Aplikace matematiky

Alexander Ženíšek

Curved triangular finite C^m-elements

Aplikace matematiky, Vol. 23 (1978), No. 5, 346–377

Persistent URL: http://dml.cz/dmlcz/103761

CURVED TRIANGULAR FINITE C^m-ELEMENTS

ALEXANDER ŽENÍŠEK

(Received February 4, 1977)

INTRODUCTION

The aim of the paper is to construct curved triangular finite C^m-elements and to apply them in solving elliptic boundary value problems of order $2(m + 1)$. The paper consists of five sections.

In Section 1 the approximation of a curved boundary used in the paper is described and a modification of the transformation first analyzed by Zlámal [9, 10, 11] is presented.

In Section 2 curved triangular finite C^m-elements are constructed. The results presented in this section modify and generalize the ideas of Mansfield [6] where curved C^1-elements were first constructed.

In Section 3 an interpolation theorem for curved finite triangular C^m-elements is presented. This theorem is a generalization of the interpolation theorem for "classical" triangles (cf. Bramble and Zlámal [1]).

In the last two sections the C^m-elements are applied. As a model problem, the Dirichlet problem is chosen. The main attention is devoted to analyzing the effect of numerical integration. The theory is a generalization of the results of Ciarlet and Raviart [2] and Ciarlet [3].

The results of the paper can be described in the case $m = 1$ as follows: The Dirichlet problem of the fourth order elliptic equation in a domain Ω with a smooth boundary Γ is solved by the finite element method. The domain Ω is triangulated and Bell's element of degree $N = 5$ is used on the interior triangles. The curved side of each boundary triangle is approximated by an arc of the third degree. The union Γ_h of these arcs is the approximation of the boundary Γ. Using a special polynomial $w^*(\xi, \eta)$ of degree $N^* = 7$ on the triangle T_0 with vertices $R_1(0, 0)$, $R_2(1, 0)$, $R_3(0, 1)$ we can create, by means of one-to-one mappings of T_0 onto the curved triangles T, elements $w_T(x, y)$ which give by piecing them together with Bell's elements a function from $C^1(\Omega_h)$. Such functions are used in solving the problem by the finite element method. In order to get numerical results the weak formulation of the problem must be expressed approximately by means of numerical integration. If the quadrature formula used for interior triangles has degree of precision $d = 6$ and the formula used for curved triangles has degree of precision $d = 10$ then the approximate solution exists and is unique and the rate of convergence is $O(h^3)$, h being the length of the greatest side in the triangulation.

In the case $m \geqq 1$ the Dirichlet problem (107), (108) is solved and on the interior triangles the generalized Bell's element of degree $N = 4m + 1$ is used. The boundary Γ is approximated piecewise by arcs of degree $2m + 1$. This implies that $w^*(\xi, \eta)$ is a polynomial of degree $N^* = N + 2m^2$ on T_0. If $d = 2N - 2(m + 1)$ for the interior triangles and $d = 2N^* - 2(m + 1)$ for the curved triangles then the approximate solution exists and is unique and the rate of convergence is $O(h^{2m+1})$.

In the paper the following notation for partial derivatives is mostly used:

$$D^\alpha u(x, y) = \partial^{|\alpha|}u/\partial x^{\alpha_1}\partial y^{\alpha_2}\,, \quad D^\alpha v(\xi, \eta) = \partial^{|\alpha|}v/\partial \xi^{\alpha_1}\partial \eta^{\alpha_2}$$

with $\alpha = (\alpha_1, \alpha_2)$, $|\alpha| = \alpha_1 + \alpha_2$.

Let $k \geqq 0$ be an integer and p any number satisfying $1 \leqq p \leqq \infty$. In the paper, the symbol $W_p^{(k)}(\Omega)$ denotes the Sobolev space which consists of those functions $v \in L_p(\Omega)$ for which all partial derivatives $D^\alpha v$ with $|\alpha| \leqq k$ belong to the space $L_p(\Omega)$. It is a Banach space with the norm

$$\|v\|_{k,p,\Omega} = \left(\sum_{j=0}^{k} |v|_{j,p,\Omega}^p\right)^{1/p}$$

where the seminorms are given by

$$|v|_{j,p,\Omega} = \left(\sum_{|\alpha|=j} \iint_\Omega |D^\alpha v|^p \,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}$$

with the standard modification for $p = \infty$:

$$|v|_{j,\infty,\Omega} = \operatorname*{vrai\,max}_{(x,y)\in\Omega,|\alpha|=j} |D^\alpha v|\,, \quad \|v\|_{k,\infty,\Omega} = \operatorname*{vrai\,max}_{(x,y)\in\Omega,|\alpha|\leqq k} |D^\alpha v|\,.$$

When $p = 2$ we shall use the notation

$$W_2^{(k)}(\Omega) = H^k(\Omega)\,; \quad \|\cdot\|_{k,2,\Omega} = \|\cdot\|_{k,\Omega}\,; \quad |\cdot|_{j,2,\Omega} = |\cdot|_{j,\Omega}\,.$$

1. APPROXIMATION OF A CURVED BOUNDARY

Let Ω be a bounded domain in the x,y-plane with a boundary Γ which is piecewise of class C^{q+1} with q sufficiently large to fulfil our requirements. Then the boundary Γ

can be divided into a finite number of arcs each of which has a parametric representation

$$(1) \qquad x = \varphi(s), \quad y = \psi(s), \quad a \leqq s \leqq b$$

with functions $\varphi(s)$, $\psi(s)$ belonging to C^{q+1} and such that at least one of the derivatives $\varphi'(s)$, $\psi'(s)$ is different from zero on $[a, b]$.

Let us triangulate the domain Ω, i.e., let us divide it into a finite number of triangles (the sides of which can be curved) in such a way that two arbitrary triangles are either disjoint, or have a common vertex, or a common side. Let the triangulation have the property that each interior triangle (i.e. a triangle having at most one point common with the boundary) has straight sides and each boundary triangle has at most one curved side. This side lies then on the boundary. Further, we assume that the domain Ω is triangulated in such a way that the curved side of each boundary triangle lies on one arc of the type (1) from which the boundary Γ consists.

With every triangulation τ we associate two parameters h and ϑ defined by

$$(2) \qquad h = \max_{T \in \tau} h_T, \quad \vartheta = \min_{T \in \tau} \vartheta_T$$

where h_T and ϑ_T are the length of the greatest side and the smallest angle, respectively, of the triangle (with straight sides) which has the same vertices as the triangle T. We restrict ourselves to such triangulations that ϑ is bounded away from zero as $h \to 0$, i.e.

$$(3) \qquad \vartheta \geqq \vartheta_0, \quad \vartheta_0 = \text{const} > 0.$$

Let $\overline{T}$ be a curved boundary triangle and P_1, P_2, P_3 a local notation of its vertices. Let $P_i(x_i, y_i)$, $P_j(x_j, y_j)$ be the end points of the curved side of $\overline{T}$ and let

$$(4a) \qquad \Phi_{ij}(t) = [\varphi(s_j + \bar{s}_{ij}t) - x_j - \bar{x}_{ij}t]/(1 - t),$$

$$(4b) \qquad \Psi_{ij}(t) = [\psi(s_j + \bar{s}_{ij}t) - y_j - \bar{y}_{ij}t]/(1 - t)$$

where

$$(5) \qquad \bar{x}_{ij} = x_i - x_j, \quad \bar{y}_{ij} = y_i - y_j, \quad \bar{s}_{ij} = s_i - s_j.$$

The symbols s_k $(k = i, j)$ denote the values of the parameter s for which

$$(6) \qquad x_k = \varphi(s_k), \quad y_k = \psi(s_k), \quad k = i, j.$$

For the sake of brevity, it is convenient to set

$$(7) \qquad \bar{x}_i = \bar{x}_{i1}, \quad \bar{y}_i = \bar{y}_{i1}, \quad \bar{s}_i = \bar{s}_{i1}.$$

In what follows we shall use the notation $\bar{x}_i$, $\bar{y}_i$ even if one of the vertices lies inside of Ω.

Finally, let $\overline{T}_0$ denote the triangle (with straight sides) which lies in the ξ, η-plane and has the vertices $R_1(0, 0)$, $R_2(1, 0)$, $R_3(0, 1)$. Now we are ready to formulate the following theorem.

Theorem 1. *Let Γ be of class C^{q+1}, $q \geqq 1$. Let the local notation of the vertices of $\overline{T}$ be chosen in such a way that P_2P_3 is the curved side of the triangle $\overline{T}$. If (3) holds and h_T is sufficiently small then each of the transformations*

$$(8a) \qquad x = x(\xi, \eta) \equiv x_1 + \bar{x}_2\xi + \bar{x}_3\eta + \xi\,\Phi_{32}(\eta)\,,$$

$$(8b) \qquad y = y(\xi, \eta) \equiv y_1 + \bar{y}_2\xi + \bar{y}_3\eta + \xi\,\Psi_{32}(\eta)\,,$$

$$(9a) \qquad x = x(\xi, \eta) \equiv x_1 + \bar{x}_2\xi + \bar{x}_3\eta + \eta\,\Phi_{23}(\xi)\,,$$

$$(9b) \qquad y = y(\xi, \eta) \equiv y_1 + \bar{y}_2\xi + \bar{y}_3\eta + \eta\,\Psi_{23}(\xi)$$

maps $\overline{T}_0$ one-to-one onto $\overline{T}$. The Jacobian $J(\xi, \eta)$ of each of these mappings is different from zero on $\overline{T}_0$, the sides R_1R_2 and R_1R_3 are linearly mapped onto the straight sides P_1P_2 and P_1P_3, respectively, and the side R_2R_3 is mapped onto the arc P_2P_3. The mappings (8), (9) as well as their inverse mappings are of class C^q. In addition,

$$(10) \qquad c_1 h_T^2 \leqq |J(\xi, \eta)| \leqq c_2 h_T^2\,, \quad c_i = \text{const} > 0\,,$$

$$(11) \qquad D^\alpha x(\xi, \eta) = O(h_T^{|\alpha|})\,, \quad D^\alpha y(\xi, \eta) = O(h_T^{|\alpha|})\,, \quad 1 \leqq |\alpha| \leqq q\,,$$

$$(12) \qquad D^\alpha \xi(x, y) = O(h_T^{-1})\,, \quad D^\alpha \eta(x, y) = O(h_T^{-1})\,, \quad 1 \leqq |\alpha| \leqq q\,.$$

Proof. Except for (12), the proof is a slight modification of the proof of [10, Th. 1]. Thus we omit it. To prove (12) let us differentiate the relations $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ with respect to x and y. Solving the equations obtained we find

$$(13) \qquad \frac{\partial \xi}{\partial x} = J^{-1}\frac{\partial y}{\partial \eta}\,, \quad \frac{\partial \eta}{\partial x} = -J^{-1}\frac{\partial y}{\partial \xi}\,, \quad \frac{\partial \xi}{\partial y} = -J^{-1}\frac{\partial x}{\partial \eta}\,, \quad \frac{\partial \eta}{\partial y} = J^{-1}\frac{\partial x}{\partial \xi}\,.$$

The relations (10), (11), (13) imply (12) in the case $|\alpha| = 1$.

Let us suppose that we proved (12) for $|\alpha| \leqq n$. We shall prove (12) for $|\alpha| = n + 1 \leqq q$. Each of the derivatives $D^\alpha \xi(x, y)$, $D^\alpha \eta(x, y)$ with $|\alpha| = n$ is a linear combination of expressions of the type

$$(14) \qquad [J(\xi, \eta)]^{-k}\,[D^\alpha J(\xi, \eta)]^p\,[D^\beta \zeta(x, y)]^r\,[D^\gamma z(\xi, \eta)]$$

where $|\alpha| < n$, $|\beta| < n$, $|\gamma| \leqq n$ and $\zeta = \xi$ or η, $z = x$ or y. It holds

$$[J(\xi, \eta)]^{-k} = O(h_T^{-2k})\,, \quad [D^\alpha J(\xi, \eta)]^p = O(h_T^{(2+|\alpha|)p})\,,$$

$$[D^\beta \zeta(x, y)]^r = O(h_T^{-r})\,, \quad D^\gamma z(\xi, \eta) = O(h_T^{|\gamma|})\,.$$

We can easily find

$$\frac{\partial}{\partial w}\{[J(\xi,\eta)]^{-k}\} = O(h_T^{-2k}), \quad \frac{\partial}{\partial w}\{[D^\alpha J(\xi,\eta)]^p\} = O(h_T^{(2+|\alpha|)p}),$$

$$\frac{\partial}{\partial w}\{[D^\beta \zeta(x,y)]^r\} = O(h_T^{-r}), \quad \frac{\partial}{\partial w}\{D^\gamma z(\xi,\eta)\} = O(h_T^{|\gamma|})$$

where $w = x$ or y. Thus the estimate of the first derivatives of (14) is the same as the estimate of (14). This proves (12).

Remark 1. In [9], [10], [11], where the curved triangles were first introduced, another local notation of vertices of $\overline{T}$ is used: the curved side of $\overline{T}$ is denoted there by P_1P_3. It should be noted that each of the possible local notations allow a construction of curved triangular C^m-elements. However, if we use the same notation as in Theorem 1 then the expressions defining C^m-elements are less cumbersome than in other cases.

