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ON NUMERICAL SOLUTION OF A VARIATIONAL INEQUALITY OF THE 4th ORDER BY FINITE ELEMENT METHOD

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INTRODUCTION

Let us solve the problem of a thin elastic clamped plate which is subjected to an external loading f. Let us suppose that its deflection u is limited below by a rigid obstacle. Mathematically it means that u is greater than or equal to a prescribed function, describing the obstacle. This problem leads to the minimization problem for energy functional over a certain convex set.

In [4] the dualisation of constraints and the algorithm of Uzawa is used for solving this problem. In the present paper another way was chosen. Using the finite element technique, the original energy functional is transformed into a quadratic function in E_n. We are led to procedures of quadratic programming. The proof of convergence for Ahlin's and Ari-Adini's elements is given and the concrete algorithm for numerical solution is proposed.

1. SETTING OF THE PROBLEM

Let Q be a bounded polygonal domain in E_2, the sides of which are parallel to the coordinate axes Ox, Oy. Let us denote the set of all continuous functions with compact support in Q and derivatives of all orders continuous in Q by $\mathscr{D}(Q)$. $H^k(Q)$ ($k \geqq 0$ integer) will denote the space of functions, the generalized derivatives of which up to the order k are elements of $L^2(Q) = H^0(Q)$, i.e. square integrable in Q. By $H_0^k(Q)$ ($k \geqq 0$ integer) we denote the completion of $\mathscr{D}(Q)$ under the seminorm

$$|u|_{k,Q} = \left(\int_Q \sum_{|\alpha|=k} |D^\alpha u|^2 \, dx \, dy \right)^{1/2}, \tag{1.1}$$

where $D^\alpha u = \partial^{|\alpha|} u / \partial x^{\alpha_1} \partial y^{\alpha_2}$, $\alpha = (\alpha_1, \alpha_2)$, α_i are non-negative integers, $|\alpha| = \alpha_1 + \alpha_2$ and $D^0 u = u$.

Let us set

$$\mathcal{J}(v) = a(v; v) - 2\int_Q fv \, dx \, dy ,$$

where $a(u; v) = \int_Q \sum_{|\alpha|=2} D^\alpha u D^\alpha v \, dx \, dy$ and $f \in L^2(Q)$.

The problem to be solved is defined in the following manner:

find $u \in K$ *such that*

$$(\mathcal{P}) \qquad \mathcal{J}(u) = \min_{v \in K} \mathcal{J}(v) ,$$

where

$$\mathcal{K} = \{v \in H_0^2(Q) : v \geqq \psi \text{ a.e. in } Q\}$$

and $\psi \in C(\bar{Q})$ is a given function, $\psi \leqq 0$ on ∂Q.

Theorem 1.1. *Problem* $(\mathcal{P})$ *has precisely one solution* u *for* $\forall f \in L^2(Q)$ *and this solution is characterized through the relation*

$$(1.2) \qquad a(u; v - u) \geqq \int_Q f(v - u) \, dx \, dy \quad \forall v \in \mathcal{K} .$$

Proof. $\mathcal{K}$ is a closed convex subset of $H_0^2(Q)$, $\mathcal{J}$ is a convex quadratic coercive[1]) functional on $H_0^2(Q)$. The rest of the proof follows immediately from [1 – Th. 0.4, p. 126].

2. APPROXIMATION OF $(\mathcal{P})$

Numerically, $(\mathcal{P})$ can be solved by minimizing $\mathcal{J}$ over *"a finite dimensional approximations* $\mathcal{K}_h$" of the original convex set $\mathcal{K}$. By u_h we denote such an element from $\mathcal{K}_h$ that

$$(\mathcal{P}_h) \qquad \mathcal{J}(u_h) = \min_{v \in K_h} \mathcal{J}(v) .$$

u_h will be called *the Ritz approximation* of u on $\mathcal{K}_h$.

We present two possible constructions of $\mathcal{K}_h$, based on the decomposition of Q into rectangles and on a suitable choice of finite elements.

