\Omega_h)$, $\forall t \in (0, T]$ be the Ritz approximation of the function $\tilde{u}(x, t)$. Then there exists a constant $c$ independent of $h$ and $t$ such that*

$$(68) \qquad \|\tilde{u} - \eta\|_{i,\Omega_h} \leqq ch^{k+1-i}\|u\|_{k+3,\Omega} \,, \quad i = 0, 1 \,.$$

Proof. The inequality (68) follows immediately from (63), (64) and (67).

Note 3. In Theorem 1 the number $k + 3$ can be replaced by the number $k + 2$ in all places. Nevertheless in what follows we shall need the above formulation.

Now we derive some estimates of errors due to isoparametric integration which was defined in (10)–(15). First, we give some technical lemmas.

**Lemma 3.** *For any functions $\varphi$ and $\psi$ from the class $C^{(\alpha_1,\dots\alpha_n)}$ the following inequalities are true*

$$|D^{\alpha_1,\dots\alpha_n}(\varphi\psi)| \leqq c_1 \sum_{\substack{\beta_1=0,\dots,\alpha_1\\ \beta_n=0,\dots,\alpha_n}} |D^{(\beta_1,\dots,\beta_n)}\varphi|\,|D^{(\alpha_1-\beta_1,\dots,\alpha_n-\beta_n)}\psi|, \tag{69}$$

$$|\varphi\psi|^2_{i,\hat{T}} \leqq c_2 \sum_{j=0}^{i} \{|\varphi|^2_{j,\hat{T}} \,.\, \sup_{\hat{T}} \max_{|\alpha|=i-j} (D^\alpha\psi)^2\}, \tag{70}$$

*where $c_1$ and $c_2$ are constants independent of $h$ and $t$.*

Proof. The proof of inequality (69) is trivial. The inequality (70) follows from (69) by simple calculation.

**Lemma 4.** *For polynomials $r, s$ on the reference set $\hat{T}$ the following inequalities are true*

$$\max_{\hat{T}} |D^\alpha r| \leqq c_1 |r|_{|\alpha|,\hat{T}}, \tag{71}$$

$$|r|^2_{j,\hat{T}} \leqq c_2 |r|^2_{i,\hat{T}} \quad \text{for} \quad j \geqq i \geqq 0, \tag{72}$$

$$|rs|^2_{i,\hat{T}} \leqq c_3 \sum_{j=0}^{i} |r|^2_{j,\hat{T}}\, |s|^2_{i-j,\hat{T}}. \tag{73}$$

Proof. For the proof of (71) see [7], p. 356. The inequality (72) is an immediate consequence of (70).

**Lemma 5.** *Let $\mathscr{C}_h$ be a k-regular triangulation of $\Omega_h$. Let $J_e$ be the Jacobian of the transformation $e = F_e(\hat{T})$, $e \in \mathscr{C}_h$. Let $J_e^{(l,m)}$ be a cofactor of $J_e$. Then*

$$D^\alpha J_e = O(h^{|\alpha|+n}), \tag{74}$$

$$D^\alpha J_e^{(l,m)} = O(h^{|\alpha|+n-1}), \tag{75}$$

$$D^\alpha\left(\frac{1}{J_e}\right) = O(h^{|\alpha|-n}). \tag{76}$$

Proof. Using the mathematical induction the following assertion can be proved: Let $D^\beta\varphi_s = O(h^{|\beta|+\varkappa_s}$ for $s = 1, \dots, r$, $|\beta| = 0, 1, \dots, |\alpha|$. Then

$$D^\alpha(\varphi_1\varphi_2 \cdots \varphi_r) = O(h^{|\alpha|+\varkappa_1+\dots+\varkappa_s}). \tag{77}$$

The relations (74) and (75) follow immediately from (77). The relation (76) can be easily proved by the mathematical induction.

**Lemma 6.** *Let* $\tau^*(\hat{x}, t) \in H^{k+1}(\hat{T})$, $\tau(x, t) \in H^{k+1}(e)$, $\forall t \in (0, T]$, $e \in \mathscr{C}_h$, *let* $\mathscr{C}_h$ *be a* k-*regular triangulation. Then there exists a constant c such that*

$$(78) \qquad |\tau^*|_{k+1,\hat{T}} \leqq ch^{-n/2+k+1} \|\tau\|_{k+1,e} .$$

Proof. Lemma is an immediate consequence of Lemma 1 from [1], p. 427.

We introduce the notation

$$(79) \qquad E(\Phi) = \sum_{e \in \mathscr{C}_h} E_e(\Phi) \quad (\text{for the definition of } E_e(\Phi) \text{ see } (14)) ,$$

$$(80) \qquad [\Phi]_{s,\hat{T}} = \sqrt{\left(\sum_{i=0}^{s} h^{-2i} |\Phi|^2_{i,\hat{T}}\right)} .$$

**Lemma 7.** *Let*

$$(81) \qquad \psi(\hat{x}) \in H^{k+2}(\hat{T}) ,$$

$$(82) \qquad \tau(\hat{x}) \textit{ be a polynomial of degree } \leqq k ,$$

$$\delta(\hat{x}) \in c^{k+1}(\hat{T}) \textit{ be a function such that },$$

$$(83) \qquad D^\alpha \delta = 0(h^{|\alpha|+\varkappa}) \quad \textit{for} \quad 0 \leqq |\alpha| \leqq k+1 , \quad \varkappa \ldots \textit{integer} .$$

*Let d be the order of a quadrature formula on the reference set $\hat{T}$. Then there exist constants $c_1$ and $c_2$ such that*

$$(84) \qquad |\hat{E}(\psi\tau\delta)| \leqq c_1 h^\varkappa \{h^{k+1} \|\tau\|_{0,\hat{T}} (h^{-(k+1)} |\psi|_{k+1,\hat{T}} + \|\psi\|_{k+1,\hat{T}}) + h^{d+1} [\tau]_{k,\hat{T}} (h^{-k} |\psi|_{k+1,\hat{T}} + [\psi]_{k,\hat{T}})\} ,$$

$$(85) \qquad \left|\hat{E}\left(\frac{\partial \psi}{\partial \hat{x}_i} \frac{\partial \tau}{\partial \hat{x}_j} \delta\right)\right| \leqq c_2 h^\varkappa \{h^{k+1} |\tau|_{1,\hat{T}} (h^{-(k+1)} |\psi|_{k+2,\hat{T}} + \sum_{j=0}^{k+1} |\psi|_{j+1,\hat{T}} + h^{d+2-k} |\tau|_{1,\hat{T}} (h^{-k} |\psi|_{k+2,\hat{T}} + h[\psi]_{k+1,\hat{T}})\} .$$

*Provided that*

$$(81') \qquad \psi(\hat{x}) \textit{ is a polynomial of degree } \leqq k ,$$

*the inequalities* (84) *and* (85) *reduce to*

$$(84') \qquad |\hat{E}(\psi\tau\delta)| \leqq c_1 h^\varkappa \{h^{k+1} \|\tau\|_{0,\hat{T}} \|\psi\|_{0,\hat{T}} + h^{d+1} [\tau]_{k,\hat{T}} [\psi]_{k,\hat{T}}\} ,$$

$$(85') \qquad \left|\hat{E}\left(\frac{\partial \psi}{\partial \hat{x}_i} \frac{\partial \tau}{\partial \hat{x}_j} \delta\right)\right| \leqq c_2 h^\varkappa \{h^{k+1} |\tau|_{1,\hat{T}} |\psi|_{1,\hat{T}} + h^{d+3-k} |\tau|_{1,\hat{T}} [\psi]_{k,\hat{T}}\} .$$