Remark 2. The arithmetical mean of the right-hand sides of (8) and (9) gives the transformation

$$x = x_1 + \bar{x}_2\xi + \bar{x}_3\eta + \tfrac{1}{2}[\xi\, \Phi_{32}(\eta) + \eta\, \Phi_{23}(\xi)], \tag{15a}$$

$$y = y_1 + \bar{y}_2\xi + \bar{y}_3\eta + \tfrac{1}{2}[\xi\, \Psi_{32}(\eta) + \eta\, \Psi_{23}(\xi)]. \tag{15b}$$

This is in another notation the transformation (3.5) of [6].

The transformations (8) and (9) (or the transformation (15)) do not allow to construct curved finite C^m-elements ($m \geqq 1$). In order to obtain such a transformation we must approximate the functions $\varphi(s)$, $\psi(s)$ by polynomials and change the definition of the functions Φ, Ψ.

In what follows we restrict ourselves to the mapping (8). Let $\varphi^*(t)$ and $\psi^*(t)$ be polynomials of degree at most n which satisfy (cf. (6))

$$\varphi^*(0) = x_2, \quad \varphi^*(1) = x_3, \quad \psi^*(0) = y_2, \quad \psi^*(1) = y_3, \tag{16}$$

$P_2(x_2, y_2)$, $P_3(x_3, y_3)$ being the end-points of the curved side of $\overline{T}$. Then

$$\varphi^*(t) = x_2 + \bar{x}_{32}t + t(1-t)\, p_1(t), \tag{17}$$

$$\psi^*(t) = y_2 + \bar{y}_{32}t + t(1-t)\, p_2(t). \tag{18}$$

The polynomials $p_1(t)$, $p_2(t)$ depend on the form of approximation of the curved side of $\overline{T}$. We restrict ourselves to the case $n = 2r + 1$ ($r \leqq q + 1$) and require $\varphi^*(t)$, $\psi^*(t)$ to be Hermite interpolation polynomials of the functions $\varphi(s_2 + \bar{s}_{32}t)$, $\psi(s_2 + \bar{s}_{32}t)$ uniquely determined by the function values and all derivatives up to order r inclusively at the points $t_2 = 0$ and $t_3 = 1$. This leads, with respect to (16), to the following additional conditions for $\varphi^*(t)$ and $\psi^*(t)$:

$$(19) \qquad \bar{s}_{32}^k\, \varphi^{(k)}(s_i) = \varphi^{*(k)}(t_i)\,, \quad k = 1, \ldots, r\,; \quad i = 2, 3 \;(t_2 = 0,\; t_3 = 1)\,,$$

$$(20) \qquad \bar{s}_{32}^k\, \psi^{(k)}(s_i) = \psi^{*(k)}(t_i)\,, \quad k = 1, \ldots, r\,; \quad i = 2, 3$$

which form $2r$ conditions for the polynomials $p_1(t)$ and $p_2(t)$ of degree $2r - 1$, respectively.

If the curved side P_2P_3 lies on a curve $y = f(x)$ then the parametric equations of P_2P_3 are $x = x_2 + \bar{x}_{32}t$, $y = f(x_2 + \bar{x}_{32}t)$, $t \in [0, 1]$. In this case $\bar{s}_{32} = \bar{x}_{32}$, $\varphi^*(t) = x_2 + \bar{x}_{32}t$ and the conditions (19) are automatically satisfied.

It should be noted that the curve

$$(21) \qquad x = \varphi^*(t)\,, \quad y = \psi^*(t)\,, \quad t \in [0, 1]$$

has the same derivatives d^jy/dx^j $(j = 1, \ldots, r)$ at the points P_2, P_3 as the curve (1).

Let $\Phi_{32}^*(t)$ and $\Psi_{32}^*(t)$ denote the functions obtained by replacing the functions $\varphi(s_2 + \bar{s}_{32}t)$ and $\psi(s_2 + \bar{s}_{32}t)$ in (4a) and (4b) by their interpolation polynomials $\varphi^*(t)$ and $\psi^*(t)$, respectively. With respect to (17) and (18), we can write

$$(22) \qquad \Phi_{32}^*(t) = t\, p_1(t)\,, \quad \Psi_{32}^*(t) = t\, p_2(t)\,.$$

If we replace in (8) the functions $\Phi_{32}(\eta)$, $\Psi_{32}(\eta)$ by the functions (22) we obtain the transformation

$$(23a) \qquad x = x^*(\xi, \eta) \equiv x_1 + \bar{x}_2\xi + \bar{x}_3\eta + \xi\eta\, p_1(\eta)\,,$$

$$(23b) \qquad y = y^*(\xi, \eta) \equiv y_1 + \bar{y}_2\xi + \bar{y}_3\eta + \xi\eta\, p_2(\eta)\,.$$

Theorem 2. *Let $\bar{T}^*$ be a curved triangle with straight sides P_1P_2, P_1P_3 and a curved side P_2P_3 the parametric equations of which are given by (21). If (3) holds and h_T is sufficiently small then the transformation (23) maps $\bar{T}_0$ one-to-one onto $\bar{T}^*$. The Jacobian $J^*(\xi, \eta)$ of the mapping (23) is different from zero on $\bar{T}_0$, the sides R_1R_2 and R_1R_3 are linearly mapped onto the straight sides P_1P_2 and P_1P_3, respectively, and the side R_2R_3 is mapped onto the arc P_2P_3. In addition,*

$$(24) \qquad c_1h_T^2 \leqq |J^*(\xi, \eta)| \leqq c_2h_T^2\,, \quad c_i = \text{const} > 0\,,$$

$$(25) \qquad D^\alpha x^*(\xi, \eta) = O(h_T^{|\alpha|})\,, \quad D^\alpha y^*(\xi, \eta) = O(h_T^{|\alpha|})\,, \quad |\alpha| = 1, 2, \ldots$$

$$(26) \qquad D^\alpha \xi^*(x, y) = O(h_T^{-1})\,, \quad D^\alpha \eta^*(x, y) = O(h_T^{-1})\,, \quad |\alpha| = 1, 2, \ldots$$

where

$$(27) \qquad \xi = \xi^*(x, y)\,, \quad \eta = \eta^*(x, y)$$

is the inverse mapping to the mapping (23).

The proof is again omitted because it follows the same lines as the corresponding considerations in [11].

2. CONSTRUCTION OF CURVED C^m-ELEMENTS

In this section we shall use the transformation (23) for constructing curved triangular finite C^m-elements. The definition of a curved finite C^m-element depends on the choice of the C^m-element which is used on the interior triangles of the given triangulation of Ω. Let us decide to use the generalized Bell's C^m-element [5] on these triangles.

Let us substitute the curved triangles $\bar{T}$ of the triangulation of Ω by the curved triangles $\bar{T}^*$ described in Section 1 and denote such a changed domain by Ω_h. Our aim is to construct on the curved triangles $\bar{T}^*$ functions which give, by piecing them together with generalized Bell's elements, functions from $C^m(\Omega_h)$.

The generalized Bell's element is a triangular element with a polynomial of degree $4m + 1$ uniquely determined by the following parameters:

$$D^\alpha w(P_i)\,, \quad |\alpha| \leqq 2m\,, \quad i = 1, 2, 3\,, \tag{28}$$

$$D^\alpha w(P_0)\,, \quad |\alpha| \leqq m - 2 \tag{29}$$

where P_1, P_2, P_3 denote the vertices of the triangle in a local notation and P_0 is the centre of gravity of the triangle. On the contrary to [12] no parameters are prescribed on the sides of the triangle. These parameters are substituted by the requirement that the k-th normal derivative $\partial^k w/\partial n_{ij}^k$ be a polynomial of degree $4m + 1 - 2k$ $(k = 1, \ldots, m)$ along the side P_iP_j $(i < j, i = 1, 2;\ j = 2, 3)$. Thus we want to construct on $\bar{T}^*$ a function $w(x, y)$ with the following properties:

(a) The function $w(x, y)$ is uniquely determined by the parameters (28) prescribed at the vertices of $\bar{T}^*$ and (if necessary) by some parameters prescribed in the interior T^* of $\bar{T}^*$.

(b) The function w and its k-th normal derivative $\partial^k w/\partial n_{1j}^k$ $(k = 1, \ldots, m)$ are polynomials of degree $4m + 1$ and $4m + 1 - 2k$, respectively, in one variable along the straight side P_1P_j $(j = 2, 3)$. These polynomials are uniquely determined on each straight side by the parameters prescribed at the end-points of this side.

(c) The function $w^*(\xi, \eta)$ given by the relation

$$w^*(\xi, \eta) = w(x^*(\xi, \eta)\,, y^*(\xi, \eta)) \tag{30}$$

is a polynomial; here the functions $x^*(\xi, \eta)$, $y^*(\xi, \eta)$ are defined by (23).

Let us suppose for a moment that there exists a function $w(x, y)$ with properties (a)–(c) and analyze the properties of the polynomial $w^*(\xi, \eta)$. The basic result of this analysis will be the relations (51), (57). We sketch how to obtain them.

Let the set of normals in the triangulation τ_h of Ω_h be given in such a way that the unit normal vector $\boldsymbol{n}_{1j}$ to each straight side P_1P_j of each curved triangle $\bar{T}^*$ is defined by

$$\boldsymbol{n}_{1j} = (\bar{y}_j/l_{1j}, \quad -\bar{x}_j/l_{1j}) \quad (j = 2, 3) \tag{31}$$

where l_{1j} is the length of the segment P_1P_j. Let us define the functions

$$f_k(\xi) = \left(\left(\bar{y}_2 \frac{\partial}{\partial x} - \bar{x}_2 \frac{\partial}{\partial y}\right)^k w\right)(x^*(\xi, 0), y^*(\xi, 0)), \tag{32}$$

$$g_k(\eta) = \left(\left(\bar{y}_3 \frac{\partial}{\partial x} - \bar{x}_3 \frac{\partial}{\partial y}\right)^k w\right)(x^*(0, \eta), y^*(0, \eta)) \tag{33}$$

where $k = 0, 1, \ldots, m$. The expression on the right-hand side of (32) must be interpreted in the following way: First we apply the operator $(\bar{y}_2\, \partial/\partial x - \bar{x}_2\, \partial/\partial y)^k$ on the function $w(x, y)$ and then we set $x = x^*(\xi, 0) \equiv x_1 + \bar{x}_2\xi$, $y = y^*(\xi, 0) \equiv y_1 + \bar{y}_2\xi$. A similar rule holds for the right-hand side of (33). Using (31) we can express $f_k(\xi)$ and $g_k(\eta)$ in the form

$$f_k(\xi) = l_{12}^k \frac{\partial^k w}{\partial n_{12}^k}(x_1 + \bar{x}_2\xi, y_1 + \bar{y}_2\xi), \quad k = 0, 1, \ldots, m, \tag{34}$$

$$g_k(\eta) = l_{13}^k \frac{\partial^k w}{\partial n_{13}^k}(x_1 + \bar{x}_3\eta, y_1 + \bar{y}_3\eta), \quad k = 0, 1, \ldots, m. \tag{35}$$

Owing to the requirement (b) the expressions (34), (35) show that the functions $f_k(\xi)$, $g_k(\eta)$ are polynomials of degree $4m + 1 - 2k$ ($k = 0, 1, \ldots, m$). The parameters $f_k^{(j)}(0)$, $f_k^{(j)}(1)$ and $g_k^{(j)}(0)$, $g_k^{(j)}(1)$ ($j = 0, 1, \ldots, 2m - k$) uniquely determining $f_k(\xi)$ and $g_k(\eta)$, respectively, are linear combinations of the parameters (28). We obtain them from (32), (33) by means of the rule of differentiation of a composite function.