Let $\{\mathcal{R}_h\}$, $h \to 0+$ be *a regular system* of rectangulations of $\bar{Q}$. This means that $\bar{Q}$ is expressed in the form of a union of rectangles R_i $(i = 1, \ldots, N(h))$, each

[1]) I.e. $\mathcal{J}(v) \to \infty$ if $|v|_{2,Q} \to +\infty$.

two of which are either disjoint, or have one vertex or one side in common, $\max_i \operatorname{diam}(R_i) \leqq h$ and there exists a constant $\alpha > 0$ such that

$$\frac{h_{\min}}{h_{\max}} \geqq \alpha .$$

$h_{\min}$, $h_{\max}$ are the minimum and the maximum respectively of lengths of all sides of $R_i \in \mathscr{R}_h$.

Let $\mathscr{N}_h$ be the set of all vertices (nodes of $\mathscr{R}_h$) of rectangles in the rectangulation $\mathscr{R}_h$. We suppose that the following condition is satisfied:

(i) $$\mathscr{N}_{h_1} \subset \mathscr{N}_{h_2} \quad \text{if} \quad h_1 > h_2 .$$

Construction I

Let $Q_3(R_i)$ be the set of bicubic polynomials defined in R_i, i.e.

$$q \in Q_3(R_i) \Leftrightarrow q(x, y) = \sum_{0 \leqq i, j \leqq 3} \alpha_{ij} x^i y^j , \quad [x, y] \in R_i .$$

Let V_h be the finite-dimensional subspace of $H_0^2(Q)$ defined by

$$V_h = \{v \in C^1(\bar{Q}) : v|_{R_i} \in Q_3(R_i) \; \forall R_i \in \mathscr{R}_h , \; i = 1, \ldots, N(h) ; \\ v = \partial v / \partial n = 0 \text{ on } \partial Q\} ,$$

i.e., V_h contains those functions which are continuous and continuously differentiable in Q and piecewise bicubic in each R_i. Then $\mathscr{K}_h$ is defined in the following manner:

(2.1) $$\mathscr{K}_h = \{v \in V_h : v(\mathring{A}_i^h) \geqq \psi(\mathring{A}_i^h) , \text{ where } \mathring{A}_i^h \in \mathscr{N}_h \cap Q \text{ are interior nodes of } \mathscr{R}_h\} .$$

Construction II

Let $\tilde{Q}_3(R_i)$ be the set of all functions defined in R_i of the form:

$$q \in \tilde{Q}_3(R_i) \Leftrightarrow q(x, y) = \sum_{0 \leqq i+j \leqq 3} \alpha_{ij} x^i y^j + \alpha_{13} x y^3 + \alpha_{31} x^3 y , \quad [x, y] \in R_i .$$

Let

$$S_h = \{v \in C(\bar{Q}) : v|_{R_i} \in \tilde{Q}_3(R_i) \; \forall R_i \in \mathscr{R}_h, \; i = 1, \ldots, N(h); \; v = 0 \text{ on } \partial Q\}$$

and

(2.2) $$\mathscr{U}_h = \{v \in S_h : v(\mathring{A}_i^h) \geqq \psi(\mathring{A}_i^h), \; \mathring{A}_i^h \in \mathscr{N}_h \cap Q\} .$$

In this section we establish convergence of Ritz approximations u_h to the exact solution u of $\mathscr{P}$. We shall consider both the cases $\mathscr{K}_h$, $\mathscr{U}_h$ separately.

I.

Let $\mathscr{K}_h$ be defined by (2.1).

Theorem 3.1. *For* $\forall h > 0$ *there exists a unique solution* $u_h \in \mathscr{K}_h$ of $(\mathscr{P}_h)$ *and this solution is characterized through the relation*

$$(3.1) \qquad a(u_h; v - u_h) \geqq \int_Q f(v - u_h)\,\mathrm{d}x\,\mathrm{d}y \quad \forall v \in \mathscr{K}_h\,.$$

Proof is the same as in Th. 1.1.

We establish the convergence of Ritz approximations in the $H_0^2(Q)$-norm, i.e. $|u - u_h|_{2,Q} \to 0$ for $h \to 0+$ under the additional restriction on the obstacle ψ. In the sequel we assume that

$$(\text{ii}) \qquad \psi < 0 \quad \text{on} \quad \partial Q\,.$$

The proof of convergence is based on the following lemmas.