Proof. Obviously

$$(86)\qquad |\hat{E}(\psi\tau\delta)| \leqq |\hat{E}(\psi\tau(\delta - \hat{\pi}\delta))| + |\hat{E}(\psi\tau\hat{\pi}\delta)|\,,$$

where $\hat{\pi}\delta$ is the $\hat{T}$-interpolate of the function $\delta(\hat{x})$ (i.e. $\hat{\pi}\delta \in \hat{P}(k)$, $\hat{\pi}\,\delta(\hat{a}_i) = \delta(\hat{a}_i)$). The first Sobolev theorem (we suppose (66) to be true), the Bramble-Hilbert lemma (shortly the B.-H. lemma) and (83) imply

$$(87)\quad |\hat{E}(\psi\tau(\delta - \hat{\pi}\delta))| = \left|\int_{\hat{T}} \psi\tau(\delta - \hat{\pi}\delta)\,\mathrm{d}\hat{x} - \sum_r \hat{\omega}_r[\psi(\hat{b}_r)\,\tau(\hat{b}_r)\,(\delta(\hat{b}_r) - \hat{\pi}\,\delta(\hat{b}_r))]\right| \leqq$$
$$\leqq \|\psi\|_{0,\hat{T}}\,\|\delta - \hat{\pi}\delta\|_{0,\hat{T}} \max_{\hat{T}} |\tau| + c \max_{\hat{T}} |\tau| \sup_{\hat{T}} |\psi| \sup_{\hat{T}} |\delta - \hat{\pi}\delta| \leqq$$
$$\leqq c\|\tau\|_{0,\hat{T}}\,\|\psi\|_{k+1,\hat{T}}\,\|\hat{\delta} - \hat{\pi}\delta\|_{k+1,\hat{T}} \leqq c\|\tau\|_{0,\hat{T}}\|\psi\|_{k+1,\hat{T}}|\delta|_{k+1,\hat{T}} \leqq$$
$$\leqq ch^{\varkappa+k+1}\|\tau\|_{0,\hat{T}}\,\|\psi\|_{k+1,\hat{T}}\,.$$

Further,

$$(88)\qquad |\hat{E}(\psi\tau\hat{\pi}\delta)| \leqq |\hat{E}(\tau\hat{\pi}\delta(\psi - \hat{\pi}\psi))| + |\hat{E}(\tau\hat{\pi}\delta\hat{\pi}\psi)|\,.$$

In the same way as in (87) we get

$$(89)\qquad |\hat{E}(\tau\hat{\pi}\delta(\psi - \hat{\pi}\psi)) \leqq c\|\tau\|_{0,\hat{T}}\,\|\hat{\pi}\delta\|_{k+1,\hat{T}}\,|\psi|_{k+1,\hat{T}}\,.$$

From the B.-H. lemma and from (83) it follows that

$$|\hat{\pi}\delta|_{j,\hat{T}} \leqq |\hat{\pi}\delta - \delta|_{j,\hat{T}} + |\delta|_{j,\hat{T}} \leqq c|\delta|_{k+1,\hat{T}} \leqq c\{^{\varkappa+k+1} + h^{\varkappa+j}\}\,, \quad 0 \leqq j \leqq k+1\,.$$

Hence

$$(90)\qquad |\hat{\pi}\delta|_{j,\hat{T}} \leqq ch^{\varkappa+j}\,.$$

Substituting (90) into (89) we get

$$(91)\qquad |\hat{E}(\tau\hat{\pi}\delta(\psi - \hat{\pi}\psi))| \leqq ch^{\varkappa}\|\tau\|_{0,\hat{T}}\,|\psi|_{k+1,\hat{T}}\,.$$

From the *B.-H.* lemma, from (73), (90) and (70) we conclude

$$(92)\qquad |\hat{E}(\tau\hat{\pi}\delta\hat{\pi}\psi)|^2 \leqq c|\tau\hat{\pi}\delta\hat{\pi}\psi|^2_{d+1,\hat{T}} \leqq$$
$$\leqq c\sum_{j=0}^{d+1} |\tau\hat{\pi}\psi|^2_{j,\hat{T}}\,|\hat{\pi}\delta|^2_{d+1-j,\hat{T}} \leqq ch^{2(\varkappa+d+1)}\sum_{j=0}^{2k} h^{-2j}\sum_{i=0}^{j} |\tau|^2_{i,\hat{T}}\,|\hat{\pi}\psi|^2_{j-i,\hat{T}}\,.$$

It is easy to verify (we use the B.-H. lemma and the notation introduced in (80)) that

$$\sum_{j=0}^{2k} h^{-2j}\sum_{i=0}^{j} |\tau|^2_{i,\hat{T}}\,|\hat{\pi}\psi|^2_{j-i,\hat{T}} = \left(\sum_{j=0}^{k} h^{-2i}|\tau|^2_{i,\hat{T}}\right)\left(\sum_{j=0}^{k} h^{-2j}|\hat{\pi}\psi|^2_{j,\hat{T}}\right) \leqq$$
$$\leqq c[\tau]^2_{k,\hat{T}}\left\{\sum_{j=0}^{k} h^{-2j}(|\psi|^2_{k+1,\hat{T}} + |\psi|^2_{j,\hat{T}})\right\}$$
$$\leqq c[\tau]^2_{k,\hat{T}}\left\{h^{-2k}|\psi|^2_{k+1,\hat{T}} + [\psi]^2_{k,\hat{T}}\right\}.$$

Substituting this inequality into (92) we get

$$|\hat{E}(\tau\hat{\pi}\delta\hat{\pi}\psi)| \leqq ch^{\varkappa+d+1}[\tau]_{k,\hat{T}}\{h^{-k}|\psi|_{k+1,\hat{T}} + [\psi]_{k,\hat{T}}\}\ . \tag{93}$$

From (86), (87), (88), (91) and (93) we obtain (84). The inequality (84′) is an immediate consequence of the inequalities (84) and (72). We prove now the inequality (85). From (84) it follows that

$$\left|\hat{E}\left(\frac{\partial\psi}{\partial\hat{x}_i}\frac{\partial\tau}{\partial\hat{x}_j}\delta\right)\right| \leqq ch^{\varkappa}\left\{h^{k+1}\left\|\frac{\partial\tau}{\partial\hat{x}_j}\right\|_{0,\hat{T}}\left(h^{-(k+1)}\left|\frac{\partial\psi}{\partial\hat{x}_i}\right|_{k+1,\hat{T}} + \left\|\frac{\partial\psi}{\partial\hat{x}_i}\right\|_{k+1,\hat{T}}\right) + \right.$$
$$\left. + h^{d+1}\left[\frac{\partial\tau}{\partial\hat{x}_j}\right]_{k,\hat{T}}\left(h^{-k}\left|\frac{\partial\psi}{\partial\hat{x}_i}\right|_{k+1,\hat{T}} + \left[\frac{\partial\psi}{\partial\hat{x}_i}\right]_{k,\hat{T}}\right)\right\}, \tag{94}$$

while (80), (82) and (72) imply

$$\left[\frac{\partial\tau}{\partial\hat{x}_j}\right]^2_{k,\hat{T}} = \sum_{i=0}^{k} h^{-2i}\left|\frac{\partial\tau}{\partial\hat{x}_j}\right|^2_{i,\hat{T}} \leqq \sum_{i=0}^{k} h^{-2i}|\tau|^2_{i+1,\hat{T}} =$$
$$= \sum_{i=0}^{k-1} h^{2i}|\tau|^2_{i+1,\hat{T}} \leqq c\sum_{i=0}^{k-1} h^{-2i}|\tau|^2_{1,\hat{T}}\ .$$

Hence

$$\left[\frac{\partial\tau}{\partial\hat{x}_j}\right]_{k,\hat{T}} \leqq ch^{-k+1}|\tau|_{1,\hat{T}}\ . \tag{95}$$

In the same way we get

$$\left[\frac{\partial\psi}{\partial\hat{x}_i}\right]_{k,\hat{T}} \leqq h[\psi]_{k+1,\hat{T}}\ . \tag{96}$$

Substituting from here and from (95) into (94) we get (85). The inequality (85′) is an immediate consequence of (85).

We can now formulate the results concerning the isoparametric integration. We remember that $\mathscr{C}_h$ is supposed to be a $k$-regular triangulation of $\Omega_h$. In the sequel we assume that (66) is satisfied.