Let us define vectors

$$\mathbf{f}_k(\xi) = (f_0^{(k)}(\xi), f_1^{(k-1)}(\xi), \ldots, f_{k-1}'(\xi), f_k(\xi))^T, \tag{36}$$

$$\mathbf{g}_k(\eta) = (g_0^{(k)}(\eta), g_1^{(k-1)}(\eta), \ldots, g_{k-1}'(\eta), g_k(\eta))^T, \tag{37}$$

$$\mathbf{u}_k(\xi) = (u_{k,0}(\xi), u_{k,1}(\xi), \ldots, u_{k,k}(\xi))^T, \tag{38}$$

$$\mathbf{v}_k(\eta) = (v_{k,0}(\eta), v_{k,1}(\eta), \ldots, v_{k,k}(\eta))^T \tag{39}$$

where the superscript T indicates transposition and where

$$u_{k,j}(\xi) = \frac{\partial^k w}{\partial x^{k-j}\, \partial y^j}(x^*(\xi, 0), y^*(\xi, 0)) \quad (j = 0, 1, \ldots, k), \tag{40}$$

(41) $$v_{k,j}(\eta) = \frac{\partial^k w}{\partial x^{k-j}\,\partial y^j}(x^*(0,\eta), y^*(0,\eta)) \quad (j = 0, 1, \ldots, k)\,.$$

Then we can write, according to (32), (33),

(42) $$\mathbf{f}_k(\xi) = \mathbf{D}_k \mathbf{u}_k(\xi) \quad (k = 1, \ldots, m)\,,$$

(43) $$\mathbf{g}_k(\eta) = \mathbf{G}_k \mathbf{v}_k(\eta) \quad (k = 1, \ldots, m)\,,$$

where $\mathbf{D}_k$ and $\mathbf{G}_k$ are square matrices with $k + 1$ columns. The entries of the matrix $\mathbf{D}_k$ are linear combinations of the monomials $\bar{x}_2^a \bar{y}_2^b$ $(a + b = k)$, and the entries of the matrix $\mathbf{G}_k$ linear combinations of the monomials $\bar{x}_3^a \bar{y}_3^b$ $(a + b = k)$. The explicit expression for the matrices $\mathbf{D}_k$, $\mathbf{G}_k$ can be obtained from (32), (33) by means of the rule of differentiation of a composite function. E.g.,

$$\mathbf{D}_2 = \begin{bmatrix} \bar{x}_2^2 & 2\bar{x}_2\bar{y}_2 & \bar{y}_2^2 \\ \bar{x}_2\bar{y}_2 & \bar{y}_2^2 - \bar{x}_2^2 & -\bar{x}_2\bar{y}_2 \\ \bar{y}_2^2 & -2\bar{x}_2\bar{y}_2 & \bar{x}_2^2 \end{bmatrix}.$$

It holds

(44) $$\mathbf{D}_k \mathbf{D}_k = (\bar{x}_2^2 + \bar{y}_2^2)^k\, \mathbf{I}_{k+1} \quad (k = 1, \ldots, m)\,,$$

(45) $$\mathbf{G}_k \mathbf{G}_k = (\bar{x}_3^2 + \bar{y}_3^2)^k\, \mathbf{I}_{k+1} \quad (k = 1, \ldots, m)$$

where $\mathbf{I}_{k+1}$ is the unit matrix with $k + 1$ columns.

Let us set

(46) $$\mathbf{M}_k = (\bar{x}_2^2 + \bar{y}_2^2)^{-k}\, \mathbf{D}_k\,, \quad \mathbf{N}_k = (\bar{x}_3^2 + \bar{y}_3^2)^{-k}\, \mathbf{G}_k\,.$$

The relations (42)–(46) then imply

(47) $$\mathbf{u}_k(\xi) = \mathbf{M}_k \mathbf{f}_k(\xi) \quad (k = 1, \ldots, m)\,,$$

(48) $$\mathbf{v}_k(\eta) = \mathbf{N}_k \mathbf{g}_k(\eta) \quad (k = 1, \ldots, m)\,.$$

Denoting

(49) $$\alpha_j = \bar{x}_j/(\bar{x}_j^2 + \bar{y}_j^2)\,, \quad \beta_j = \bar{y}_j/(\bar{x}_j^2 + \bar{y}_j^2) \quad (j = 2, 3)$$

we see from (46) that the entries of the matrix $\mathbf{M}_k$ are linear combinations of the monomials $\alpha_2^a \beta_2^b$ $(a + b = k)$, e.g.,

$$\mathbf{M}_2 = \begin{bmatrix} \alpha_2^2 & 2\alpha_2\beta_2 & \beta_2^2 \\ \alpha_2\beta_2 & \beta_2^2 - \alpha_2^2 & -\alpha_2\beta_2 \\ \beta_2^2 & -2\alpha_2\beta_2 & \alpha_2^2 \end{bmatrix},$$

and the entries of the matrix $\mathbf{N}_k$ are linear combinations of the monomials $\alpha_3^a \beta_3^b$ $(a + b = k)$.

Now we are able to express the derivative $\partial^s w^*(\xi, 0)/\partial\eta^s$ $(1 \leqq s \leqq m)$ in a form suitable for our considerations. It follows from (30) that

$$\frac{\partial^s w^*}{\partial\eta^s} = \frac{\partial^s w}{\partial x^s}\left(\frac{\partial x^*}{\partial\eta}\right)^s + \ldots + \frac{\partial w}{\partial y}\frac{\partial^s y^*}{\partial\eta^s}. \tag{50}$$

Considering (50) for the points $(\xi, \eta) = (\xi, 0)$ we can express the derivatives $\partial^k w/\partial x^{k-j}\,\partial y^j$ $(j = 0, 1, \ldots, k;\ k = 1, \ldots, s)$ by means of (47). We obtain then

$$\frac{\partial^s w^*}{\partial\eta^s}(\xi, 0) = \sum_{k=1}^{s}\sum_{j=0}^{k} \varrho_{skj}(\xi)\, f_k^{(k-j)}(\xi) \quad (1 \leqq s \leqq m) \tag{51}$$

where the functions $\varrho_{skj}(\xi)$ are of the form

$$\varrho_{skj}(\xi) = \sum_{i=1}^{k+1} \sigma_{kji}(\alpha_2, \beta_2)\, \tau_{skji}(\xi). \tag{52}$$

Each $\sigma_{kji}(\alpha_2, \beta_2)$ is a linear combination of the monomials $\alpha_2^a \beta_2^b$ $(a + b = k)$ and each $\tau_{skji}(\xi)$ a linear combination of the products

$$\left(\frac{\partial^{i_1} x^*}{\partial\eta^{i_1}}(\xi, 0)\right)^{j_1} \ldots \left(\frac{\partial^{i_p} x^*}{\partial\eta^{i_p}}(\xi, 0)\right)^{j_p} \left(\frac{\partial^{\lambda_1} y^*}{\partial\eta^{\lambda_1}}(\xi, 0)\right)^{\mu_1} \ldots \left(\frac{\partial^{\lambda_r} y^*}{\partial\eta^{\lambda_r}}(\xi, 0)\right)^{\mu_r} \tag{53}$$

where

$$i_\varkappa \geqq 1, \quad j_\varkappa \geqq 0, \quad \lambda_\varkappa \geqq 1, \quad \mu_\varkappa \geqq 0, \tag{54}$$

$$j_1 + \ldots + j_p + \mu_1 + \ldots + \mu_r = k, \tag{55}$$

$$i_1 j_1 + \ldots + i_p j_p + \lambda_1 \mu_1 + \ldots \lambda_r \mu_r = s. \tag{56}$$

As the functions $x^*(\xi, \eta)$, $y^*(\xi, \eta)$ depend on ξ only linearly (see (23)) the expression for the s-th normal derivative on the side $R_1 R_3$ is simpler:

$$\frac{\partial^s w^*}{\partial\xi^s}(0, \eta) = \sum_{j=0}^{s} \bar{\varrho}_{sj}(\eta)\, g_j^{(s-j)}(\eta) \quad (1 \leqq s \leqq m) \tag{57}$$

where

$$\bar{\varrho}_{sj}(\eta) = \sum_{i=0}^{s} \bar{\sigma}_{sji}(\alpha_3, \beta_3)\left[\frac{\partial x^*}{\partial\xi}(0, \eta)\right]^{s-i}\left[\frac{\partial y^*}{\partial\xi}(0, \eta)\right]^i. \tag{58}$$

Each $\bar{\sigma}_{sji}(\alpha_3, \beta_3)$ is a linear combination of the monomials $\alpha_3^a \beta_3^b$ $(a + b = s)$.

Finally, the relations (30), (32), (33) imply

$$w^*(\xi, 0) = f_0(\xi), \quad w^*(0, \eta) = g_0(\eta). \tag{59}$$

It follows from (23), (51), (52), (55), (57), (58), (59) that $\partial^s w^*(\xi, 0)/\partial\eta^s$ is a polynomial of degree $4m + 1$ and $\partial^s w^*(0, \eta)/\partial\xi^s$ a polynomial of degree $4m + 1 + (n - 2) s$ where $s = 0, 1, \ldots, m$ and $n = \max(n_1, n_2)$, n_1 and n_2 being the degrees of the polynomials $x^*(\xi, \eta)$ and $y^*(\xi, \eta)$, respectively. Thus, if $w^*(\xi, \eta)$ is a polynomial it is at least of degree N^*,

$$N^* = 4m + 1 + (n - 1) m . \tag{60}$$

We shall now construct a polynomial $w^*(\xi, \eta)$ of degree N^* which satisfies the relations (51), (57) and (59). This will be done by means of the following theorem which is a consequence of the theorem introduced in [4], p. 31.

Theorem 3. *A polynomial $w^*(\xi, \eta)$ of degree $4m + 1 + (n - 1) m$ is uniquely determined by the following parameters*:

$$D^\alpha w^*(R_i) , \quad |\alpha| \leqq 2m ; \quad i = 1, 2, 3 \tag{61}$$

$$w^*(R_{0j}) , \qquad j = 1, 2, \ldots, M ; \quad M = mn(mn - 1)/2 \tag{62}$$

$$\frac{\partial^k w^*}{\partial v_{ij}^k} (Q_{ij}^{(r,s)}) , \quad i = 1, 2 ; \quad j = 2, 3; ; \quad i < j ; \quad r = 1, \ldots, s ; \tag{63}$$

$$s = (n - 1) m + k ; \quad k = 0, 1, \ldots, m$$

where R_i $(i = 1, 2, 3)$ are the vertices of the triangle $\bar{T}_0$, $\partial/\partial v_{ij}$ the derivative in the direction of the normal to the side R_iR_j, $Q_{ij}^{(1,s)}, \ldots, Q_{ij}^{(s,s)}$ the points dividing the side R_iR_j into $s + 1$ equal parts and R_{0j} $(j = 1, \ldots, M)$ points lying in the interior T_0 of $\bar{T}_0$ and ordered in the same way as M integers in the Pascal triangle.

As $w^*(\xi, \eta)$ is connected with the function $w(x, y)$ by the relation (30) we can express the parameters (61) in the form of a linear combination of the prescribed parameters (28).

To get the parameters (62) let us prescribe in the interior T^* of $\bar{T}^*$ the parameters

$$w(P_{0j}) , \quad j = 1, \ldots, M \tag{64}$$

where P_{0j} is the image of the point R_{0j} obtained by the mapping (23). Then, according to (30), $w^*(R_{0j}) = w(P_{0j})$.

As the parameters uniquely determining the polynomials $f_k(\xi)$, $g_k(\eta)$ $(k = 0, 1, \ldots$ $\ldots, m)$ are known (see the text following (34), (35)) we can express $f_k(\xi)$ and $g_k(\eta)$ explicitly. The explicit expression for $\partial^k w^*(\xi, 0)/\partial\eta^k$ and $\partial^k w^*(0, \eta)/\partial\xi^k$ $(k = 0, \ldots, m)$ can be then easily obtained by means of (51), (57) and (59). Using these expressions we find the parameters (63) on the sides R_1R_2 and R_1R_3. Each of these parameters is a linear combination of the parameters (28).

As to the parameters (63) on the side R_2R_3 we are allowed to prescribe them quite arbitrarily. Let us prescribe them in such a way that they are linear combinations of the parameters (28). This means that we require the function $\partial^k w^*(1 - \eta, \eta)/ /\partial v_{23}^k$ to be a polynomial of degree $4m + 1 - 2k$ $(k = 0, 1, \ldots, m)$.

Let us order the parameters (28), (64) in a d-dimensional column vector Δ, $d = = 3(m + 1)(2m + 1) + mn(mn - 1)/2$, and the parameters (61), (62), (63) in a d^*-dimensional column vector Δ^*, $d^* = [4m + 2 + (n - 1)m][4m + 3 + + (n - 1)m]/2$. The preceding considerations imply

$$\Delta^* = \mathbf{L}\Delta \,, \tag{65}$$

where the $(d^* \times d)$-matrix $\mathbf{L}$ depends on the co-ordinates of the vertices P_i of $\bar{T}^*$, on the values of the polynomials $p_1(\eta)$, $p_2(\eta)$ and their derivatives at some discrete points and on the ordering of the parameters in the vectors Δ^*, Δ only.

Theorem 4. *Let a vector Δ be given and let $w^*(\xi, \eta)$ be the polynomial of degree $4m + 1 + (n - 1)m$ uniquely determined by the components of the vector* (65). *Then the function*

$$w(x, y) = w^*(\xi^*(x, y), \eta^*(x, y)) \tag{66}$$

has the properties (a)–(c).

Proof. To prove (a) let us set $\Delta = \mathbf{0}$. Then, according to (65), $\Delta^* = \mathbf{0}$ which implies $w^*(\xi, \eta) \equiv 0$. The relation (66) then gives $w(x, y) \equiv 0$.

The property (c) follows immediately from (66) and from the fact that the mapping (23) maps $\bar{T}_0$ one-to-one onto $\bar{T}^*$.