Lemma 3.1. *It holds:*

$$(3.2) \qquad |u - u_h|_{2,Q}^2 \leqq \{(f; u - v_h) + (f; u_h - v) + a(u_h - u; v_h - u) + a(u; v - u_h) + a(u; v_h - u)\} \quad \text{for} \quad \forall v \in \mathscr{K}, v_h \in \mathscr{K}_h\,,$$

where (;) *denotes the scalar product in* $L^2(Q)$.

Proof. Since $a(u; u) \leqq a(u; v) + (f; u - v)\,\forall v \in \mathscr{K}$ and similarly $a(u_h; u_h) \leqq a(u_h; v_h) + (f; u_h - v_h)\,\forall v_h \in \mathscr{K}_h$ we have

$$\begin{aligned} |u - u_h|_{2,Q}^2 &= a(u - u_h; u - u_h) = a(u; u) + a(u_h; u_h) - a(u; u_h) - \\ &- a(u_h; u) \leqq a(u; v) + (f; u - v) + a(u_h; v_h) + (f; u_h - v_h) - \\ &- a(u; u_h) - a(u_h; u) = a(u; v - u_h) + a(u; u_h) + \\ &+ (f; u - v) + a(u_h; v_h - u) + a(u_h; u) + (f; u_h - v_h) - \\ &- a(u; u_h) - a(u_h; u) = a(u; v - u_h) + (f; u - v_h) + (f; v_h - v) + \\ &+ a(u_h - u; v_h - u) + a(u; v_h - u) + (f; u_h - v) + (f; v - v_h)\,. \end{aligned}$$

Lemma 3.2. *For* $\forall v \in \mathscr{K}$ *there exists* $v_h \in \mathscr{K}_h$ *such that*

$$|v - v_h|_{2,Q} \to 0 \quad for \quad h \to 0+ .$$

Proof. 1° First let us assume $v \in \mathscr{K} \cap H^4(Q)$. Let $v_h \in V_h$ be an element, the restriction of which in $R_i \in \mathscr{R}_h$ is the Hermite bicubic interpolate of v. Then v_h has the required property. In fact, by definition

$$v_{h|R_i} = \Pi_{R_i} v$$

and $\Pi_{R_i} v \in Q_3(R_i)$ is determined from the following conditions:

$$\Pi_{R_i} v(A_j) = v(A_j), \quad \frac{\partial}{\partial x} \Pi_{R_i} v(A_j) = \frac{\partial}{\partial x} v(A_j),$$

$$\frac{\partial}{\partial y} \Pi_{R_i} v(A_j) = \frac{\partial}{\partial y} v(A_j), \quad \frac{\partial^2}{\partial x\, \partial y} \Pi_{R_i} v(A_j) = \frac{\partial^2}{\partial x\, \partial y} v(A_j),$$

where $A_j, j = 1, \ldots, 4$ are the vertices of R_i. From the construction of v_h and the definition of $\mathscr{K}_h$ it follows that $v_h \in \mathscr{K}_h$ and moreover [3]:

$$|v - v_h|_{2,Q} = O(h^2) \quad \text{for} \quad h \to 0+ .$$

2° Let $v \in \mathscr{K}$ be arbitrary.

Let $\Phi \in H_0^2(Q)$ be a function with the following properties:

$$|\Phi|_{2,Q} = 1, \quad \Phi > 0 \quad \text{in } Q .$$

Let $v_\varepsilon = v + \varepsilon\Phi$, $\varepsilon > 0$. Then

$$|v_\varepsilon - v|_{2,Q} = \varepsilon|\Phi|_{2,Q} = \varepsilon ,$$
$$v_\varepsilon \geqq v \quad \text{in} \quad \bar{Q}$$

and the assumption (ii) results in

$$v_\varepsilon > \psi \quad \text{in} \quad \bar{Q} \quad \text{for} \quad \forall\varepsilon > 0 . \tag{3.3}$$

The definition of $H_0^2(Q)$ implies that there exist $v_{\varepsilon H} \in \mathscr{D}(Q)$ such that

$$|v_\varepsilon - v_{\varepsilon H}|_{2,Q} \to 0 \quad \text{if} \quad H \to 0+ .$$

The imbedding theorem of $H_0^2(Q)$ into $C(\bar{Q})$ yields

$$v_{\varepsilon H} \rightrightarrows v_\varepsilon \quad \text{(uniformly)} \quad \text{in} \quad \bar{Q} .$$