**Theorem 2.** *Let $w(x, t) \in H^{k+1}(\Omega_h)$, $v(x, t) \in V_h(\Omega_h)$, $\forall t \in (0, T]$. Let the quadrature formula given on the reference set $\hat{T}$ be of a degree*

$$d \geqq 2k - 1\ . \tag{97}$$

*Then there exists a constant $c$ such that*

$$|E(wv)| \leqq ch^{k+1}\|w\|_{k+1,\Omega_h}\|v\|_{1,\Omega_h}\ . \tag{98}$$

Proof. (79) and (15) imply

$$E(wv) = \sum_{e \in \mathscr{C}_h} \hat{E}(v^* w^* J_e) . \tag{99}$$

It is easy to verify that all the assumptions of Lemma 7 are satisfied for $\varkappa = n$ (this follows from (74)). Hence, the inequality (84) may be applied. When substituting the inequalities (which can be easily derived from (78) and from the relation $[v^*]_{k,\hat{T}}^2 = |v^*|_{0,\hat{T}}^2 + \sum_{i=1}^{k} h^{-2i} |v^*|_{i,\hat{T}}^2$):

$$\|v^*\|_{0,\hat{T}} \leqq ch^{-n/2} \|v\|_{0,e} , \tag{100}$$

$$|w^*|_{k+1,\hat{T}} \leqq ch^{-n/2 + k+1} \|w\|_{k+1,e} , \tag{101}$$

$$\|w^*\|_{k+1,\hat{T}} \leqq ch^{-n/2} \|w\|_{k+1,e} , \tag{102}$$

$$[v^*]_{k,\hat{T}} \leqq ch^{-n/2-k+1} \|v\|_{1,e} , \tag{103}$$

$$[w^*]_{k,\hat{T}} \leqq ch^{-n/2} \|w\|_{k,e} \tag{104}$$

into (84) and applying (97) then we conclude by simple calculation

$$|\hat{E}(w^* v^* J_e)| \leqq ch^{k+1} \|v\|_{1,e} \, \|w\|_{k+1,e} .$$

From here and from the Schwarz inequality (98) follows.

**Theorem 3.** *Let $b(x) \in C^{k+1}(\Omega_h)$, let $u(x, t) \in H^{k+3}(\Omega)$ $\forall t \in (0, T]$ be a solution of the problem (20) and $\tilde{u}(x, t)$ be an extension of the function $u(x, t)$ satisfying the condition (23). Let $\eta(x, t) \in V_h(\Omega_h)$ be the Ritz approximation of $\tilde{u}(x, t)$. Let $v \in V_h$ and let the quadrature formula satisfy (97).*

*Then there exists a constant $c$ such that*

$$|E(b\eta v)| \leqq ch^{k+1} \|u\|_{k+3,\Omega} \|v\|_{1,\Omega_h} . \tag{105}$$

Proof. In the same way as in (99) we get

$$E(b\eta v) = \sum_{e \in \mathscr{C}_h} \hat{E}(\eta^* v^* b^* J_e) . \tag{106}$$

Evidently $D^\alpha b^* = 0(h^{|\alpha|})$ and $D^\alpha J_e = 0(h^{n+|\alpha}{}_{|})$. (77) implies $D^\alpha(b^* J_e) = 0(h^{n+|\alpha|})$. Hence, all the assumptions of Lemma 7 for $\varkappa = n$ are satisfied. From (84′) it follows that

$$|\hat{E}(b^* \eta^* v^* J_e)| \leqq ch^2 \{ h^{k+1} \|v^*\|_{0,\hat{T}} \, \|\eta^*\|_{0,\hat{T}} + h^{d+1} [v^*]_{k,\hat{T}} \, [\eta^*]_{k,\hat{T}} \} \tag{107}$$

while (78) implies

$$\|v^*\|_{0,\hat{T}} \leqq ch^{-n/2} \|v\|_{0,e} , \quad \|\eta^*\|_{0,\hat{T}} \leqq ch^{-n/2} \{ \|\eta - \tilde{u}\|_{0,e} + \|\tilde{u}\|_{0,e} \} . \tag{108}$$

It can be easily verified that

$$(109) \qquad [\eta^*]_{k,T} \leqq c\{[\eta^* - (\pi_e\tilde{u})^*]_{k,T} + [(\pi_e\tilde{u})^* - \tilde{u}^*]_{k,T} + [\tilde{u}^*]_{k,T}\} .$$

From (80), (72) and from the interpolation theorem (see (6)) we get

$$[\eta^* - (\pi_e\tilde{u})^*]_{k,T}^2 \leqq ch^{-2k}\|\eta^* - (\pi_e\tilde{u})^*\|_{0,T}^2 \leqq$$
$$\leqq ch^{-2k}\{\|\eta^* - \tilde{u}^*\|_{0,T}^2 + \|\tilde{u}^* - (\pi_e\tilde{u})^*\|_{0,T}^2\} \leqq$$
$$\leqq ch^{-2k-n}\{\|\eta - \tilde{u}\|_{0,e}^2 + h^{2(k+1)}\|\tilde{u}\|_{k+1,e}^2\} .$$

Hence

$$(110) \qquad [\eta^* - (\pi_e\tilde{u})^*]_{k,T} \leqq ch^{-n/2-k}\{\|\eta - \tilde{u}\|_{0,e} + h^{k+1}\|\tilde{u}\|_{k+1,e}\} .$$

Analogously we get

$$(111) \qquad [(\pi_e\tilde{u})^* - \tilde{u}^*]_{k,T} \leqq c\sqrt{\left(\sum_{i=0}^{k} h^{-n}\|\pi_e\tilde{u} - \tilde{u}\|_{i,e}^2\right)} \leqq ch^{-n/2+1}\|\tilde{u}\|_{k+1,e} ,$$

$$(112) \qquad [\tilde{u}]_{k,T} \leqq ch^{-n/2}\|\tilde{u}\|_{k,e} .$$

If we substitute from (110), (111) and (112) into (109) and use elementary calculation then we obtain

$$(113) \qquad [\eta^*]_{k,T} \leqq ch^{-n/2}\{h^{-k}\|\eta - \tilde{u}\|_{0,e} + \|\tilde{u}\|_{k+1,e}\} .$$

Substituting from (108), (103) and (113) into (107) and using (97) we get

$$(114) \qquad |\hat{E}(\eta^* v^* b^* J_e)| \leqq ch^{k+1}\|v\|_{1,e}\{h^{-k}\|\eta - \tilde{u}\|_{0,e} + \|\tilde{u}\|_{k+1,e}\} ,$$

which together with (106), the Schwarz inequality, Theorem on Ritz approximation (see (68)) and (67) yields

$$|E(b\eta v)| \leqq ch^{k+1}\|v\|_{1,\Omega_h}\{h^{-k}\|\eta - \tilde{u}\|_{0,\Omega_h} + \|\tilde{u}\|_{k+1,\Omega_h}\} \leqq$$
$$\leqq ch^{k+1}\|v\|_{1,\Omega_h}\{h\|u\|_{k+3,\Omega} + \|u\|_{k+3,\Omega}\}$$

and the inequality (105) is proved.

**Theorem 4.** *Let all the assumptions of Theorem* 3 *be satisfied. Then there exists a constant c such that*

$$(115) \qquad \left|E\left(b\frac{\partial\eta}{\partial x_i}\frac{\partial v}{\partial x_j}\right)\right| \leqq ch^{k+1}\|u\|_{k+3,\Omega}\,\|v\|_{1,\Omega_h} .$$

Proof. Analogously as in (106) the following relation is true

$$(116) \qquad E\left(b\frac{\partial\eta}{\partial x_i}\frac{\partial v}{\partial x_j}\right) = \sum_{e\in\mathscr{C}_h}\hat{E}\left(b^*\left(\frac{\partial\eta}{\partial x_i}\right)^*\left(\frac{\partial v}{\partial x_j}\right)^* J_e\right).$$

From the rule on the composed function derivative it follows that

$$(117) \qquad \hat{E}\left(b^*\left(\frac{\partial\eta}{\partial x_i}\right)^*\left(\frac{\partial v}{\partial x_j}\right)^* J_e\right) = \sum_{l,m=1}^{n} \hat{E}\left(\frac{\partial\eta^*}{\partial\hat{x}_l}\frac{\partial v^*}{\partial\hat{x}_m} b^* \frac{J_e^{(l,i)} J_e^{(m,j)}}{J_e}\right)$$

(75), (76) and (77) implies

$$D^\alpha(b^*(J_e^{(l,i)} J_e^{(m,j)})/J_e) = 0(h^{|\alpha|+n-2}).$$

We may apply Lemma 7 (with $\varkappa = n - 2$) to the expressions on the right hand side of (116). From (85′) we obtain in the same way as in (114)

$$(118) \quad \left|\hat{E}\left(\frac{\partial\eta^*}{\partial\hat{x}_i}\frac{\partial v^*}{\partial\hat{x}_m} b \frac{J_e^{(l,i)} J_e^{(m,j)}}{J_e}\right)\right| \leqq ch^{k+1}\|v\|_{1,e}\{h^{-k}\|\eta - \tilde{u}\|_{1,e} + \|\tilde{u}\|_{k+1,e}\}.$$

From here, from (117), (116), from the Schwarz inequality, from Theorem on Ritz approximation and from (67) the inequality (115) follows.