To prove (b) let us set

$$F_k(\xi) = \left(\left(\bar{y}_2 \frac{\partial}{\partial x} - \bar{x}_2 \frac{\partial}{\partial y}\right)^k w\right)(x^*(\xi, 0), y^*(\xi, 0)) \quad (k = 0, 1, \ldots, m)\,, \tag{67}$$

$$G_k(\eta) = \left(\left(\bar{y}_3 \frac{\partial}{\partial x} - \bar{x}_3 \frac{\partial}{\partial y}\right)^k w\right)(x^*(0, \eta), y^*(0, \eta)) \quad (k = 0, 1, \ldots, m)\,. \tag{68}$$

The considerations introduced in (36)–(58) hold even if $f_k(\xi)$, $g_k(\eta)$ are not polynomials (they do not depend on the form of the functions $f_k(\xi)$, $g_k(\eta)$). Thus it holds

$$\frac{\partial^s w^*}{\partial \eta^s}(\xi, 0) = \sum_{k=1}^{s} \sum_{j=0}^{k} \varrho_{skj}(\xi)\, F_j^{(k-j)}(\xi)\,, \tag{69}$$

$$\frac{\partial^s w^*}{\partial \xi^s}(0, \eta) = \sum_{j=0}^{s} \bar{\varrho}_{sj}(\eta)\, G_j^{(s-j)}(\eta)\,. \tag{70}$$

It follows from (66), (67), (68) and from the linearity of the transformation (23) along the sides R_1R_2, R_1R_3 that

$$F_0(\xi) = f_0(\xi)\,, \quad G_0(\eta) = g_0(\eta)\,.$$

If we set now in (69), (70) successively $s = 1, 2, \ldots, m$ and compare the obtained relations with the relations (51), (57) for $s = 1, 2, \ldots, m$ we get

$$F_j(\xi) = f_j(\xi), \quad G_j(\eta) = g_j(\eta), \quad j = 1, 2, \ldots, m$$

which was to be proved.

3. INTERPOLATION THEOREM

Let $\overline{T}$ be a curved triangle the geometry of which is described in Theorem 2 (we omit the star in this section). We say that a function $w(x, y)$ is the $C^m(T)$-interpolate of a sufficiently smooth function $u(x, y)$ if

$$(71) \qquad D^\alpha w(P_i) = D^\alpha u(P_i), \quad |\alpha| \leqq 2m, \quad i = 1, 2, 3$$

$$(72) \qquad w(P_{0j}) = u(P_{0j}), \quad j = 1, \ldots, M; \quad M = mn(mn - 1)/2$$

and if the function w is constructed according to (66) and the assumption of Theorem 4.

It should be noted that the definition of the $C^m(T)$- interpolate includes the assumption that h_T is sufficiently small.

Theorem 5. *Let $u(x, y) \in H^k(T)$ where $2m + 2 \leqq k \leqq 3m + 2$. Let $w(x, y)$ be the $C^m(T)$-interpolate of the function $u(x, y)$. Then, for $0 \leqq s \leqq k$ and sufficiently small h_T,*

$$(73) \qquad \|u - w\|_{s,T} \leqq C h_T^{k-s} \|u\|_{k,T}$$

where the constant C does not depend on the triangle $\overline{T}$ and the function $u(x, y)$.

Proof. The proof (as well as the formulation of Theorem 5) is a modification and generalization of the proof of [1, Th. 2].

a) Let us set

$$(74) \qquad u^*(\xi, \eta) = u(x^*(\xi, \eta), y^*(\xi, \eta)).$$

Then the relations (71), (72) imply

$$(75) \qquad D^\alpha w^*(R_i) = D^\alpha u^*(R_i), \quad |\alpha| \leqq 2m; \quad i = 1, 2, 3$$

$$(76) \qquad w^*(R_{0j}) = u^*(R_{0j}), \quad j = 1, 2, \ldots, M$$

where the function $w^*(\xi, \eta)$ is defined by (30). Let us consider the linear functional

$$(77) \qquad F(u^*) = (u^* - w^*, v)_{s,T_0}$$

where $(\cdot, \cdot)_{s,T_0}$ means the scalar product in $H^s(T_0)$ and v is an arbitrary function from $H^s(T_0)$. We shall prove

$$(78) \qquad F(u^*) = 0\,, \quad \forall u^* \in P(k-1)$$

where $P(k-1)$ denotes the set of all polynomials of degree at most $k-1$.

The following relations are obtained in the same way as the relations (69), (70)

$$(79) \qquad \frac{\partial^r u^*}{\partial \eta^r}(\xi, 0) = \sum_{p=1}^{r} \sum_{j=0}^{p} \varrho_{rpj}(\xi)\, l_{12}^j \frac{\mathrm{d}^{p-j}}{\mathrm{d}\xi^{p-j}} \left[\frac{\partial^j u}{\partial n_{12}^j}(x_1 + \bar{x}_2\xi, y_1 + \bar{y}_2\xi)\right],$$

$$(80) \qquad \frac{\partial^r u^*}{\partial \xi^r}(0, \eta) = \sum_{j=0}^{r} \bar{\varrho}_{rj}(\eta)\, l_{13}^j \frac{\mathrm{d}^{r-j}}{\mathrm{d}\eta^{r-j}} \left[\frac{\partial^j u}{\partial n_{13}^j}(x_1 + \bar{x}_3\eta, y_1 + \bar{y}_3\eta)\right]$$

where $\varrho_{rpj}(\xi)$ and $\bar{\varrho}_{rj}(\eta)$ are given by (52) and (58), respectively, and where $r = 1, \ldots$ $\ldots, m$. Let $u^* \in P(k-1)$. Then the left-hand sides of (79), (80) are polynomials of degree at most $k - r - 1$. The linearity of the transformation (23) along the side R_1R_j $(j = 2, 3)$ implies that $u(x_1 + \bar{x}_j\zeta, y_1 + \bar{y}_j\zeta)$ is a polynomial of degree at most $k-1$, where $\zeta = \xi$ if $j = 2$ and $\zeta = \eta$ if $j = 3$. Setting successively $r = 1, \ldots, m$ in (79) and (80) we find that $(\partial^r u/\partial n_{1j}^r)(x_1 + \bar{x}_j\zeta, y_1 + \bar{y}_j\zeta)$ is a polynomial of degree at most $k - r - 1$ $(r = 1, \ldots, m)$.

It holds $k \leqq 3m + 2$. Thus $(\partial^r u/\partial n_{1j}^r)(x_1 + \bar{x}_j\zeta, y_1 + \bar{y}_j\zeta)$ $(r = 0, 1, \ldots, m)$ is a polynomial of degree at most $3m + 1 - r = 4m + 1 - (m + r)$. As $r \leqq m$ we see that $\partial^r u/\partial n_{1j}^r$ is on P_1P_j a polynomial of degree at most $4m + 1 - 2r$ $(r = 0, 1, \ldots$ $\ldots, m)$. This polynomial is uniquely determined by the parameters (71) prescribed at the points P_1, P_j. Thus

$$(81) \qquad \frac{\partial^r u}{\partial n_{1j}^r} = \frac{\partial^r w}{\partial n_{1j}^r} \quad \text{on} \quad P_1P_j \quad (r = 0, 1, \ldots, m\,;\; j = 2, 3)\,.$$

The relations (51), (57), (79), (80), (81) imply for $r = 0, 1, \ldots, m$

$$(82) \qquad \frac{\partial^r u^*}{\partial \eta^r}(\xi, 0) = \frac{\partial^r w^*}{\partial \eta^r}(\xi, 0)\,, \quad \frac{\partial^r u^*}{\partial \xi^r}(0, \eta) = \frac{\partial^r w^*}{\partial \xi^r}(0, \eta)\,.$$

To derive a similar relation on the side R_2R_3, i.e. the relation

$$(83) \qquad \frac{\partial^r u^*}{\partial v_{23}^r}(1 - \eta, \eta) = \frac{\partial^r w^*}{\partial v_{23}^r}(1 - \eta, \eta) \quad (r = 0, 1, \ldots, m)\,,$$

is simpler: According to the definition (see the text following Theorem 3), the right-hand side of (83) is a polynomial of degree $4m + 1 - 2r$. As $k \leqq 3m + 2$, $r \leqq m$ the left-hand side of (83) is a polynomial of degree at most $4m + 1 - 2r$. Thus (83) follows from (75).

The relations (75), (76), (82), (83) and Theorem 3 imply $u^* = w^*$. Since $F(u^*)$ is of the form (77), the relation (78) holds.

b) It follows from (77) that

$$|F(u^*)| \leqq \|v\|_{s,T_0} (\|u^*\|_{k,T_0} + \|w^*\|_{k,T_0}) . \tag{84}$$

We estimate now $\|w^*\|_{k,T_0}$ by means of $\|u^*\|_{k,T_0}$. It holds

$$w^*(\xi, \eta) = \sum_{i=1}^{d^*} a_i \, r_i(\xi, \eta) \tag{85}$$

where $a_1, \ldots, a_{d^*}$ are the parameters uniquely determining the polynomial $w^*(\xi, \eta)$ of degree N^* (see Theorem 3) and $r_i(\xi, \eta)$ $(i = 1, \ldots, d^*)$ are the basis functions. As the polynomials $r_i(\xi, \eta)$ are defined on the fixed triangle $\overline{T}_0$ they are bounded together with their partial derivatives by absolute constants. Thus it holds

$$\|w^*\|_{k,T_0} \leqq \sum_{i=1}^{d^*} |a_i| \cdot \|r_i\|_{k,T_0} \leqq C \sum_{i=1}^{d^*} |a_i| . \tag{86}$$

As $k \geqq 2m + 2$ and the relations (75), (76) hold we get for the parameters a_j prescribed at the vertices R_1, R_2, R_3 and in the interior T_0 of $\overline{T}_0$ the following estimate by means of the Sobolev lemma:

$$|a_j| \leqq C \|u^*\|_{k,T_0} . \tag{87}$$

In (86), (87) and in the following text we denote by C an absolute constant not necessarily the same in any two places.

The estimate (87) holds also for all parameters which are linear combinations of the parameters (75), i.e. for the function values on all three sides $R_i R_j$ and for all normal derivatives on the side $R_2 R_3$. It remains to estimate the normal derivatives on the sides $R_1 R_j$ $(j = 2, 3)$.

According to (51), it holds

$$\left| \frac{\partial^r w^*}{\partial \eta^r} (\xi, 0) \right| \leqq \sum_{p=1}^{r} \sum_{j=0}^{p} |\varrho_{rpj}(\xi)| \cdot |f_j^{(p-j)}(\xi)| . \tag{88}$$

First we estimate $|\varrho_{rpj}(\xi)|$. The assumption (3) together with the sine theorem gives $(\bar{x}_j^2 + \bar{y}_j^2)^{1/2} \geqq h_T \sin \vartheta_0$. As $\bar{x}_j = O(h_T)$, $\bar{y}_j = O(h_T)$ we obtain from (49) $\alpha_j = O(h_T^{-1})$, $\beta_j = O(h_T^{-1})$. Thus $\sigma_{pji}(\alpha_2, \beta_2) = O(h_T^{-p})$. Using (25) and (56) we find $\tau_{rpij}(\xi) = O(h_T^r)$. Hence

$$|\varrho_{rpj}(\xi)| \leqq C h_T^{r-p} \leqq C \tag{89}$$

because $r \geqq p$ and h_T is sufficiently small. The estimates (88) and (89) give

$$\left| \frac{\partial^r w^*}{\partial \eta^r} (\xi, 0) \right| \leqq C \sum_{p=1}^{r} \sum_{j=0}^{p} |f_j^{(p-j)}(\xi)| \quad (r = 1, \ldots, m) . \tag{90}$$

In the same way we can obtain from (57) the estimate

$$\left|\frac{\partial^r w^*}{\partial \xi^r}(0, \eta)\right| \leqq C \sum_{j=0}^{r} |g_j^{(r-j)}(\eta)| \quad (r = 1, \ldots, m). \tag{91}$$

As the functions $f_0(\xi)$ and $g_0(\eta)$ are Hermite interpolation polynomials of degree $4m + 1$ uniquely determined by linear combinations of the parameters (75) it holds, according to the Sobolev lemma,

$$|f_0^{(j)}(\xi)| \leqq C\|u^*\|_{k,T_0}, \quad \xi \in [0, 1], \quad j = 1, \ldots, m, \tag{92}$$

$$|g_0^{(j)}(\eta)| \leqq C\|u^*\|_{k,T_0}, \quad \eta \in [0, 1], \quad j = 1, \ldots, m. \tag{93}$$

According to (b) and (32), $f_1(\xi)$ is an Hermite interpolation polynomial of degree $4m - 1$. In order to estimate $|f_1^{(j)}(\xi)|$ where $\xi \in [0, 1]$ and $j = 0, \ldots, m - 1$, it suffices to estimate $|f_1^{(i)}(0)|$ and $|f_1^{(i)}(1)|$ $(i = 0, 1, \ldots, 2m - 1)$. We have, according to (32) and (71),

$$f_1^{(j)}(0) = \bar{x}_2^j \bar{y}_2 \frac{\partial^{j+1} u}{\partial x^{j+1}}(P_1) + \ldots - \bar{x}_2 \bar{y}_2^j \frac{\partial^{j+1} u}{\partial y^{j+1}}(P_1), \quad j \leqq 2m - 1. \tag{94}$$