Hence $v_{\varepsilon H} > \psi$ in $\bar{Q}$ for H sufficiently small. As $v_{\varepsilon H} \in \mathscr{D}(Q) \cap \mathscr{K}$, part 1° of the proof ensures the existence of $v_h \in \mathscr{K}_h$ such that

$$|v_{\varepsilon H} - v_h|_{2,Q} \to 0, \; h \to 0+ .$$

Finally,

$$|v - v_h|_{2,Q} \leqq |v - v_\varepsilon|_{2,Q} + |v_\varepsilon - v_{\varepsilon H}|_{2,Q} + |v_{\varepsilon H} - v_h|_{2,Q} \to 0$$
$$\text{if} \quad \varepsilon, h, H \to 0+ .$$

Lemma 3.3. *Let $\{v_h\}$, $v_h \in \mathscr{K}_h$ be such that $v_h \rightharpoonup v$ (weakly) if $h \to 0+$ in $H_0^2(Q)$. Then $v \in \mathscr{K}$.*

Proof. It is sufficient to prove $v \geqq \psi$ in $\bar{Q}$. As $\delta(x) \in H^{-2}(Q)$[1]) (Dirac function concentrated at $x \in \bar{Q}$), we have

$$v_h(x) \to v(x) \quad \text{for all} \quad x \in \bar{Q} .$$

Let us suppose that there exists $x^* \in \bar{Q}$ such that

$$v(x^*) < \psi(x^*) . \tag{3.4}$$

As $v, \psi \in C(\bar{Q})$, (3.4) holds in a neighbourhood $U(x^*, \varepsilon) \cap \bar{Q}$, $\varepsilon > 0$, where $U(x^*, \varepsilon) = \{x \in E_2 : \varrho(x, x^*) \leqq \varepsilon\}$.

Further, diam $(R_i) \leqq h \; \forall R_i \in \mathscr{R}_h$ and $h \to 0+$, therefore there exists $A_i^{h_0} \in \mathscr{N}_{h_0}$ such that $A_i^{h_0} \in U(x^*, \varepsilon) \cap \bar{Q}$. The assumption (i) implies

$$A_i^{h_0} \in \mathscr{N}_h \quad \text{for} \quad \forall h \leqq h_0 .$$

As $v_h(A_i^{h_0}) \geqq \psi(A_i^{h_0})$ for $\forall h \leqq h_0$, it must be

$$v(A_i^{h_0}) = \lim_{h \to 0} v_h(A_i^{h_0}) \geqq \psi(A_i^{h_0}) ,$$

which is a contradiction with the above considerations.

Theorem 3.2. *Let (i), (ii) hold. Then*

$$|u - u_h|_{2,Q} \to 0 \quad \text{for} \quad h \to 0+ .$$

Proof. Lemma 3.2 ensures the existence of $v_h^* \in \mathscr{K}_h$ such that $v_h^* \to u$ in $H_0^2(Q)$. Further,

$$\mathscr{J}(u_h) \leqq \mathscr{J}(v_h^*), \mathscr{J}(v_h^*) \to \mathscr{J}(u) \quad \text{if} \quad h \to 0+ .$$

This and the coerciveness of $\mathscr{J}$ implies the boundedness of u_h in the $H_0^2(Q)$-norm. By virtue of boundedness there exist an element $v^* \in H_0^2(Q)$ and a subsequence $\{u_{h'}\} \in \{u_h\}$ such that

$$u_{h'} \rightharpoonup v^* \quad \text{in} \quad H_0^2(Q) .$$

[1]) $H^{-2}(Q)$ denotes the space of linear, bounded functionals on $H_0^2(Q)$.

By virtue of Lemma 3.3, v^* belongs to $\mathscr{K}$ and (3.2) yields

$$(3.5) \qquad |u - u_{h'}|_{2,Q}^2 \leqq \{(f; u - v_{h'}^*) + (f; u_{h'} - v^*) + a(u_{h'} - u; v_{h'}^* - u) + \\ + a(u; v^* - u_{h'}) + a(u; v_{h'}^* - u)\} \to 0 \quad \text{if} \quad h' \to 0+ .$$

As the limit (3.5) does not depend on the choice of the subsequence $u_{h'}$, we obtain $u_h \to u$ if $h \to 0+$.