Before proving the next theorem, we formulate Lemma 8 which is an obvious consequence of Theorem 3 from [1], p. 436.

**Lemma 8.** *If the quadrature formula on the reference set $\hat{T}$ is of a degree $d \geqq 2k - 2$ then there exists a constant $c$ such that*

$$(119) \qquad \sum_r \omega_{r,e} \sum_{i=1}^{n}\left(\frac{\partial p}{\partial x_i}(b_{r,e})\right)^2 \geqq c|p|_{1,e}^2, \quad \forall p \in P_e, \quad e \in \mathscr{C}_h$$

*($P_e$ is defined in (3)).*

We introduce the notation

$$(120) \qquad |v|_h^2 = (g(x)\, v, v)_h, \quad \|v\|_h^2 = a_h(v, v)$$

where the forms $(\cdot, \cdot)_h$, $a_h(\cdot, \cdot)$ are defined in (31) and (32).

**Theorem 5.** *There exist constants $c_1$ and $c_2$ such that*

$$(121) \qquad \text{a)} \quad c_1\|v\|_{0,\Omega_h} \leqq |v|_h, \quad \forall v \in V_h$$

*provided the quadrature formula on the reference set $\hat{T}$ is of a degree $d \geqq 2k$.*

$$(122) \qquad \text{b)} \quad c_2|v|_{1,\Omega_h} \leqq \|v\|_h, \quad \forall v \in V_h$$

*provided the quadrature formula on the reference set $\hat{T}$ is of a degree $d \geqq 2k - 2$.*

Proof. From (31), (16), (12) and from the fact that $\hat{v}^2$ is a polynomial of degree $\leqq 2k$ on $\hat{T}$ we obtain

$$|v|_h^2 \geqq g_0 \sum_{e\in\mathscr{C}_h} \min_{\hat{T}} J_e \sum_r \hat{\omega}_r (v^*)^2 (\hat{b}_r) = g_0 \sum_{e\in\mathscr{C}_h} \min_{\hat{T}} J_e \int_{\hat{T}} (v^*)^2\, d\hat{x} \geqq$$

$$\geqq g_0 \sum_{e\in\mathscr{C}_h} \frac{\min\limits_{\hat{T}} J_e}{\max\limits_{\hat{T}} J_e} \int_e (v^*)^2 J_e \, d\hat{x} = g_0 \sum_{e\in\mathscr{C}_h} \frac{\min\limits_{\hat{T}} J_e}{\max\limits_{\hat{T}} J_e} \int_e v^2 \, dx .$$

From (2) it follows $\min\limits_{\hat{T}} J_e / \max\limits_{\hat{T}} J_e \geqq 1/c_0^2$. Hence

$$|v|_h^2 \geqq \frac{g_0}{c_0^2} \sum_{e\in\mathscr{C}_h} \int_e v^2 \, dx = \frac{g_0}{c_0^2} \|v\|_{0,\Omega_h}^2$$

and (121) is proved.

From (32), (18) and (12) it follows that

$$\|v\|_h^2 \geqq g_1 \sum_{e\in\mathscr{C}_h} \sum_r \omega_{r,e} \sum_{i=1}^n \left[\frac{\partial v}{\partial x_i}(b^s{}_{,e})\right]^2 .$$

This together with (119) proves the inequality (122).

## 5. ERROR ESTIMATES IN ONE AND TWO-STEPS $A$-STABLE METHODS

Let us recall the notation introduced in (28). As we said in (34) the aim of our paper is to derive error bounds for

(123) $$\|\lambda\|_{0,\Omega\cap\Omega_h} = \|u^j - u_h^j\|_{0,\Omega\cap\Omega_h}, \quad 1 \leqq j \leqq \frac{T}{\Delta t},$$

where $u$ and $u_h$ are solutions of the problems (20) and (33) respectively.

Let

(124) $$\tilde{u}^j = \eta^j + \xi^j$$

where $\eta^j = \eta(x, j\,\Delta t)$ is the Ritz approximation of the function $\tilde{u}^j = \tilde{u}(x, j\,\Delta t)$ (we remind that $\tilde{u}$ is an extension of $u$ satisfying the inequality (23)).

From Theorem on Ritz approximation we get

(125) $$\|\xi^j\|_{0,\Omega_h} = \|\tilde{u}^j - \eta^j\|_{0,\Omega_h} \leqq ch^k \|u\|_{k+3,\Omega} .$$

It is evident that

(126) $$\|u^j - u_h^j\|_{0,\Omega\cap\Omega_h} \leqq \|\tilde{u}^j - u_h^j\|_{0,\Omega_h} \leqq \|\xi^j\|_{0,\Omega_h} + \|\eta^j - u_h^j\|_{0,\Omega_h} .$$

Hence it is sufficient to estimate error-bounds for

(127) $$\varepsilon^j = \eta^j - u_h^j .$$

From (35) we get

(128) $$\tilde{a}(\eta^m, v) = (\tilde{f}^m, v)_{0,\Omega_h} - (\tilde{g}\tilde{u}_t^m, v)_{0,\Omega_h}, \quad \forall v \in V_h .$$

Hence

$$\Delta t\ \tilde{a}(\sum_{j=0}^{\nu} \beta_j \eta^{m+j}, v) = \Delta t(\sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}, v)_{0,\Omega_h} - \Delta t(\tilde{g} \sum_{j=0}^{\nu} \beta_j \tilde{u}_t^{m+j}, v)_{0,\Omega_h}.$$

If we add $(\tilde{g} \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}, v)_{0,\Omega_h}$ to both sides of this identity and apply (124) then we get

$$(129) \qquad (\tilde{g} \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}, v)_{0,\Omega_h} + \Delta t\ \tilde{a}(\sum_{j=0}^{\nu} \beta_j \eta^{m+j}, v) =$$

$$= \Delta t(\sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}, v)_{0,\Omega_h} + (\pi_\nu^m - \omega_\nu^m, v)_{0,\Omega_h}$$

where

$$(130) \qquad \pi_\nu^m = \tilde{g} \sum_{j=0}^{\nu} (\alpha_j \tilde{u}^{m+j} - \Delta t\ \beta_j \tilde{u}_t^{m+j}), \quad \omega_\nu^m = \tilde{g} \sum_{j=0}^{\nu} \alpha_j \xi^{m+j}.$$

From (31), (32) and (79) it follows that

$$(131) \quad (z, v)_{0,\Omega_h} - (z, v)_h = E(zv), \quad \tilde{a}(z, v) - a_h(z, v) = E\left(\sum_{i,j=1}^{n} \tilde{g}_{ij} \frac{\partial z}{\partial x_i} \frac{\partial v}{\partial x_j}\right).$$

From here and from (129) we get

$$(132) \qquad (g \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}, v)_h + E(\tilde{g} v \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}) + \Delta t\ a_h(\sum_{j=0}^{\nu} \beta_j \eta^{m+j}, v) +$$

$$+ \Delta t\ E\left(\sum_{i,j=1}^{n} \tilde{g}_{ij} \frac{\partial v}{\partial x_j} \left(\sum_{l=0}^{\nu} \beta_l \frac{\partial \eta^{m+l}}{\partial x_i}\right)\right) =$$

$$= \Delta t(\sum_{j=0}^{\nu} \beta_j f^{m+j}, v)_h + \Delta t\ E(v \sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}) + (\pi_\nu^m - \omega_\nu^n, v)_{0,\Omega_h}.$$

If we subtract (33) from (132) and take (127) into account then we come to the identity

$$(133) \qquad (g \sum_{j=0}^{\nu} \alpha_j \varepsilon^{m+j}, v)_h + \Delta t\ a_h(\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}, v) =$$

$$= (\pi_\nu^m - \omega_\nu^m, v)_{0,\Omega_h} + \Delta t\ E(v \sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}) - E(\tilde{g} v \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}) -$$

$$- \Delta t\ E\left(\sum_{i,j=1}^{n} \tilde{g}_{ij} \frac{\partial v}{\partial x_j} \sum_{l=0}^{\nu} \beta_l \frac{\partial \eta^{m+l}}{\partial x_i}\right), \quad \forall v \in V_h.$$