If we express the derivatives $D^\alpha u(P_1)$ $(|\alpha| = j + 1)$ by means of the derivatives $D^\alpha u^*(R_1)$ $(1 \leqq |\alpha| \leqq j + 1)$, i.e. if we use relations of the type

$$\frac{\partial^{j+1} u}{\partial x^i \partial y^{j+1-i}}(P_1) = \frac{\partial^{j+1} u^*}{\partial \xi^{j+1}}(R_1)\left(\frac{\partial \xi^*}{\partial x}(P_1)\right)^i \left(\frac{\partial \xi^*}{\partial y}(P_1)\right)^{j+1-i} + \ldots +$$
$$+ \frac{\partial u^*}{\partial \eta}(R_1) \frac{\partial^{j+1} \eta^*}{\partial x^i \partial y^{j+1-i}}(P_1)$$

we obtain from (94), according to (26) and $\bar{x}_j = O(h_T)$, $\bar{y}_j = O(h_T)$,

$$|f_1^{(j)}(0)| \leqq C\{\sum_{i=1}^{j+1} h_T^{j+1-i} \sum_{|\alpha|=i} |D^\alpha u^*(R_1)|\} \tag{95}$$

where $j = 0, \ldots, 2m - 1$. In the same way we obtain

$$|f_p^{(j)}(0)| \leqq C\{\sum_{i=1}^{j+p} h_T^{j+p-i} \sum_{|\alpha|=i} |D^\alpha u^*(R_1)|\}, \quad 0 \leqq j \leqq 2m - p. \tag{96}$$

As h_T is sufficiently small we get from (96) by means of the Sobolev lemma

$$|f_p^{(j)}(0)| \leqq C\|u^*\|_{k,T_0}, \quad (j = 0, 1, \ldots, 2m - p). \tag{97}$$

In the same way we can estimate

$$|f_p^{(j)}(1)| \leqq C\|u^*\|_{k,T_0}, \quad (j = 0, 1, \ldots, 2m - p). \tag{98}$$

The relations (97), (98) give for the polynomial $f_p(\xi)$ of degree $4m + 1 - 2p$ and its derivatives the following estimate:

$$(99) \qquad |f_p^{(j)}(\xi)| \leqq C\|u^*\|_{k,T_0}, \quad \xi \in [0, 1], \quad j = 0, 1, \ldots, 2m - p, \quad p = 1, \ldots, m.$$

The estimates (90), (92) and (99) imply

$$(100) \qquad \left|\frac{\partial^r w^*}{\partial \eta^r}(\xi, 0)\right| \leqq C\|u^*\|_{k,T_0}, \quad \xi \in [0, 1]; \quad r = 1, \ldots, m.$$

Similarly,

$$(101) \qquad \left|\frac{\partial^r w^*}{\partial \xi^r}(0, \eta)\right| \leqq C\|u^*\|_{k,T_0}, \quad \eta \in [0, 1]; \quad r = 1, \ldots, m.$$

It follows from (100) and (101) that the estimate (87) holds also for the normal derivatives on the sides $R_1 R_j$ ($j = 2, 3$). The estimates (84), (86), (87) then imply

$$(102) \qquad |F(u^*)| \leqq C\|v\|_{s,T_0} \|u^*\|_{k,T_0}.$$

c) Now we use the Bramble-Hilbert lemma which we formulate for the purpose of this paper as follows (cf. [3, Ch. 8, Th. 3]):

Lemma 1. *Let Ω be an open subset of E_n with a Lipschitz-continuous boundary. Let f be a continuous linear functional on the space $W_p^{(k)}(\Omega)$ $(1 \leqq p \leqq \infty)$ with the property*

$$f(v) = 0, \quad \forall v \in P(k - 1).$$

Then there exists a constant $C(\Omega)$ such that

$$|f(v)| \leqq C(\Omega) \|f\|^*_{k,p,\Omega} |v|_{k,p,\Omega}, \quad \forall v \in W_p^{(k)}(\Omega)$$

where $\|\cdot\|^*_{k,p,\Omega}$ is the norm in the dual space of $W_p^{(k)}(\Omega)$.

According to (102), the linear functional $F(u^*)$ is continuous with the norm equal to or less than $C\|v\|_{r,T_0}$. As (78) holds Lemma 1 gives

$$(103) \qquad |F(u^*)| \leqq C\|v\|_{s,T_0} |u^*|_{k,T_0}, \quad \forall u^* \in H^k(T_0).$$

Choosing $v = u^* - w^*$ we get from (77) and (103)

$$(104) \qquad \|u^* - w^*\|_{s,T_0} \leqq C|u^*|_{k,T_0}.$$

The theorem on transformation of multiple integrals and Theorem 2 imply the estimates

(105) $$\|u - w\|_{s,T} \leqq C h_T^{1-s} \|u^* - w^*\|_{s,T_0},$$

(106) $$|u^*|_{k,T_0} \leqq C h_T^{k-1} \|u\|_{k,T}.$$

Combining the inequalities (104), (105), (106) we obtain the estimate (73). Theorem 5 is proved.

4. THE MODEL PROBLEM. V_h-INTERPOLATES

In this section we use essentially the approach of Ciarlet and Raviart [2] and Ciarlet [3] which we generalize to the case of elliptic equations of order $2(m + 1)$.

We shall consider as a model problem the Dirichlet problem

(107) $$Au \equiv (-1)^{m+1} \sum_{|\alpha|,|\beta|=m+1} D^\alpha(a_{\alpha\beta} D^\beta u) = f \quad \text{in} \quad \Omega,$$

(108) $$D^\alpha u|_\Gamma = 0, \quad |\alpha| \leqq m$$

where Γ is a smooth boundary of Ω and $a_{\alpha\beta}(= a_{\beta\alpha})$, f are sufficiently smooth functions (the smoothness will be specified later). The weak solution of the problem (107), (108) is a function $u \in V = H_0^{m+1}(\Omega) \equiv \mathring{W}_2^{(m+1)}(\Omega)$ satisfying

(109) $$a(u, v) = l(v), \quad \forall v \in V$$

where

(110) $$a(u, v) = \sum_{|\alpha|,|\beta|=m+1} \iint_\Omega a_{\alpha\beta}(D^\alpha u)(D^\beta v)\, dx\, dy,$$

(111) $$l(v) = \iint_\Omega fv\, dx\, dy.$$

We assume that there exists a constant $\mu > 0$ such that the inequality

(112) $$\sum_{|\alpha|,|\beta|=m+1} a_{\alpha\beta}(x, y)\, \xi_\alpha \xi_\beta \geqq \mu \sum_{|\alpha|=m+1} \xi_\alpha^2$$

holds for arbitrary $(x, y) \in \bar{\Omega}$ and for arbitrary values of ξ_α. Using the Friedrichs inequality, we see from (110) and (112) that the form $a(u, v)$ is V-elliptic.

Piecing together in the triangulation τ_h of Ω_h the generalized Bell's C^m-elements and the curved C^m-elements described in Section 2 let us construct a finite dimensional space V_h which satisfies

(113) $$V_h \subset H_0^{m+1}(\Omega_h).$$

The natural way of defining the discrete problem associated with the space V_h consists in finding a function $\tilde{u}_h \in V_h$ such that

$$\tilde{a}_h(\tilde{u}_h, v_h) = \tilde{l}_h(v_h), \quad \forall v_h \in V_h \tag{114}$$

with

$$\tilde{a}_h(v, w) = \iint_{\Omega_h} \sum_{|\alpha|,|\beta|=m+1} \tilde{a}_{\alpha\beta}(D^\alpha v)(D^\beta w) \, dx \, dy, \tag{115}$$

$$\tilde{l}_h(v) = \iint_{\Omega_h} \tilde{f} v \, dx \, dy \tag{116}$$

where the functions $\tilde{a}_{\alpha\beta}$ and $\tilde{f}$ are extensions of the functions $a_{\alpha\beta}$ and f to the domain Ω_h, when $\Omega_h \not\subset \Omega$. In other words, we are actually approximating the solution of the problem

$$\tilde{A}_h u \equiv (-1)^{m+1} \sum_{|\alpha|,|\beta|=m+1} D^\alpha(\tilde{a}_{\alpha\beta} D^\beta u) = \tilde{f} \quad \text{in} \quad \Omega_h, \tag{117}$$

$$D^\alpha u|_{\Gamma_h} = 0, \quad |\alpha| \leqq m \tag{118}$$

where Γ_h is the boundary of Ω_h.

As we shall consider a family of discrete problems, with the defining parameter h approaching zero in the limit, we shall make a natural assumption that there exists a bounded domain $\tilde{\Omega}$ such that

$$\Omega \subset \tilde{\Omega} \quad \text{and} \quad \Omega_h \subset \tilde{\Omega} \quad \text{for all } h. \tag{119}$$

By the Tietze-Urysohn theorem the functions $a_{\alpha\beta}$ and f have (non unique) continuous extension to E_2. The extension $\tilde{f}$ of f will be defined by (131). As to $\tilde{a}_{\alpha\beta}$ we shall use in (115) those of the extensions of $a_{\alpha\beta}$ which extend also the validity of (112) to a domain $\hat{\Omega}^1 \supset \bar{\Omega}$ (the reason will be explained in Section 5, Corollary 1). It is irrelevant that these extensions are not unique because the results will not depend upon them. (This will be a consequence of using approximate integration with integration points lying in $\bar{\Omega}$.) So let us assume that a certain choice of the extensions $\tilde{a}_{\alpha\beta}$ has has been made once for all. Then the bilinear form $\tilde{a}_h(v, w)$ defined over $H^{m+1}(\Omega_h) \times H^{m+1}(\Omega_h)$ by (115) is continuous and moreover, by (119) there exists a constant $\tilde{M}$ depending on $\tilde{\Omega}$ only such that

$$|\tilde{a}_h(u, v)| \leqq \tilde{M} \|u\|_{m+1,\Omega_h} \|v\|_{m+1,\Omega_h}, \quad \forall u, v \in H^{m+1}(\Omega_h). \tag{120}$$

If $\Omega_h \not\subset \Omega$ we do not know the values of $\tilde{a}_{\alpha\beta}, \tilde{f}$ in $\Omega_h - \Omega$. In this case it is impossible to evaluate the integrals in (115) and (116). To avoid this difficulty we use numerical integration and make the basic assumption that for all h the integration points B_i lie in the set $\bar{\Omega}$. Then

$$\tilde{a}_{\alpha\beta}(B_i) = a_{\alpha\beta}(B_i), \quad \tilde{f}(B_i) = f(B_i). \tag{121}$$

Let us have at our disposal a numerical quadrature scheme over the unit triangle $\overline{T}_0$ (e.g., some of the conical product formulas which are known for arbitrary degree of precision [8])

$$\iint_{T_0} \varphi^*(\xi, \eta)\, d\xi\, d\eta \sim \sum_{i=1}^{I} \omega_i^* \varphi^*(B_i^*), \tag{122}$$

where ω_i^* are the coefficients and B_i^* the integration points of the formula. According to the theorem on the transformation of multiple integrals, we have

$$\iint_T \varphi(x, y)\, dx\, dy = \iint_{T_0} \varphi(x^*(\xi, \eta), y^*(\xi, \eta)) \left|J^*(\xi, \eta)\right| d\xi\, d\eta = \tag{123}$$

$$= \iint_{T_0} \varphi^*(\xi, \eta) \left|J^*(\xi, \eta)\right| d\xi\, d\eta .$$

The relations (122) and (123) imply

$$\iint_T \varphi(x, y)\, dx\, dy \sim \sum_{i=1}^{I} \omega_{i,T}\, \varphi(B_{i,T}) \tag{124}$$

with

$$\omega_{i,T} = \omega_i^* \left|J^*(B_i^*)\right|, \quad B_{i,T} = \left(x^*(B_i^*), y^*(B_i^*)\right). \tag{125}$$

Let us approximate the bilinear form (115) and the linear form (116) by means of (124), i.e., let us define a bilinear form $a_h(v, w)$ and a linear form $l_h(v)$ by

$$a_h(v, w) = \sum_{T \in \tau_h} \sum_{i=1}^{I_T} \omega_{i,T} \sum_{|\alpha|, |\beta| = m+1} \left(a_{\alpha\beta} D^\alpha v D^\beta w\right)(B_{i,T}), \tag{126}$$

$$l_h(v) = \sum_{T \in \tau_h} \sum_{i=1}^{I_T} \omega_{i,T} (fv)(B_{i,T}). \tag{127}$$

In (126) and (127) we use (121) and write $a_{\alpha\beta}$ and f instead of $\tilde{a}_{\alpha\beta}$ and $\tilde{f}$, respectively. The symbol I_T expresses that we may use different numerical quadrature schemes (124) on different triangles $T \in \tau_h$. (For more details see Section 5.)

We replace now the discrete problem (114) by the following discrete problem: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = l_h(v_h), \quad \forall v_h \in V_h . \tag{128}$$

In applications we always solve the problem (128). As all integration points $B_{i,T}$ belong to $\overline{\Omega}$ the discrete problem (128) is independent of the choice of the extensions $\tilde{a}_{\alpha\beta}, \tilde{f}$. Now we are ready to formulate an abstract error theorem.