II.

Let $\mathscr{U}_h$ be defined by (2.2).

It is easy to see that $\mathscr{U}_h \not\subset H_0^2(Q)$ but only $\mathscr{U}_h \subset H_0^1(Q)$. By $a_h(u; v)$ we denote the bilinear form defined on $S_h \times S_h$ through the relation

$$a_h(u; v) = \sum_{R_i \in \mathscr{R}_h} \int_{R_i} \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \, \partial y} \frac{\partial^2 v}{\partial x \, \partial y} + \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 v}{\partial y^2} \right) \mathrm{d}x \, \mathrm{d}y .$$

Let us set $|v|_{2,h} = a_h(v; v)^{1/2}$ for $\forall v \in S_h$. In order to define Ritz approximations, we introduce the functional

$$\mathscr{J}_h(v) = a_h(v; v) - 2 \int_Q fv \, \mathrm{d}x \, \mathrm{d}y , \quad v \in S_h .$$

Analogously to $(\mathscr{P}_h)$ we define the problem $(\mathscr{P}'_h)$ in the following manner:

$$(\mathscr{P}'_h) \qquad \begin{gathered} \textit{find } u_h \in \mathscr{U}_h \textit{ such that} \\ \mathscr{J}_h(u_h) = \min_{v \in \mathscr{U}_h} \mathscr{J}_h(v) . \end{gathered}$$

Theorem 3.3. *For $\forall h > 0$ there exists a unique solution $u_h \in \mathscr{U}_h$ of $(\mathscr{P}'_h)$, characterized through the relation*

$$(3.6) \qquad a_h(u_h; v - u_h) \geqq \int_Q f(v - u_h) \, \mathrm{d}x \, \mathrm{d}y \quad \forall v \in \mathscr{U}_h .$$

Proof. It is readily seen that $|v|_{2,h}$ defines a norm on $S_h \times S_h$. $\mathscr{J}_h$ is a convex function which is coercive on S_h and $\mathscr{U}_h$ is a closed convex subset of S_h. Hence the existence and the uniqueness of the solution of $(\mathscr{P}'_h)$ follows.

Our aim is to prove that $|u - u_h|_{2,h} \to 0$ if $h \to 0+$. First we prove some auxiliary lemmas.

Lemma 3.4. *It holds*:

(3.7) $$|u - u_h|_{2,h}^2 \leqq \{(f; u - v_h) + (f; u_h - v) + a_h(u_h - u; v_h - u) + + a_h(u; v - u_h) + a_h(u; v_h - u)\}$$

for $\forall v_h \in \mathscr{U}_h$, $v \in \mathscr{K}$.

Proof. Taking into account the fact that

$$a_h(u; v) = a(u; v) \quad \forall u, v \in H_0^2(Q), h \in (0, 1)$$

and repeating the proof of Lemma 3.1, we obtain (3.7).

Lemma 3.5. *For* $\forall v \in \mathscr{K}$ *there exist* $\{v_h\}$, $v_h \in \mathscr{U}_h$ *such that*

(3.8) $$|v - v_h|_{2,h} \to 0 \quad \text{if} \quad h \to 0+ .$$

Proof. 1° First, let us assume $v \in \mathscr{K} \cap H^4(Q)$. Choosing $v_h \in S_h$ as the Hermite interpolate of v by Ari-Adini's element over $R_i \in \mathscr{R}_h$, we obtain the assertion of our lemma. Indeed, by definition

$$v_h = \Pi_{R_i} v \quad \text{in} \quad R_i \in \mathscr{R}_h ,$$

where $\Pi_{R_i} v \in \tilde{Q}_3(R_i)$ is uniquely determined from the values of v, $\partial v/\partial x$, $\partial v/\partial y$ at the vertices of R_i. Hence $v_h \in \mathscr{U}_h$. Moreover, the approximation has the following order:

$$|v - v_h|_{2,h} = O(h^2) \quad \text{for} \quad h \to 0+ .$$

2° Let $v \in \mathscr{K}$ be arbitrary. Using the same approach as in the second part of the proof of Lemma 3.2 we obtain (3.8).