Let us denote

$$(134) \qquad A_\nu^m = (g \sum_{j=0}^{\nu} \alpha_j \varepsilon^{m+j}, \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j})_h,$$

$$(135)\qquad B_\nu^m = a_h\Big(\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j},\ \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}\Big)\,,$$

$$(136)\qquad D_\nu^m = \Big(\pi_\nu^m - \omega_\nu^m,\ \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}\Big)_{0,\Omega_h}\,,$$

$$(137)\qquad F_\nu^m = E\Big(\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+j}\Big)\,,$$

$$(138)\qquad G_\nu^m = E\Big(\tilde{g} \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \sum_{j=0}^{\nu} \alpha_j \eta^{m+j}\Big)\,,$$

$$(139)\qquad H_\nu^m = E\left(\sum_{i,j=1}^{n} \tilde{g}_{ij}\left[\sum_{l=0}^{\nu} \beta_l \frac{\partial \varepsilon^{m+l}}{\partial x_j} \sum_{l=0}^{\nu} \beta_l \frac{\partial \eta^{m+l}}{\partial x_i}\right]\right),$$

$$(140)\qquad Q_\nu^m = \Delta t\, F_\nu^m - G_\nu^m - \Delta t\, H_\nu^m\,.$$

The identity (133) is true for all $v \in V_h$. Hence it is also true for

$$(141)\qquad v = \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}\,.$$

From this and from (134)–(140) we get for any $s$ such that $s\,\Delta t \leqq T$, $s \geqq \nu$ the following basic identity:

$$(142)\qquad \sum_{m=0}^{s-\nu} A_\nu^m + \Delta t \sum_{m=0}^{s-\nu} B_\nu^m = \sum_{m=0}^{s-\nu} D_\nu^m + \sum_{m=0}^{s-\nu} Q_\nu^m\,.$$

Now, we estimate the expressions in (142) when one and two-step $A$-stable methods are used. From [11] and [12] it follows:

a) for one-step $A$-stable methods

(143) $\nu = 1,\ \alpha_1 = 1,\ \alpha_0 = -1,\ \beta_1 = 1 - \theta,\ \beta_0 = \theta,\ \theta \leqq \frac{1}{2}$ is any real number.

If $\theta = \frac{1}{2}$ then the method is of the order $q = 2$, in all other cases the method is of the order $q = 1$.

b) for two-step $A$-stable methods

$$(144)\qquad \nu = 2\,,\quad \alpha_2 = \theta\,,\quad \alpha_1 = 1 - 2\theta\,,\quad \alpha_0 = -1 + \theta\,,\quad \beta_2 = \tfrac{1}{2}\theta + \delta\,,$$
$$\beta_1 = \tfrac{1}{2} - 2\delta\,,\quad \beta_0 = \tfrac{1}{2} - \tfrac{1}{2}\theta + \delta\,,\quad \theta \geqq \tfrac{1}{2}\,,\quad \delta > 0\,.$$

From (134), (143) and (120) it follows that

$$\begin{aligned}
A_1^m &= \big(g(\varepsilon^{m+1} - \varepsilon^m),\ (1-\theta)\,\varepsilon^{m+1} + \theta\varepsilon^m\big)_h = \\
&= (1-\theta)\,|\varepsilon^{m+1}|_h^2 - (1-2\theta)\,(g\varepsilon^{m+1}, \varepsilon^m)_h - \theta|\varepsilon^m|_h^2 \geqq \\
&\geqq (1-\theta)\,|\varepsilon^{m+1}|_h^2 - \tfrac{1}{2}(1-2\theta)\,|\varepsilon^{m+1}|_h^2 - \tfrac{1}{2}(1-2\theta)\,|\varepsilon^m|_h^2 - \theta|\varepsilon^m|_h^2 = \\
&= \tfrac{1}{2}|\varepsilon^{m+1}|_h^2 - \tfrac{1}{2}|\varepsilon^m|_h^2\,.
\end{aligned}$$

Hence

$$\sum_{m=0}^{s-1} A_1^m \geqq \tfrac{1}{2}|\varepsilon^s|_h^2 - \tfrac{1}{2}|\varepsilon^0|_h^2 .$$

From here and from (121) we have

$$\sum_{m=0}^{s-1} A_1^m \geqq c\|\varepsilon^s\|_{0,\Omega_h}^2 - \tfrac{1}{2}|\varepsilon^0|_h^2 . \tag{145}$$

For $v = 2$ we find out by simple calculation that (see [11])

$$\begin{aligned} A_2^m &= \big(g(\theta\varepsilon^{m+2} + (1-2\theta)\,\varepsilon^{m+1} + (-1+\theta)\,\varepsilon^m),\ (\tfrac{1}{2}\theta + \delta)\,\varepsilon^{m+2} + \\ &\quad + (\tfrac{1}{2} - 2\delta)\,\varepsilon^{m+1} + (\tfrac{1}{2} - \tfrac{1}{2}\theta + \delta)\,\varepsilon^m\big)_h = \\ &= \tfrac{1}{2}(\theta^2 + \delta)\,|\varepsilon^{m+2}|_h^2 - (\theta - \tfrac{1}{2})\,|\varepsilon^{m+1}|_h^2 - \tfrac{1}{2}[(\theta-1)^2 + \delta]\,|\varepsilon^m|_h^2 - \\ &\quad - [\theta(\theta-1) + \delta]\,[(g\varepsilon^{m+2}, \varepsilon^{m+1})_h - (g\varepsilon^{m+1}, \varepsilon^m)_h] + \\ &\quad + \delta(\theta - \tfrac{1}{2})\,|\varepsilon^{m+2} - 2\varepsilon^{m+1} + \varepsilon^m|_h^2 . \end{aligned}$$

Hence

$$\begin{aligned} \sum_{m=0}^{s-2} A_2^m &\geqq \sum_{m=0}^{s-2} \{\tfrac{1}{2}(\theta^2 + \delta)\,|\varepsilon^{m+2}|_h^2 - (\theta - \tfrac{1}{2})\,|\varepsilon^{m+1}|_h^2 - \\ &\quad - \tfrac{1}{2}[(\theta-1)^2 + \delta]\,|\varepsilon^m|_h^2 - \\ &\quad - [\theta(\theta-1) + \delta]\,(g\varepsilon^{m+2}, \varepsilon^{m+1})_h + [\theta(\theta-1) + \delta]\,(g\varepsilon^{m+1}, \varepsilon^m)_h\} = \\ &= \sum_{m=2}^{s-2} \{\tfrac{1}{2}(\theta^2 + \delta) - (\theta - \tfrac{1}{2}) - \tfrac{1}{2}[(\theta-1)^2 + \delta]\}\,|\varepsilon^m|_h^2 - \\ &\quad - \sum_{m=1}^{s-2} \{[\theta(\theta-1) + \delta] - [\theta(\theta-1) + \delta]\}\,(g\varepsilon^{m+1}, \varepsilon^m)_h + \\ &\quad + \tfrac{1}{2}(\theta^2 + \delta)\,[|\varepsilon^{s-1}|_h^2 + |\varepsilon^s|_h^2] - (\theta - \tfrac{1}{2})\,[|\varepsilon^{s-1}|_h^2 + |\varepsilon^1|_h^2] - \\ &\quad - \tfrac{1}{2}[(\theta-1)^2 + \delta]\,[|\varepsilon^0|_h^2 + |\varepsilon^1|_h^2] - [\theta(\theta-1) + \delta]\,(g\varepsilon^s, \varepsilon^{s-1})_h + \\ &\quad + [\theta(\theta-1) + \delta]\,(g\varepsilon^1, \varepsilon^0)_h \geqq M - c_1[|\varepsilon^0|_h^2 + |\varepsilon^1|_h^2] \end{aligned} \tag{146}$$

where

$$\begin{aligned} M &= \tfrac{1}{2}(\theta^2 + \delta)\,|\varepsilon^s|_h^2 + \tfrac{1}{2}[(\theta-1)^2 + \delta]\,|\varepsilon^{s-1}|_h^2 - \\ &\quad - |(\sqrt{(g)}\,\varepsilon^s, \sqrt{(g)}\,[\theta(\theta-1) + \delta]\,\varepsilon^{s-1})_h| . \end{aligned} \tag{147}$$

Let us suppose first $\theta(\theta - 1) + \delta = 0$. Then (147) implies

$$M \geqq \tfrac{1}{2}(\theta^2 + \delta)\,|\varepsilon^s|_h^2 . \tag{148}$$

Let $\theta(\theta - 1) + \delta \neq 0$ now. Then using the inequality $|ab| \leqq \frac{1}{2}\tau a^2 + b^2/2\tau$ for $\tau = [\theta(\theta - 1) + \delta]^2/[(\theta - 1)^2 + \delta]$ we get

$$M \geqq \frac{1}{2}\left\{\theta^2 + \delta - \frac{[\theta(\theta - 1) + \delta]^2}{(\theta - 1)^2 + \delta}\right\} |\varepsilon^s|_h^2 =$$

$$= \frac{1}{2}(\theta^2 + \delta)\left\{1 - \frac{[\theta(\theta - 1) + \delta]^2}{[\theta(\theta - 1) + \delta]^2 + \delta}\right\} |\varepsilon^s|_h^2 .$$

Hence the inequality (148) is again true.