Theorem 6. *Let a family of discrete problems* (128) *be given. We assume that* (119) *holds and that there exists a constant* $\gamma > 0$ *independent of* h *such that*

$$(129) \qquad a_h(v_h, v_h) \geqq \gamma \|v_h\|^2_{m+1,\Omega_h}, \quad \forall v_h \in V_h, \quad \forall h .$$

Then the unique solution u_h *of the discrete problem* (128) *satisfies the inequality*

$$(130) \qquad \|\tilde{u} - u_h\|_{m+1,\Omega_h} \leqq C \left[\sup_{w_h \in V_h} \frac{|\tilde{a}_h(\tilde{u}, w_h) - l_h(w_h)|}{\|w_h\|_{m+1,\Omega_h}} + \right.$$

$$\left. + \inf_{v_h \in V_h} \left\{ \|\tilde{u} - v_h\|_{m+1,\Omega_h} + \sup_{w_h \in V_h} \frac{|\tilde{a}_h(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_{m+1,\Omega_h}} \right\} \right]$$

where $\tilde{u}$ *is any function in* $H^{m+1}(\tilde{\Omega})$ *and* C *is a constant independent of* $\tilde{u}$ *and* h.

We omit the proof because it follows the same lines as the proof of [2, Th. 1] and [3, Ch. 11, Th. 1]. We make now some comments to Theorem 6.

The assumption (129) is a "uniform" ellipticity assumption (i.e. independent of h) made upon the discrete problems. A sufficient condition for (129) will be given in Section 5.

When $\Omega_h = \Omega$ (and thus $\tilde{\Omega} = \Omega$) we set $\tilde{u} = u$ where u is the solution of the problem (107), (108). When $\Omega_h \not\subset \Omega$ a natural candidate for the function $\tilde{u}$ of Theorem 6 will be any extension $\tilde{u} \in H^{m+1}(\tilde{\Omega})$ of the function u. Notice that in this case

$$(131) \qquad \tilde{A}_h \tilde{u} = f \quad \text{in} \quad \Omega$$

so that $\tilde{A}_h \tilde{u}$ is an extension of f to Ω_h. This fact enables us to explain the meaning of the first term on the right-hand side of (130): It represents the contribution to the error of the numerical integration used on the right-hand side of (128). To prove it let us write, according to (116), (117) and (131),

$$(132) \quad \tilde{l}_h(w_h) = \iint_{\Omega_h} (\tilde{A}_h \tilde{u})\, w_h \,\mathrm{d}x\,\mathrm{d}y \equiv \iint_{\Omega_h} [(-1)^{m+1} \sum_{|\alpha|,|\beta|=m+1} D^\alpha(\tilde{a}_{\alpha\beta} D^\beta \tilde{u})] \,. \\ . \, w_h \,\mathrm{d}x\,\mathrm{d}y \,.$$

Let $w_h \in V_h$. Then, using Green's theorem and the relation $\tilde{a}_{\alpha\beta} = \tilde{a}_{\beta\alpha}$, we can write (132) in the form

$$(133) \qquad \tilde{l}_h(w_h) = \tilde{a}_h(\tilde{u}, w_h) \quad (w_h \in V_h) \,.$$

Thus

$$(134) \qquad \tilde{a}(\tilde{u}, w_h) - l_h(w_h) = \tilde{l}_h(w_h) - l_h(w_h)$$

which was to be proved.

The second term on the right-hand side of (130), i.e. $\inf \|\tilde{u} - v_h\|$, is a generalization of the usual term of the approximation theory: if $\Omega = \Omega_h$, $\tilde{u} = u$ and no numerical integration is used we obtain the well-known upper bound for the error.

The third term on the right-hand side of (130) represents the contribution to the error of the numerical integration used on the left-hand side of (128).

All three terms on the right-hand side of (130) will be estimated in Section 5. At the end of this section we mention the so called V_h-interpolates which will play an important role in Section 5.

The function $w(x, y)$ is called a $C^m(\Omega_h)$-interpolate of the function $u(x, y)$ if $w(x, y) \in \in C^m(\Omega_h)$ and is pieced together from the generalized Bell's C^m-elements and curved C^m-elements which on each triangle of τ_h interpolate the function $u(x, y)$.

The function $w(x, y)$ is called a V_h-interpolate of the function $u(x, y)$ if $w(x, y)$ belongs to V_h and is a $C^m(\Omega_h)$-interpolate of $u(x, y)$.

We want to know how to construct the domain Ω_h (or the finite dimensional space V_h) to be sure that every function satisfying the boundary conditions (108) and being sufficiently smooth has a V_h-interpolate. To this end let us consider an arbitrary boundary triangle $\bar{T}$ of the triangulation τ of Ω. As usual, we denote its curved side by P_2P_3. If follows from (108) that

$$D^\alpha u(\varphi(s), \psi(s)) = 0\,, \quad |\alpha| \leqq m\,, \quad s \in [s_2, s_3]\,. \tag{135}$$

The relations (135) imply

$$\frac{d^k}{ds^k} D^\alpha u(\varphi(s), \psi(s)) = 0\,, \quad |\alpha| = m\,; \quad k = 1, \ldots, m\,. \tag{136}$$

E.g., in the case $m = 1$ the relations (136) have the form

$$\varphi'(s) \frac{\partial^2 u}{\partial x^2}(\varphi(s), \psi(s)) + \psi'(s) \frac{\partial^2 u}{\partial x\, \partial y}(\varphi(s), \psi(s)) = 0\,,$$

$$\varphi'(s) \frac{\partial^2 u}{\partial x\, \partial y}(\varphi(s), \psi(s)) + \psi'(s) \frac{\partial^2 u}{\partial y^2}(\varphi(s), \psi(s)) = 0\,.$$

It should be noted that owing to (135) the relations

$$\frac{d^k}{ds^k} D^\alpha u(\varphi(s), \psi(s)) = 0\,, \quad |\alpha| < m\,; \quad k = 1, \ldots, \quad 2m - |\alpha|\,,$$

where $|\alpha| + k > m$, are linear combinations of the relations (136).

Let $r \geqq m$ in (19), (20). Let us set $s = s_i$ $(i = 2, 3)$ in (135), (136) and use the relations (19), (20) and the fact that $w(x, y)$, the $C^m(\Omega_h)$-interpolate of $u(x, y)$, satisfies

$$D^\alpha w(P_i) = D^\alpha u(P_i)\,, \quad |\alpha| \leqq 2m \quad (i = 2, 3)\,. \tag{137}$$

Then we obtain

$$D^\alpha w(P_i) = 0\,, \quad |\alpha| \leqq m\,, \quad i = 2, 3 \tag{138}$$

$$\frac{d^k}{dt^k} D^\alpha w(\varphi^*(t)\,, \psi^*(t))|_{t=t_2,t_3} = 0\,, \quad |\alpha| = m\,, \quad k = 1, \ldots, m\,. \tag{139}$$

E.g., in the case $m = 1$ the relations (139) have the form

$$\varphi^{*\prime}(t_i)\frac{\partial^2 w}{\partial x^2}(P_i) + \psi^{*\prime}(t_i)\frac{\partial^2 w}{\partial x\, \partial y}(P_i) = 0\,, \quad (i = 2, 3)\,,$$

$$\varphi^{*\prime}(t_i)\frac{\partial^2 w}{\partial x\, \partial y}(P_i) + \psi^{*\prime}(t_i)\frac{\partial^2 w}{\partial y^2}(P_i) = 0\,, \quad (i = 2, 3)\,.$$

The relations (138), (139) imply

$$(140) \qquad \frac{\partial^{k+j} w^*}{\partial \tau^k\, \partial v_{23}^j}(R_i) = 0\,, \quad k = 0, \ldots, 2m - j\,; \quad j = 0, \ldots, m\,; \quad i = 2, 3$$

where $\partial/\partial\tau$ is the derivative in the direction R_2R_3. (Details are even in the case $m = 1$ cumbersome and we do not introduce them.) As $\partial^j w^*/\partial v_{23}^j$ is a polynomial of degree $4m + 1 - 2j$ on R_2R_3 ($j = 0, 1, \ldots, m$) it follows from (140) that $\partial^j w^*/\partial v_{23}^j = 0$ on R_2R_3 ($j = 0, 1, \ldots, m$) and thus

$$(141) \qquad D^\alpha w^*(\xi, \eta) = 0\,, \quad |\alpha| \leqq m \quad \text{on} \quad R_2R_3\,.$$

Using Theorem 2 we obtain from (141) that $D^\alpha w(x, y) = 0$, $|\alpha| \leqq m$ on the curved side P_2P_3 of the triangle $\overline{T}^*$ which has the same vertices as the triangle $\overline{T}$. Thus we have obtained

Lemma 2. *Let $n \geqq 2m + 1$, i.e. let $r \geqq m$ in the relations* (19), (20). *Let a function $u(x, y)$ belong to $C^{2m}(\Omega)$ and satisfy the boundary conditions* (108). *Then the $C^m(\Omega_h)$-interpolate of $u(x, y)$ is the V_h-interpolate of $u(x, y)$.*

5. THE EFFECT OF NUMERICAL INTEGRATION

First we define the error functionals by

$$(142) \qquad E_T(\varphi) = \iint_T \varphi(x, y)\, \mathrm{d}x\, \mathrm{d}y - \sum_{i=1}^{I} \omega_{i,T}\, \varphi(B_{i,T})\,,$$

$$(143) \qquad E^*(\varphi^*) = \iint_{T_0} \varphi^*(\xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta - \sum_{i=1}^{I} \omega_i^*\, \varphi^*(B_i^*)\,.$$

According to (122)–(125), it holds

$$(144) \qquad E_T(\varphi) = E^*(\varphi^* J^*)\,.$$

Taking into account (115), (121), (126) and (142), we can write

$$(145) \qquad \tilde{a}_h(v_h, w_h) - a_h(v_h, w_h) = \sum_{T \in \tau_h} E_T\Big(\sum_{|\alpha|, |\beta| = m+1} \tilde{a}_{\alpha\beta} D^\alpha v_h D^\beta w_h\Big)\,.$$

We have with respect to (144)

$$(146)\qquad E_T\Big(\sum_{|\alpha|,|\beta|=m+1} \tilde{a}_{\alpha\beta} D^\alpha v_h D^\beta w_h\Big) = E^*\Big(\sum_{|\alpha|,|\beta|=m+1} \tilde{a}^*_{\alpha\beta} (D^\alpha v_h)^* (D^\beta w_h)^* J^*\Big) .$$

Our aim is now to express $(D^\alpha v)^*$ by means of $D^\gamma v^*$. According to the rule of differentiation of a composite function we have

$$(147)\qquad \frac{\partial^{\alpha_1+\alpha_2} w}{\partial x^{\alpha_1}\,\partial y^{\alpha_2}} = \frac{\partial^{\alpha_1+\alpha_2} w^*}{\partial \xi^{\alpha_1+\alpha_2}} \left(\frac{\partial \xi^*}{\partial x}\right)^{\alpha_1} \left(\frac{\partial \xi^*}{\partial y}\right)^{\alpha_2} + \ldots + \frac{\partial w^*}{\partial \eta}\,\frac{\partial^{\alpha_1+\alpha_2} \eta^*}{\partial x^{\alpha_1}\,\partial y^{\alpha_2}}$$

or in a more concise form

$$(148)\qquad D^\alpha w(x, y) = \sum_{1 \leqq |\gamma| \leqq m+1} b_{\alpha\gamma}(x, y)\, D^\gamma w^*(\xi, \eta)\,, \quad |\alpha| = m + 1$$

where the explicit expressions for $b_{\alpha\gamma}(x, y)$ can be obtained by comparing the right-hand sides of (147) and (148). If we express the derivatives $D^\alpha \xi^*(x, y)$, $D^\alpha \eta^*(x, y)$ in the same way as we did in the proof of Theorem 1 (see (13) etc.) we can transform the functions $b_{\alpha\gamma}(x, y)$ to the functions $b^*_{\alpha\gamma}(\xi, \eta)$. Then we obtain from (148):

$$(149)\qquad (D^\alpha w)^* = \sum_{1 \leqq |\gamma| \leqq m+1} b^*_{\alpha\gamma} D^\gamma w^*\,, \quad |\alpha| = m + 1\,.$$

Inserting (149) into (146) we have

$$(150)\qquad E_T\Big(\sum_{|\alpha|,|\beta|=m+1} \tilde{a}_{\alpha\beta} D^\alpha v_h D^\beta w_h\Big) =$$

$$= E^*\Big(\sum_{\substack{|\alpha|,|\beta|=m+1\\ 1\leqq|\gamma|,|\delta|\leqq m+1}} \tilde{a}^*_{\alpha\beta} b^*_{\alpha\gamma} b^*_{\beta\delta} J^* D^\gamma v^*_h D^\delta w^*_h\Big)\,.$$

Lemma 3. *It holds*

$$(151)\qquad D^\mu(b^*_{\alpha\gamma} b^*_{\beta\delta} J^*) = O(h_T^{2-|\gamma|-|\delta|+|\mu|})\,.$$

The relation (151) is a generalization of the relation (5.9) of [6] and can be proved by means of (24), (25) and (26). The following Lemma 4 is proved in [3, Ch. 8] and Lemma 5 in [11].