We know that $S_h \not\subset H_0^2(Q)$. The question is, how closely can an arbitrary function $\varphi \in S_h$ be approximated by members of $H_0^2(Q)$. The answer is given in

Lemma 3.6. *For* $\forall \varphi \in S_h$ *there exists a function* $r_h\varphi \in H_0^2(Q)$ *such that*

(3.9) $$\varphi = r_h\varphi , \quad \frac{\partial r_h \varphi}{\partial v_{ih}} = L_{ih}\left(\frac{\partial \varphi}{\partial v_{ih}}\right) \quad \text{on} \quad \partial R_i, R_i \in \mathscr{R}_h$$

(3.10) $$|\varphi - r_h\varphi|_{2,h} \leqq c|\varphi|_{2,h}$$

(3.11) $$\|\varphi - r_h\varphi\|_{0,Q} = ch^2|\varphi|_{2,h} ,$$

where $L_{ih}\varphi$ *denotes the linear Lagrange interpolate of* φ *on* ∂R_i, $\partial/\partial v_{ih}$ *is the normal derivative on* ∂R_i *and* $c > 0$ *is an absolute constant.*

Proof. Using the extension theorem from [5], one can construct the function $r_h\varphi$ over $R_i \in \mathscr{R}_h$, satisfying (3.9). A detailed proof of (3.9)–(3.11) can be found in [6].

Lemma 3.7. *It holds*

$$\|\varphi\|_{0,Q} \leqq c|\varphi|_{2,h} \quad \text{for} \quad \forall \varphi \in S_h\,, \tag{3.12}$$

where $c > 0$ is an absolute constant.

Proof.

$$\|\varphi\|_{0,Q} \leqq \|\varphi - r_h\varphi\|_{0,Q} + \|r_h\varphi\|_{0,Q} \leqq ch^2|\varphi|_{2,h} + c|r_h\varphi|_{2,Q} =$$
$$= ch^2|\varphi|_{2,h} + c|r_h\varphi|_{2,h} \leqq ch^2|\varphi|_{2,h} + c|r_h\varphi - \varphi|_{2,h} + c|\varphi|_{2,h} \leqq$$
$$\leqq c|\varphi|_{2,h}\,,$$

where (3.10), (3.11) and Friedrich's inequality in $H_0^2(Q)$ have been used.

The main result of this part is

Theorem 3.4. *Let (i), (ii) hold. Then*

$$|u - u_h|_{2,h} \to 0 \quad \text{for} \quad h \to 0+\,.$$

Proof. The sequence $\{u_h\}$ is bounded. Indeed, using (3.12) we have

$$\mathscr{J}_h(v_h) \to +\infty \quad \text{if} \quad |v_h|_{2,h} \to +\infty\,, \quad v_h \in S_h\,. \tag{3.13}$$

On the other hand, there exists a sequence $\{v_h^*\}$, $v_h^* \in \mathscr{U}_h$ such that (see Lemma 3.5)

$$|u - v_h^*|_{2,h} \to 0 \quad \text{for} \quad h \to 0+$$

and from the definition of u_h:

$$\mathscr{J}_h(u_h) \leqq \mathscr{J}_h(v_h^*) \to \mathscr{J}(u) \quad \text{if} \quad h \to 0+\,. \tag{3.14}$$

From (3.13) and (3.14) the boundedness of $\{u_h\}$ follows. Let $r_hu_h \in H_0^2(Q)$ be functions with the properties given in (3.9)–(3.11). As

$$|r_hu_h - u_h|_{2,h} \leqq c|u_h|_{2,h}\,, \tag{3.15}$$

$$\|r_hu_h - u_h\|_{0,Q} \leqq ch^2|u_h|_{2,h}\,, \tag{3.16}$$

the sequence $\{r_hu_h\}$ is bounded in the $H_0^2(Q)$-norm. Thus there exist $v^* \in H_0^2(Q)$ and a subsequence $\{r_{h'}u_{h'}\} \in \{r_hu_h\}$ such that

$$r_{h'}u_{h'} \rightharpoonup v^* \quad \text{in} \quad H_0^2(Q)\,. \tag{3.17}$$