From (148), (146) and (121) we get

$$\sum_{m=0}^{s-2} A_2^m \geqq c_2\|\varepsilon^s\|_{0,\Omega_h}^2 - c_1[|\varepsilon^0|_h^2 + |\varepsilon^1|_h^2] . \tag{149}$$

Both the inequalities (145) and (149) can be written in the same way as

$$\sum_{m=0}^{s-\nu} A_\nu^m \geqq c_2\|\varepsilon^s\|_{0,\Omega_h}^2 - c_1[|\varepsilon^0|_h^2 + |\varepsilon^{\nu-1}|_h^2] , \quad \nu = 1, 2 . \tag{150}$$

From (135) and (120) it follows that

$$B_\nu^m = \|\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}\|_h^2 .$$

From here and from (122) we get

$$\sum_{m=0}^{s-\nu} B_\nu^m \geqq c_3 \sum_{m=0}^{s-\nu} |\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}|_{1,\Omega_h}^2 , \quad \nu = 1, 2 . \tag{151}$$

Substituting (150) and (151) into (142) we get

$$c_2\|\varepsilon^s\|_{0,\Omega_h}^2 + c_3\, \Delta t \sum_{m=0}^{s-\nu} |\sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j}|_{1,\Omega_h}^2 \leqq \tag{152}$$

$$\leqq \sum_{m=0}^{s-\nu} |D_\nu^m| + \sum_{m=0}^{s-\nu} |Q_\nu^m| + c_1[|\varepsilon^0|_h^2 + |\varepsilon^{\nu-1}|_h^2] , \quad \nu = 1, 2 .$$

From (136) and (130) it follows that

$$|D_1^m| \leqq \|\pi_1^m - \omega_1^m\|_{0,\Omega_h} \|\sum_{j=0}^{1} \beta_j \varepsilon^{m+j}\|_{0,\Omega_h} \leqq \tag{153}$$

$$\leqq [\|\pi_1^m\|_{0,\Omega_h} + \|\omega_1^m\|_{0,\Omega_h}] \|\sum_{j=0}^{1} \beta_j \varepsilon^{m+j}\|_{0,\Omega_h}$$

where

$$\pi_1^m = \tilde{g}[\tilde{u}^{m+1} - \tilde{u}^m - \Delta t((1 - \theta)\, \tilde{u}_t^{m+1} + \theta \tilde{u}_t^m)] , \tag{154}$$

$$\omega_1^m = \tilde{g}(\xi^{m+1} - \xi^m) .$$

In the sequel we suppose additionally that

$$(155)\qquad \frac{\partial^l u(x, t)}{\partial t^l} \in L^\infty(H^{k+3}(\Omega)), \quad l = 1, \ldots, q + 1$$

where $q$ is the order of the method and $u(x, t)$ is a solution of (20). From the Calderon theorem and from (154) we get

$$(156)\qquad \|\pi_1^m\|_{0,\Omega_h} \leqq c \left\| u^{m+1} - u^m - \Delta t\left((1 - \theta)\frac{\partial u^{m+1}}{\partial t} + \theta \frac{\partial u^m}{\partial t}\right)\right\|_{k+3,\Omega}.$$

a) Let $q = 1$ (i.e. $\theta \neq \frac{1}{2}$). Then the Taylor theorem yields

$$u^{m+1} - u^m - \Delta t\left((1 - \theta)\frac{\partial u^{m+1}}{\partial t} + \theta \frac{\partial u^m}{\partial t}\right) = \frac{1}{2}\Delta t^2 \frac{\partial^2 u^{\varkappa_1}}{\partial t^2} - (1 - \theta)\Delta t^2 \frac{\partial^2 u^{\varkappa_2}}{\partial t^2}.$$

Hence

$$(157)\qquad \|\pi_1^m\|_{0,\Omega_h} \leqq c\,\Delta t^2.$$

b) Let $q = 2$ (i.e. $\theta = \frac{1}{2}$). Then the Taylor theorem yields

$$u^{m+1} - u^m - \Delta t\left((1 - \theta)\frac{\partial u^{m+1}}{\partial t} + \theta \frac{\partial u^m}{\partial t}\right) = \frac{1}{6}\Delta t^3 \frac{\partial^3 u^{\varkappa_3}}{\partial t^3} - \frac{1}{4}\Delta t^3 \frac{\partial^3 u^{\varkappa_4}}{\partial t^3}.$$

Hence

$$(158)\qquad \|\pi_1^m\|_{0,\Omega_h} \leqq c\,\Delta t^3.$$

From (157) and (158) we conclude

$$(159)\qquad \|\pi_1^m\|_{0,\Omega_h} \leqq c\Delta t^{q+1}.$$

From (154) and (124) it follows that

$$\omega_1^m = \tilde{g}[\tilde{u}^{m+1} - \tilde{u}^m - (\eta^{m+1} - \eta\|)].$$

Obviously the function $\eta^{m+1} - \eta^m$ is the Ritz approximation of $\tilde{u}^{m+1} - \tilde{u}^m$. Theorem on Ritz approximation yields

$$(160)\qquad \|\omega_1^m\|_{0,\Omega_h} \leqq ch^{k+1}\|u^{m+1} - u^m\|_{k+3,\Omega} = ch^{k+1}\,\Delta t \left\|\frac{\partial u^{\varkappa_5}}{\partial t}\right\|_{k+3,\Omega}.$$

Hence

$$(161)\qquad \|\omega_1^m\|_{0,\Omega_h} \leqq ch^{k+1}\,\Delta t.$$

(153), (159) and (161) imply

$$|D_1^m| \leqq c\,\Delta t(\Delta t^q + h^{k+1})\,\Big\|\sum_{j=0}^{1}\beta_j \varepsilon^{m+j}\Big\|_{0,\Omega_h}.$$

In the same way an analogous inequality may be derived for $\nu = 2$. Hence

$$\sum_{m=0}^{s-\nu} \left|D_\nu^m\right| \leqq c\,\Delta t(\Delta t^q + h^{k+1}) \sum_{m=0}^{s-\nu} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{0,\Omega_h}, \quad \nu = 1, 2\,. \tag{162}$$

From (140) we have

$$\sum_{m=0}^{s-\nu} \left|Q_\nu^m\right| \leqq \Delta t \sum_{m=0}^{s-\nu} \left|F_\nu^m\right| + \sum_{m=0}^{s-\nu} \left|G_\nu^m\right| + \Delta t \sum_{m=0}^{s-\nu} \left|H_\nu^m\right|. \tag{163}$$

From (137) and (98) (supposing (97)) we obtain

$$\left|F_\nu^m\right| \leqq c h^{k+1} \Big\| \sum_{j=0}^{\nu} \beta_j \tilde{f}^{m+1} \Big\|_{k+1,\Omega_h} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{1,\Omega_h}.$$

From here, from (24), (23) and (22) we get

$$\sum_{m=0}^{s-\nu} \left|F_\nu^m\right| \geqq c h^{k+1} \sum_{m=0}^{s-\nu} \Big\| \sum_{+j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{1,\Omega_h}. \tag{164}$$

(138) and (105) (under the assumption (97)) imply

$$\left|G_\nu^m\right| \leqq c h^{k+1} \Big\| \sum_{j=0}^{\nu} \alpha_j u^{m+j} \Big\|_{k+3,\Omega} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{1,\Omega_h}. \tag{165}$$