Lemma 4. *Let $\varphi \in W_q^{(k)}(T)$, $w \in W_\infty^{(k)}(T)$. Then the function $\varphi w \in W_q^{(k)}(T)$ satisfies*

$$(152)\qquad |\varphi w|_{k,q,T} \leqq C \sum_{j=0}^{k} |\varphi|_{k-j,q,T}\, |w|_{j,\infty,T}$$

where C is a constant depending only upon the integers k and q.

Lemma 5. *Let k be a given integer. There exists a constant C independent of $v^* \in P(k)$ such that*

$$(153) \qquad |v^*|_{j,T_0} \leqq C|v^*|_{i,T_0}, \quad 0 \leqq i \leqq j, \quad \forall v^* \in P(k),$$

$$(154) \qquad |v^*|_{j,\infty,T_0} \leqq C|v^*|_{j,T_0}, \quad j \geqq 0, \qquad \forall v^* \in P(k),$$

$P(k)$ being the space of all polynomials of degree not greater than k.

The proofs of the following two theorems are generalizations of the proof of [3, Ch. 11, Th. 4].

Theorem 7. *Let*

$$(155) \qquad E^*(\varphi^*) = 0, \quad \forall \varphi^* \in P(2N^* - 2(m + 1))$$

where $N^* = 4m + 1 + (n - 1)m$ for the curved elements (cf. (60)) and $N^* = 4m + 1$ for the interior elements. Then

$$(156) \qquad |E_T(\sum_{|\alpha|,|\beta|=m+1} \tilde{a}_{\alpha\beta} D^\alpha v_h D^\beta w_h)| \leqq Ch_T(\sum_{|\alpha|,|\beta|=m+1} \|\tilde{a}_{\alpha\beta}\|_{N^*-m,\infty,T}) \cdot \|v_h\|_{m+1,T} \|w_h\|_{m+1,T}$$

where C is a constant independent of h_T, v_h and w_h.

Theorem 8. *Let $r \geqq m + 1$ be a given integer and let*

$$(157) \qquad E^*(\psi^*) = 0, \quad \forall \psi^* \in P(r + N^* - (m + 2))$$

where N^* is the same as in Theorem 7. Then

$$(158) \qquad |E_T(\sum_{|\alpha|,|\beta|=m+1} \tilde{a}_{\alpha\beta} D^\alpha v_h D^\beta w_h)| \leqq \leqq Ch_T^r(\sum_{|\alpha|,|\beta|=m+1} \|\tilde{a}_{\alpha\beta}\|_{r,\infty,T}) \|v_h\|_{r+m+1,T} \|w_h\|_{m+1,T}$$

where C is a constant independent of h_T, v_h and w_h.

Proof of Theorems 7 and 8. A typical term on the right-hand side of (150) is of the form

$$(159) \qquad E^*(c^* D^\gamma v_h^* D^\delta w_h^*)$$

with $|\gamma| = |\delta| = m + 1$ for the interior elements and with $1 \leqq |\gamma|, |\delta| \leqq m + 1$ for the boundary elements. The function c^* is given by

$$(160) \qquad c^* = \tilde{a}_{\alpha\beta}^* b_{\alpha\gamma}^* b_{\beta\delta}^* J^* .$$

As v_h^*, $w_h^* \in P(N^*)$ it holds $D^\gamma v_h^* \in P(N^* - |\gamma|)$, $D^\delta w_h^* \in P(N^* - |\delta|)$.

Let us consider the form

$$(161) \qquad E^*(\varphi^* u^*)\,, \quad \varphi^* \in W_\infty^{(s)}(T_0)\,, \quad u^* \in P(N^* - |\delta|)$$

where in the case of Theorem 7

$$(162) \qquad s = N^* + |\delta| - 2m - 1$$

and in the case of Theorem 8

$$(163) \qquad s = r + |\delta| - m - 1\,.$$

It holds, according to (143) and (161),

$$(164) \qquad |E^*(\varphi^* u^*)| \leqq C|\varphi^*|_{0,\infty,T_0}\,|u^*|_{0,\infty,T_0}\,.$$

Using (154) and the inequality $|\varphi^*|_{0,\infty,T_0} \leqq \|\varphi^*\|_{s,\infty,T_0}$ we get from (164)

$$(165) \qquad |E^*(\varphi^* u^*)| \leqq C\|\varphi^*\|_{s,\infty,T_0}\,|u^*|_{0,T_0}\,.$$

For a given $u^* \in P(N^* - |\delta|)$ let us define a linear form $f(\varphi^*)$ on $W_\infty^{(s)}(T_0)$ by

$$(166) \qquad f(\varphi^*) = E^*(\varphi^* u^*)\,, \quad \forall \varphi^* \in W_\infty^{(s)}(T_0)\,.$$

The linear functional $f(\varphi^*)$ is continuous with the norm less than or equal to $C|u^*|_{0,T_0}$ on the one hand, and vanishes over $P(s - 1)$ on the other hand, by virtue of the assumptions (155), (157) and the expressions (162), (163). Therefore, using Lemma 1 we obtain

$$(167) \qquad |E^*(\varphi^* u^*)| \leqq C\,|\varphi^*|_{s,\infty,T_0}\,|u^*|_{0,T_0}$$
$$\forall \varphi^* \in W_\infty^{(s)}(T_0)\,, \quad \forall u^* \in P(N^* - |\delta|)\,.$$

Let us set $\varphi^* = g^* p^*$ with $g^* \in W_\infty^{(s)}(T_0)$, $p^* \in P(N^* - |\gamma|)$ and use Lemma 4 and Lemma 5. We obtain

$$(168) \qquad |\varphi^*|_{s,\infty,T_0} \leqq C \sum_{j=0}^{s} |g^*|_{s-j,\infty,T_0}\,|p^*|_{j,T_0}\,.$$

Let us set $g^* = c^*$, $p^* = D^\gamma v_h^*$, $u^* = D^\delta w_h^*$. Then, according to (167) and (168),

$$(169) \qquad |E^*(c^* D^\gamma v_h^* D^\delta w_h^*)| \leqq C\Big(\sum_{j=0}^{s} |c^*|_{s-j,\infty,T_0}\,|v_h^*|_{j+|\gamma|,T_0}\Big)\,|w_h^*|_{|\delta|,T_0}\,.$$

It holds

$$(170) \qquad |w_h^*|_{|\delta|,T_0} \leqq C|w_h^*|_{m+1,T_0} \leqq Ch_T^m\|w_h\|_{m+1,T}\,.$$

The first inequality (170) follows in the case of boundary elements from (108) and the Friedrichs inequality; in the case of interior elements we have $|\delta| = m + 1$. The second inequality (170) follows from (106).

Now we estimate $|c^*|_{s-j,\infty,T_0}$. It holds, according to (160) and lemmas 4 and 3,

$$|c^*|_{s-j,\infty,T_0} \leqq C \sum_{i=0}^{s-j} |\tilde{a}^*_{\alpha\beta}|_{s-j-i,\infty,T_0} |b^*_{\alpha\gamma} b^*_{\beta\delta} J^*|_{i,\infty,T_0} \leqq \tag{171}$$

$$\leqq C \sum_{i=0}^{s-j} h_T^{2-|\gamma|-|\delta|+i} |\tilde{a}^*_{\alpha\beta}|_{s-j-i,\infty,T_0} .$$

The estimates (25) imply

$$|\tilde{a}^*_{\alpha\beta}|_{s-j-i,\infty,T_0} \leqq C h_T^{s-j-i} \|\tilde{a}_{\alpha\beta}\|_{s-j-i,\infty,T} . \tag{172}$$

Combining the inequalities (171) and (172) we obtain

$$|c^*|_{s-j,\infty,T_0} \leqq C h_T^{2-|\gamma|-|\delta|+s-j} \|\tilde{a}_{\alpha\beta}\|_{s-j,\infty,T} . \tag{173}$$

The inequalities (169), (170) and (173) give

$$|E^*(c^* D^\gamma v_h^* D^\delta w_h^*)| \leqq \tag{174}$$

$$\leqq C h_T^{s+m+2-|\gamma|-|\delta|} \|\tilde{a}_{\alpha\beta}\|_{s,\infty,T} \|w_h\|_{m+1,T} \sum_{j=0}^{s} h_T^{-j} |v_h^*|_{j+|\gamma|,T_0} .$$

In the proof of Theorem 7 we shall distinguish two cases:

(i) If $|\gamma| = |\delta| = m + 1$ then $|v_h^*|_{s+|\gamma|} = |v_h^*|_{N^*+1} = 0$, according to (162), and we obtain, using (153) and (106),

$$\sum_{j=0}^{s} h_T^{-j} |v_h^*|_{j+|\gamma|,T_0} \leqq C h_T^{m+1-s} \|v_h\|_{m+1,T} . \tag{175}$$

Taking into account (150), (159) and (160), we see that the inequalities (174), (175) imply (156).

(ii) The situation $|\gamma| + |\delta| \leqq 2m + 1$ occurs only in the case of boundary elements. In this case we can use, besides (153) and (106), the Friedrichs inequality and obtain

$$\sum_{j=0}^{s} h_T^{-j} |v_h^*|_{j+|\gamma|,T_0} \leqq C h_T^{m-s} \|v_h\|_{m+1,T} . \tag{176}$$

The inequalities (174), (176) imply (156) because in this case $s \leqq N^* - m$. Theorem 7 is proved.

Now we prove Theorem 8. According to (106) we have

$$|v_h^*|_{j+|\gamma|,T_0} \leqq C h_T^{j+|\gamma|-1} \|v_h\|_{j+|\gamma|,T} \leqq C h_T^{j+|\gamma|-1} \|v_h\|_{j+m+1,T} . \tag{177}$$

Inserting (177) into (174) and using (163) we get (158) because in this case $s \leqq r$. Theorem 8 is proved.

Corollary 1. *Let* $\tilde{a}_{\alpha\beta} \in W_{\infty}^{(N^*-m)}(\tilde{\Omega})$ $(|\alpha|, |\beta| = m + 1)$ *where* N^* *is given by* (60). *Let the inequality* (112) *hold for arbitrary* $(x, y) \in \bar{\Omega}$ *and for arbitrary values of* ξ_α *and let the assumption* (155) *be satisfied. Then the inequality* (129) *holds for sufficiently small* h.

Proof. Let $\{\hat{\Omega}\}$ be the set of domains of the following properties: $\hat{\Omega} \supset \bar{\Omega}$; there exists a constant $\hat{\mu} > 0$ (depending on $\hat{\Omega}$) such that the inequality

$$\sum_{|\alpha|,|\beta|=m+1} \hat{a}_{\alpha\beta}(x, y)\, \xi_\alpha \xi_\beta \geqq \hat{\mu} \sum_{|\alpha|=m+1} \xi_\alpha^2 \tag{178}$$

holds for arbitrary $(x, y) \in \hat{\Omega}$ and for arbitrary values of ξ_α. The functions $\hat{a}_{\alpha\beta}$ are continuous extensions of the functions $a_{\alpha\beta}$ to the domain $\hat{\Omega}$. The existence of such domains $\hat{\Omega}$ follows from the assumptions of Corollary 1.

If $h_1 > 0$ is sufficiently small we can find $\hat{\Omega}^1 \in \{\hat{\Omega}\}$ such that

$$\Omega_h \subset \Omega^1 = \hat{\Omega}^1 \cap \tilde{\Omega}\,, \quad \forall h < h_1\,. \tag{179}$$

Let $S = \{(x, y) : |x| < a, |y| < a\}$ be such a square that $\Omega^1 \subset S$. Then, according to (179) and [7, pp. 13–14], it holds

$$|v|_{0,\Omega_h}^2 \leqq 4a^2 |v|_{1,\Omega_h}^2\,, \quad \forall v \in H_0^1(\Omega_h)\,, \quad \forall h < h_1\,. \tag{180}$$

In Section 4 the extensions $\tilde{a}_{\alpha\beta}$ were chosen in such a way that $\tilde{a}_{\alpha\beta} = \hat{a}_{\alpha\beta}$ in Ω^1. Thus (115), (178) and (180) imply

$$\tilde{a}_h(v, v) \geqq K \|v\|_{m+1,\Omega_h}^2\,, \quad \forall v \in H_0^{m+1}(\Omega_h)\,, \quad \forall h < h_1 \tag{181}$$

where the constant $K > 0$ is independent of v, Ω_h and h.

For the sake of brevity, let us set

$$B_1 = \sum_{|\alpha|,|\beta|=m+1} \|\tilde{a}_{\alpha\beta}\|_{N^*-m,\infty,\Omega^1}\,. \tag{182}$$

The inequality (156) then gives

$$\Big|E_T\Big(\sum_{|\alpha|,|\beta|=m+1} \tilde{a}_{\alpha\beta} D^\alpha v_h D^\beta v_h\Big)\Big| \leqq C B_1 h_T \|v_h\|_{m+1,T}^2 \tag{183}$$

which implies

$$-\sum_{T\in\tau_h} E_T\Big(\sum_{|\alpha|,|\beta|=m+1} \tilde{a}_{\alpha\beta} D^\alpha v_h D^\beta v_h\Big) \geqq -C B_1 h \|v_h\|_{m+1,\Omega_h}^2\,. \tag{184}$$

The relations (145), (181) and (184) give

$$a_h(v_h, v_h) \geqq (K - C B_1 h)\, \|v_h\|_{m+1,\Omega_h}^2\,. \tag{185}$$

Let us choose $h_2 = K/2CB_1$. Then the inequality (129) is satisfied with $\gamma = K/2$ for $h < \min(h_1, h_2)$. Corollary 1 is proved.