We estimate now $\| \sum_{j=0}^{\nu} \alpha_j u^{m+j} \|_{k+3,\Omega}$ for $\nu = 1$. From (143) and (22) it follows that

$$\Big\| \sum_{j=0}^{1} \alpha_j u^{m+j} \Big\|_{k+3,\Omega} = \left\|u^{m+1} - u\right\|_{k+3,\Omega} = \Delta t \left\| \frac{\partial u^{\varkappa_6}}{\partial t} \right\|_{k+3,\Omega} \leqq c\,\Delta t\,. \tag{166}$$

In the same way for $\nu = 2$ it follows from (144)

$$\begin{aligned} \Big\| \sum_{j=0}^{2} \alpha_j u^{m+j} \Big\|_{k+3,\Omega} &= \left\|\theta u^{m+2} + (1 - 2\theta)\, u^{m+1} + (-1 + \theta)\, u^m\right\|_{k+3,\Omega} \leqq \\ &\leqq \theta \left\|u^{m+2} - u^{m+1}\right\|_{k+3,\Omega} + \left|1 - \theta\right| \left\|u^{m+1} - u^m\right\|_{k+3,\Omega} \leqq c\,\Delta t\,. \end{aligned} \tag{167}$$

Substituting (166) and (167) into (165) we get

$$\left|G_\nu^m\right| \leqq c\,\Delta t\, h^{k+1} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{1,\Omega_h}, \quad \nu = 1, 2\,.$$

Hence

$$\sum_{m=0}^{s-\nu} \left|G_\nu^m\right| \leqq c\,\Delta t\, h^{k+1} \sum_{m=0}^{s-\nu} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{1,\Omega_h}, \quad \nu = 1, 2\,. \tag{168}$$

From (139) and (115) it follows that

$$\left|H_\nu^m\right| \leqq c h^{k+1} \Big\| \sum_{j=0}^{\nu} \beta_j u^{m+j} \Big\|_{k+3,\Omega} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{1,\Omega_h}.$$

Hence

$$\sum_{m=0}^{s-\nu} |H_\nu^m| \leqq ch^{k+1} \sum_{m=0}^{s-\nu} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{1,\Omega_h} . \tag{169}$$

Substituting (164), (168) and (169) into (163) we get

$$\sum_{m=0}^{s-\nu} |Q_\nu^m| \leqq c\, \Delta t\, h^{k+1} \sum_{m=0}^{s-\nu} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+1} \Big\|_{1,\Omega_h} . \tag{170}$$

From (152), (162) and (170) it follows that

$$\begin{aligned} \|\varepsilon^s\|_{0,\Omega_h}^2 &+ \Delta t \sum_{m=0}^{s-\nu} \Big| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big|_{1,\Omega_h}^2 \leqq \\ &\leqq c\Big\{\Delta t(\Delta t^q + h^{k+1}) \sum_{m=0}^{s-\nu} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{0,\Omega_h} + \\ &+ \Delta t\, h^{k+1} \sum_{m=0}^{s-\nu} \Big\| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big\|_{1,\Omega_h} + |\varepsilon^0|_h^2 + |\varepsilon^{\nu-1}|_h^2\Big\} \leqq \\ &\leqq c\Big\{\Delta t(\Delta t^q + h^{k+1}) \sum_{m=0}^{s} \|\varepsilon^m\|_{0,\Omega_h} + \\ &+ \Delta t\, h^{k+1} \sum_{m=0}^{s-\nu} \Big| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big|_{1,\Omega_h} + |\varepsilon^0|_h^2 + |\varepsilon^{\nu+1}|_h^2\Big\} . \end{aligned} \tag{171}$$

Using the inequality

$$|ab| \leqq \frac{1}{2} \tau a^2 + \frac{1}{2\tau} b^2 \tag{172}$$

for $1/2\tau = 1/c$ we get

$$\begin{aligned} c\, \Delta t\, h^{k+1} \sum_{m=0}^{s-\nu} \Big| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big|_{1,\Omega_h} &\leqq c \sum_{m=0}^{s-\nu} \left[ \frac{1}{2} \tau\, \Delta t\, h^{2(k+1)} + \frac{1}{2\tau} \Delta t \Big| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big|_{1,\Omega_h}^2 \right] \leqq \\ &\leqq \frac{c^2 T}{4} h^{2(k+1)} + \Delta t \sum_{m=0}^{s-\nu} \Big| \sum_{j=0}^{\nu} \beta_j \varepsilon^{m+j} \Big|_{1,\Omega_h}^2 . \end{aligned}$$

If we substitute this inequality into (171) then we obtain

$$\begin{aligned} \|\varepsilon^s\|_{0,\Omega_h}^2 \leqq c\Big\{&\Delta t(\Delta t^q + h^{k+1}) \Big[\|\varepsilon^s\|_{0,\Omega_h} + \sum_{m=0}^{s-1} \|\varepsilon^m\|_{0,\Omega_h}\Big] + \\ &+ h^{2(k+1)} + |\varepsilon^0|_h^2 + |\varepsilon^{\nu-1}|_h^2\Big\} . \end{aligned} \tag{173}$$

Using the inequality (172) for $\tau = c$ we get

$$c\, \Delta t(\Delta t^q + h^{k+1}) \|\varepsilon^s\|_{0,\Omega_h} \leqq \frac{c^2}{2} \Delta t^2 (\Delta t^q + h^{k+1})^2 + \frac{1}{2} \|\varepsilon^s\|_{0,\Omega_h}^2 .$$

If we substitute this inequality into (173) then we obtain

$$(174)\qquad \|\varepsilon^s\|^2_{0,\Omega_h} \leqq c\{\Delta t(\Delta t^q + h^{k+1}) \sum_{m=0}^{s-1} \|\varepsilon^m\|_{0,\Omega_h} + \Delta t^2(\Delta t^q + h^{k+1})^2 + $$
$$+ h^{2(k+1)} + |\varepsilon^0|^2_h + |\varepsilon^{\nu-1}|^2_h\} .$$

When using the inequality (172) for $\tau = c/2$ we get

$$c\, \Delta t(\Delta t^q + h^{k+1}) \sum_{m=0}^{s-1} \|\varepsilon^m\|_{0,\Omega_h} \leqq c \sum_{m=0}^{s-1} \left\{\frac{1}{2}\tau\, \Delta t(\Delta t^q + h^{k+1})^2 + \frac{1}{2\tau}\, \Delta t\, \|\varepsilon^m\|^2_{0,\Omega_h}\right\} \leqq$$
$$\leqq \frac{c^2 T}{4}(\Delta t^q + h^{k+1})^2 + \Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|^2_{0,\Omega_h} .$$

From here and from (174) it follows that

$$(175)\qquad \|\varepsilon^s\|^2_{0,\Omega_h} \leqq c(\Delta t^{2q} + h^{2(k+1)} + |\varepsilon^0|^2_h + |\varepsilon^{\nu-1}|^2_h) + \Delta t \sum_{m=0}^{s-1} \|\varepsilon^m\|^2_{0,\Omega_h} .$$

In [9] (see Lemma 2.1, p. 396) the following lemma is formulated:

**Lemma 9.** *Let $\Phi$, $\phi$, $\chi$ be nonnegative functions defined for $t = j\, \Delta t$, $j = 0, 1, \ldots$ $\ldots, M$ and let $\chi$ be nondecreasing. If*

$$(176)\qquad \Phi^s + \phi^s \leqq \chi^s + c\, \Delta t \sum_{m=0}^{s-1} \Phi^m , \quad s = 0, 1, \ldots, M ,$$

*where $c$ is a positive constant then*

$$(177)\qquad \Phi^s + \phi^s \leqq \chi^s e^{cs\Delta t} , \quad s = 0, 1, \ldots, M .$$

For the proof see [10].
Applying (176) onto (175) we get

$$\|\varepsilon^s\|^2_{0,\Omega_h} \leqq c(\Delta t^{2q} + h^{2(k+1)} + |\varepsilon^0|^2_h + |\varepsilon^{\nu-1}|^2_h)\, e^{s\Delta t} .$$

Hence

$$(178)\qquad \|\varepsilon^s\|_{0,\Omega_h} \leqq c(\Delta t^q + h^{k+1} + |\varepsilon^0|_h + |\varepsilon^{\nu-1}|_h) .$$

Now we are able to formulate the result

**Theorem 6.** *Let $u(x, t)$ be a solution of the problem (20) such that $u, \partial^l u/\partial t^l \in$ $\in L^\infty(H^{k+3}(\Omega))$, $l = 1, \ldots, q$. Let $\mathscr{C}_h$ be a $k$-regular triangulation of the set $\Omega_h$ where $k$ is a positive integer such that $k > n/2 - 1$. Let a quadrature formula on the reference set $\hat{T}$ for calculation of the forms $(\cdot, \cdot)_{0,\Omega_h}$ and $\tilde{a}(\cdot, \cdot)$ be of degrees $d \geqq 2k$ and $d \geqq 2k - 1$, respectively. Let a given $\nu$-step time discretization method be A-stable and of an order $q$. Let $\nu = 1$ or $2$.*

*Then the discrete problem* (33) *has one and only one solution* $u_h(x, t)$ *and there exists a constant* $c$ *independent of* $t$ *and* $h$ *such that*

$$(179) \qquad \|u^s - u_h^s\|_{0,\Omega\cap\Omega_h} \leqq c(\Delta t^q + h^{k+1} + |\varepsilon^0|_h + |\varepsilon^{\nu-1}|_h)\,.$$

Proof. The existence and uniqueness of the solution $u_h$ is a consequence of $A$-stability and Theorem 5. The inequality (179) is a consequence of the inequalities (126), (125), (123) and (178).

Remark 1. From (179) we see that $L_2$-norm of the error is of a magnitude of the order $\Delta t^q$ ($q = 1, 2$) with respect to $\Delta t$ and of the order $h^{k+1}$ with respect to $h$.

Remark 2. According to our result, for 1-regular triangulation (i.e. for linear isoparametric elements) the quadrature formula on the reference set $\hat{T}$ for calculation of the forms $(\cdot, \cdot)_{0,\Omega}$ and $a(\cdot, \cdot)$ must be, in general, of degree 2 and 1, respectively. It can be proved that using the quadrature formula

$$(180) \qquad \int_{\hat{T}} \varphi(\hat{x})\,\mathrm{d}\hat{x} \approx \frac{\operatorname{mes} \hat{T}}{n}\,[\varphi(0, 0, \ldots, 0) + \varphi(0, 1, \ldots, 0) + \varphi(0, 0, \ldots, 1)]$$

(which is of degree 1) for calculation of the form $(\cdot, \cdot)_{0,\Omega}$ we obtain the same estimate as in (179). In this case the mass matrix is diagonal. In the ingeneering literature this effect is called the *mass lumping*.

Remark 3. For the three-dimensional space the simplicial curved elements have no practical use. For such case the theory using quadrilateral elements must be developed. We are working on this problem now.

### *Literature*


[1] *P. G. Ciarlet* and *A. P. Raviart:* The combined effect of curved boundaries and numerical integration in isoparametric finite element methods. In A. K. Aziz: The mathematical foundations of the finite element method with applications to partial differential equations. Academic Press. New York and London. 1972.

[2] *P. A. Raviart:* The use of numerical integration in finite element methods for solving parabolic equations. Lecture presented at the Conference on Numerical Analysis. Royal Irish Academy. Dublin, August 14—18, 1972.

[3] *Jindřich Nečas:* Les Méthodes Directe en Théorie des Equations Elliptiques. Mason. Paris. 1967.

[4] *V. J. Smirnov:* Kurs vyššej matěmatiki, tom V. Gosudarstvěnnoje izdatělstvo fiziko-matěmatičeskoj litěratury. Moskva. 1960.

[5] *Miloš Zlámal:* Finite Element Multistep Discretizations of Parabolic Boundary Value Problems. Mathematics of Computation, 29, Nr 130 (1975), 350—359.

[6] *Miloš Zlámal:* Curved Elements in the Finite Element Method I. SIAM J. Numer. Anal., 10, No 1 (1973), 229—240.

[7] *Miloš Zlámal:* Curved Elements in the Finite Element Methods II. SIAM J. Numer. Anal., 11, No 2 (1974), 347—362.

[8] *Miloš Zlámal:* Finite Element Methods for Parabolic Equations. Mathematics of Computation, 28, No 126 (1974), 393—404.

[9] *T. Dupont, G. Fairweather, J. P. Johnson:* Three-level Galerkin Methods for Parabolic Equations. SIAM J. Numer. Anal., 11, No 2 (1974).

[10] *M. Lees:* A priori estimates for the solutions of difference approximations to parabolic differential equations. Duke Math. J., 27 (1960), 287—311.

[11] *Miloš Zlámal:* Finite element methods for nonlinear parabolic equations. R.A.I.R.O. Analyse numérique/Numerical Analysis, 11, No 1 (1977), 93—107.

[12] *W. Liniger:* A criterion for A-stability of linear multistep integration formulae. Computing, 3 (1968), 280—285.

## Souhrn

## ŘEŠENÍ PARABOLICKÝCH ROVNIC METODOU KONEČNÝCH PRVKŮ

JOSEF NEDOMA

V dosud publikovaných pracích o řešení parabolických rovnic metodou konečných prvků jsou odvozeny odhady chyb buďto pro případ kdy sjednocení konečných elementů (rovných nebo křivých) přesně pokrývá danou oblast (tak je tomu např. ve Zlámalových pracích) nebo pro případ kdy sjednocení konečných elementů (křivých) pokrývá danou oblast jen přibližně (tak je tomu např. v Ciarlet-Raviartových pracích). V prvém případě jsou odhady odvozeny pro plně diskretizovaná (tj. v prostoru i čase) přibližná řešení. Není však brána v úvahu chyba způsobená numerickou integrací. Ve druhém případě je sice uvažována i chyba způsobená numerickou integrací, odhady jsou však odvozeny pouze pro semidiskretní (nediskretizovaná v čase) přibližná řešení.

V této práci jsou odvozeny odhady chyb pro plně diskretizovaná řešení a pro libovolné křivé oblasti. Jsou odvozeny požadavky na stupeň přesnosti kvadraturních formulí tak, aby numerická integrace nezhoršovala optimální řád konvergence v $L_2$ normě daný metodou konečných prvků. Obdržený výsledek je následující: Jestliže metoda konečných prvků je řádu $k + 1$, potom k tomu, aby se numerickou integrací řád nesnížil, je třeba použít pro výpočet integrálů, ve kterých vystupují samotné funkce, kvadraturních formulí stupně přesnosti aspoň $2k$ a pro výpočet integrálů, ve kterých vystupují derivace funkcí kvadraturních formulí stupně přesnosti aspoň $2k — 1$.

Práce je rozdělena do pěti částí. Vprvé části je konstruován prostor konečných elementů. Vychází se zde z isoparametrické třídy simpliciálních elementů definované Ciarletem a Raviartem. Důvodem k tomu je, že bylo tak možné beze změny převzít interpolační teorém odvozený těmito autory. Obdržené výsledky lze odvodit i pro jiné elementy, pokud ovšem zůstane zachována platnost interpolačního teorému ve tvaru uvedeném v (6).

Ve druhé části je formulován problém a cíl celé práce. Základní parabolický problém je definován ve (20) a to přímo ve variačním tvaru. Předpokládá se, že čtenáři je dostatečně známo, jak tento variační parabolický model souvisí s modelem klasickým. Z metodických důvodů je v (26) formulován příslušný semidiskretní model, tj. model diskretizovaný metodou konečných prvků vzhledem k prostorovým proměnným. Ve (33) je formulován plně diskretní model, tj. model, který vznikne ze semidiskretního modelu diskretizací vzhledem k časové proměnné. Pro tuto diskretizaci je použito $\nu$-krokových lineárních metod.

Ve třetí části je odvozena věta, kterou jsme nazvali větou o Ritzově aproximaci. Ritzova aproximace se ukázala být velmi silným prostředkem v dalších důkazech a proto je jí věnována tak velká pozornost.

Ve čtvrté části jsou odvozeny odhady chyb při použití isoparametrické integrace. Přesto, že výsledky jsou formulovány s přihlédnutím k jejich dalšímu využití pro naše účely, mají obecnější charakter a platnost.

Konečně v páté, závěrečné části jsou provedeny odhady chyb v případě, kdy pro časovou diskretizaci jsou použity $A$-stabilní jedno nebo dvoukrokové metody. Podmínka $A$-stability je zde podstatná. Finální výsledek celé práce je pak formulován v závěru páté části ve větě 6.

*Author's address:* RNDr. Ing. *Josef Nedoma,* CSc., Laboratoř počítacích strojů VUT, Třída Obránců míru 21, 602 00 Brno.