Corollary 2. *Let $\tilde{a}_{\alpha\beta} \in W_\infty^{(r)}(\tilde{\Omega})$ and let the extension $\tilde{u}$ of the solution of the problem* (107), (108) *belong to $H^{r+m+1}(\tilde{\Omega})$ where $m + 1 \leqq r \leqq 2m + 1$ is a given integer. Let* (157) *hold and $n \geqq 2m + 1$. Then*

$$\inf_{v_h \in V_h} \left\{ \sup_{w_h \in V_h} \frac{|\tilde{a}_h(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|_{m+1,\Omega_h}} + \right. \tag{186}$$

$$\left. + \|\tilde{u} - v_h\|_{m+1,\Omega_h} \right\} \leqq C(1 + B_2)\, h^r \|\tilde{u}\|_{r+m+1,\tilde{\Omega}}$$

where

$$B_2 = \sum_{|\alpha|,|\beta|=m+1} \|\tilde{a}_{\alpha\beta}\|_{r,\infty,\tilde{\Omega}} . \tag{187}$$

Proof. Let $\Pi_h\tilde{u}$ be the V_h-interpolate of the function $\tilde{u}$ which exists according to Lemma 2. Theorem 5 and a similar theorem for generalized Bell's elements [1, p. 819] imply

$$\|\tilde{u} - \Pi_h\tilde{u}\|_{m+1,\Omega_h} \leqq Ch^r \|\tilde{u}\|_{r+m+1,\tilde{\Omega}} . \tag{188}$$

Next, according to (145), (187) and Theorem 8, it holds for all $w_h \in V_h$

$$|\tilde{a}_h(\Pi_h\tilde{u}, w_h) - a_h(\Pi_h\tilde{u}, w_h)| \leqq CB_2 h^r \sum_{T \in \tau_h} \|\Pi_h\tilde{u}\|_{r+m+1,T}\, \|w_h\|_{m+1,T} . \tag{189}$$

Using Theorem 5 for $s = k = r + m + 1$ we get

$$\|\Pi_h\tilde{u}\|_{r+m+1,T} \leqq \|\tilde{u}\|_{r+m+1,T} + \|\tilde{u} - \Pi_h\tilde{u}\|_{r+m+1,T} \leqq C\|\tilde{u}\|_{r+m+1,T} . \tag{190}$$

According to (190) and the Schwarz inequality, it holds

$$\sum_{T \in \tau_h} \|\Pi_h\tilde{u}\|_{r+m+1,T}\, \|w_h\|_{m+1,T} \leqq C\|\tilde{u}\|_{r+m+1,\tilde{\Omega}}\, \|w_h\|_{m+1,\Omega_h} . \tag{191}$$

As $\Pi_h\tilde{u} \in V_h$ the relations (188), (189) and (191) imply (186). Corollary 2 is proved.

The following theorem is a slight modification and generalization of [3, Ch. 11, Th. 6]. The proof is therefore omitted.

Theorem 9. *Let the assumptions of Theorem 8 be satisfied. Then*

$$|E_T(\tilde{f}w_h)| \leqq Ch_T^r \|\tilde{f}\|_{r,T}\, \|w_h\|_{m+1,T} \tag{192}$$

where C is a constant independent of h_T, f and w_h.

The main result of the paper is formulated in the following theorem where the results of this section are summarized.

Theorem 10. *Let the inequality* (112) *hold for arbitrary $(x, y) \in \bar{\Omega}$ and for arbitrary values of ξ_α. Let the extensions $\tilde{u}$ and $\tilde{a}_{\alpha\beta}$ of u, the solution of the problem*

(107), (108), *and the coefficients* $a_{\alpha\beta}$, *respectively, to the domain* $\Omega^1 = \hat{\Omega}^1 \cap \tilde{\Omega}$ *satisfy*

$$\tilde{u} \in H^{3m+2}(\Omega^1), \tag{193}$$

$$\tilde{a}_{\alpha\beta} \in W_\infty^{(2m^2+3m+1)}(\Omega^1), \quad |\alpha|, |\beta| = m + 1, \tag{194}$$

$$\tilde{A}\tilde{u} \equiv (-1)^{m+1} \sum_{|\alpha|,|\beta|=m+1} D^\alpha(\tilde{a}_{\alpha\beta} D^\beta \tilde{u}) \in H^{2m+1}(\Omega^1). \tag{195}$$

Let the degree n of arcs from which Γ_h *consists be equal to* $2m + 1$. *Let the numerical quadrature schemes over the unit triangle* T_0 *preserve polynomials of degree* $2(n + 2) m$,

$$E^*(\varphi^*) = 0, \quad \forall \varphi^* \in P(2(n + 2)m) \tag{196}$$

with $n = 1$ *for generalized Bell's* C^m*-elements and* $n = 2m + 1$ *for curved triangular* C^m*-elements. Then for sufficiently small h the solution* u_h *of the discrete problem* (128) *exists and is unique and the following estimate holds*:

$$\|\tilde{u} - u_h\|_{m+1,\Omega_h} \leqq Ch^{2m+1}[\|\tilde{A}\tilde{u}\|_{2m+1,\Omega^1} + \\ + \|\tilde{u}\|_{3m+2,\Omega^1}(1 + \sum_{|\alpha|,|\beta|=m+1} \|\tilde{a}_{\alpha\beta}\|_{2m+1,\infty,\Omega^1})] \tag{197}$$

where C is a constant independent of h, $\tilde{u}$ *and* $\tilde{a}_{\alpha\beta}$.

Proof. According to (60), it holds $2N^* - 2(m + 1) = 2(n + 2) m$ and $N^* - m = (n + 2) m + 1 = 2m^2 + 3m + 1$. Thus, according to Corollary 1 the assumptions (194), (196) together with (112) imply for sufficiently small h the inequality (129). Thus the solution u_h of (128) exists and is unique.

As $P(2(n + 2) m) \supset P(r + N^* - m - 2)$ for $r \leqq N^* - m$ the assumption (157) holds with $r = 2m + 1$. Using (116) with $\tilde{f} = \tilde{A}\tilde{u}$, (127), (131), (134), (142) and Theorem 9 with $r = 2m + 1$ we obtain

$$\begin{aligned} |\tilde{a}_h(\tilde{u}, w_h) - l_h(w_h)| &\leqq \sum_{T \in \tau_h} |E_T(\tilde{A}\tilde{u} w_h)| \leqq \\ &\leqq Ch^{2m+1} \sum_{T \in \tau_h} \|\tilde{A}\tilde{u}\|_{2m+1,T} \|w_h\|_{m+1,T} \leqq \\ &\leqq Ch^{2m+1} \|\tilde{A}\tilde{u}\|_{2m+1,\Omega^1} \|w_h\|_{m+1,\Omega_h}. \end{aligned} \tag{198}$$

Inspecting the proof of Corollary 2 we see that Corollary 2 remains true if we replace $\tilde{\Omega}$ by Ω^1 in it. Thus the estimate (197) follows from Theorem 6, the estimate (198) and Corollary 2 with $r = 2m + 1$. Theorem 10 is proved.

Remark. There is a disproportion between the assumptions (155) and (157) which increases with m. This disproportion does not arise in two cases: (i) if Ω is a polygonal domain and we use triangular C^m-elements with full polynomials of an arbitrary degree $N \geqq 4m + 1$ (for their definition see [12]), (ii) if $m = 0$ and we use full polynomials of an arbitrary degree N on the interior triangles, while the curved boundary C^0-elements are of the same accuracy (cf. [3, Ch. 11]). In both cases the order of accuracy is $r = N - m$ and $N^* = N$ where N^* is the degree of the corresponding polynomial on T_0. Thus $2N^* - 2(m + 1) = r + N^* - (m + 2)$.

References

[1] *Bramble J. H., Zlámal M.:* Triangular elements in the finite element method. Math. Comp. 24 (1970), 809—820.

[2] *Ciarlet P. G., Raviart P. A.:* The combined effect of curved boundaries and numerical integration in isoparametric finite element methods. In: The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A. K. Aziz, Editor), pp. 409—474, Academic Press, New York 1972.

[3] *Ciarlet P. G.:* Numerical Analysis of the Finite Element Method. Université de Montréal, 1975.

[4] *Holuša L., Kratochvíl J., Zlámal M., Ženíšek A.:* The Finite Element Method. Technical Report. Computing Center of the Technical University of Brno, 1970. (In Czech.)

[5] *Kratochvíl J., Ženíšek A., Zlámal M.:* A simple algorithm for the stiffness matrix of triangular plate bending finite elements. Int. J. numer. Meth. Engng. 3 (1971), 553—563.

[6] *Mansfield L.:* Approximation of the boundary in the finite element solution of fourth order problems. SIAM J. Numer. Anal. 15 (1978), the June issue.

[7] *Nečas J.:* Les méthodes directes en théorie des équations elliptiques. Academia, Prague, 1967.

[8] *Stroud A. H.:* Approximate Calculation of Multiple Integrals. Prentice-Hall., Englewood Cliffs, N. J., 1971.

[9] *Zlámal M.:* The finite element method in domains with curved boundaries. Int. J. numer. Meth. Engng. 5 (1973), 367—373.

[10] *Zlámal M.:* Curved elements in the finite element method. I. SIAM J. Numer. Anal. 10 (1973), 229—240.

[11] *Zlámal M.:* Curved elements in the finite element method. II. SIAM J. Numer. Anal. 11 (1974), 347—362.

[12] *Ženíšek A.:* Interpolation polynomials on the triangle. Numer. Math. 15 (1970), 283—296.

Souhrn

ZAKŘIVENÉ TROJÚHELNÍKOVÉ KONEČNÉ C^m-PRVKY

ALEXANDER ŽENÍŠEK

V první části článku (odst. 1 a 2) je uvedena konstrukce funkcí $w(x, y)$ s těmito vlastnostmi:

(a) Funkce $w(x, y)$ je jednoznačně určena parametry (28) a (64).

(b) Na přímce P_1P_j $(j = 2, 3)$ je funkce $\partial^k w/\partial n_{1j}^k$ polynomem stupně $4m + 1 - 2k$ $(k = 0, 1, \ldots, m)$ jedné proměnné uvažované ve směru P_1P_j $(\partial/\partial n_{1j}$ je derivace podle normály k $P_1P_j)$.

(c) Funkce $w^*(\xi, \eta) = w(x^*(\xi, \eta), y^*(\xi, \eta))$ je polynomem stupně $N^* = 4m + + 1 + (n - 1)\,m$.

Přitom $x^*(\xi, \eta)$, $y^*(\xi, \eta)$ jsou polynomy stupně n (viz (23)) a $x = x^*(\xi, \eta)$, $y = y^*(\xi, \eta)$ je vzájemně jednoznačné zobrazení trojúhelníka $\bar{T}_0$ s vrcholy $R_1(0, 0)$, $R_2(1, 0)$, $R_3(0, 1)$, který leží v rovině ξ, η, na zakřivený trojúhelník $\bar{T}^*$, jehož strany P_1P_2, P_1P_3 jsou přímé a P_2P_3 oblouk stupně n daný rovnicemi (21).

Zakřivený trojúhelník $\bar{T}^*$ spolu s funkcí $w(x, y)$ tvoří tedy C^m-prvek, který je možno napojit na zobecněný Bellův C^m-prvek stupně $N = 4m + 1$.

V druhé části článku (odst. 3) je uveden interpolační teorém pro zakřivené C^m-prvky. Přesnost aproximace je stejná jako v případě zobecněných Bellových C^m-prvků, tj. $O(h^{3m+2-s})$ v normě prostoru $H^s(T^*)$.

V třetí části článku (odst. 4 a 5) jsou zakřivené trojúhelníkové C^m-prvky aplikovány na řešení Dirichletova problému (107), (108) eliptické rovnice řádu $2(m + 1)$. Hranice Γ oblasti Ω je aproximována křivkou Γ_h, která je sjednocením stran zakřivených trojúhelníkových C^m-prvků. Křivé strany těchto trojúhelníků jsou oblouky stupně $n = 2m + 1$, které mají v uzlových bodech triangulace tytéž derivace až do řádu m jako křivka Γ. To stačí k tomu, aby interpolant funkce, která splňuje okrajové podmínky (108), splňoval okrajové podmínky (118).

Bilineární a lineární forma diskrétního problému (128) jsou definovány pomocí kvadraturních formulí. Je-li stupeň přesnosti těchto formulí $2N - 2(m + 1)$ pro zobecněné Bellovy C^m-prvky a $2N^* - 2(m + 1)$ pro zakřivené C^m-prvky, potom existuje právě jedno řešení diskrétního problému (128) a rychlost konvergence k přesnému řešení problému (107), (108) je $O(h^{2m+1})$ v normě prostoru $H^{m+1}(\Omega_h)$.

Author's address: Doc. RNDr. *Alexander Ženíšek*, CSc., Laboratoř počítacích strojů VUT, Obránců míru 21, 602 00 Brno.