The definition of r_hu_h implies that $r_hu_h(A_i^h) \geqq \psi(A_i^h)$, $A_i^h \in \mathscr{N}_h$. Hence v^* belongs to $\mathscr{K}$ (the proof is the same as in Lemma 3.3). Finally, we use (3.7) and we obtain

$$|u - u_{h'}|_{2,h'}^2 \leqq \{(f; u - v_{h'}^*) + (f; u_{h'} - v^*) + a_{h'}(u_{h'} - u; v_{h'}^* - u) +$$
$$+ a_{h'}(u; v^* - u_{h'}) + a_{h'}(u; v_{h'}^* - u)\}\,.$$

As $|v^*_{h'} - u|_{2,h'} \to 0$ for $h' \to 0+$, we have

$$(3.18) \qquad (f; u - v^*_{h'}) \to 0\,, \quad a_{h'}(u; u - v^*_{h'}) \to 0 \quad \text{for} \quad h' \to 0+\,.$$

Further,

$$(3.19) \qquad |a_{h'}(u_{h'} - u; v^*_{h'} - u)| \leqq c\varepsilon |u_{h'} - u|^2_{2,h'} + \frac{c}{\varepsilon} |v^*_{h'} - u|^2_{2,h'}$$

for every $\varepsilon > 0$ and

$$(f; u_{h'} - v^*) = (f; u_{h'} - r_{h'}u_{h'}) + (f; r_{h'}u_{h'} - v^*) \to 0 \quad \text{for} \quad h' \to 0+$$

by virtue of (3.16) and (3.17). It remains to estimate the term $a_{h'}(u; v^* - u_{h'})$. Let

$$Q_2(\mathscr{R}_h) = \{v \in L^2(Q) : v|_{R_i} \text{ is a quadratic function}\}\,.$$

We can write

$$\begin{aligned} a_{h'}(u; v^* - u_{h'}) &= a_{h'}(u; v^* - r_{h'}u_{h'}) + a_{h'}(u; r_{h'}u_{h'} - u_{h'}) = \\ &= a_{h'}(u; v^* - r_{h'}u_{h'}) + a_{h'}(u - p; r_{h'}u_{h'} - u_{h'}) + \\ &+ a_{h'}(p; r_{h'}u_{h'} - u_{h'}) \quad \text{for} \quad \forall p \in Q_2(\mathscr{R}_h)\,. \end{aligned}$$

Ari-Adini's element satisfies the criterion of "the patch test" (cf. [2], [6]), i.e.

$$(3.20) \qquad a_{h'}(p; r_{h'}u_{h'} - u_{h'}) = 0 \quad \forall p \in Q_2(\mathscr{R}_{h'})\,, \quad h' > 0\,.$$

Let $p_{h'} \in Q_2(\mathscr{R}_{h'})$ be a piecewise quadratic Lagrange interpolate of u on Q. Then

$$|u - p_{h'}|_{2,h'} \to 0 \quad \text{if} \quad h' \to 0+\,.$$

This and (3.15) yields

$$(3.21) \qquad a_{h'}(u - p_{h'}; u_{h'} - r_{h'}u_{h'}) \to 0 \quad \text{if} \quad h' \to 0+\,.$$

Finally,

$$(3.22) \qquad a_{h'}(u; v^* - r_{h'}u_{h'}) = a(u; v^* - r_{h'}u_{h'}) \to 0 \quad \text{if} \quad h' \to 0+$$

by virtue of (3.17). Using (3.18)–(3.22) we obtain

$$(3.23) \qquad |u - u_{h'}|_{2,h'} \to 0 \quad \text{if} \quad h' \to 0+\,.$$

As the limit (3.23) does not depend on the choice of the subsequence $\{u_{h'}\}$, we obtain

$$|u - u_h|_{2,h} \to 0 \quad \text{if} \quad h \to 0+\,.$$

To find the solution of $(\mathscr{P}_h)$, $(\mathscr{P}'_h)$ respectively, we can apply various procedures of the quadratic programming. We restrict ourselves to $(\mathscr{P}_h)$ only.

Let $\varphi_1, \ldots, \varphi_M, \ldots, \varphi_R$ be the interpolating basis of V_h and let the first M functions correspond to the value of the interpolated function, i.e.

$$(3.24) \qquad \varphi_j(\mathring{A}_i) = \delta_{ij}\,, \quad \frac{\partial}{\partial x}\,\varphi_j(\mathring{A}_i) = \frac{\partial}{\partial y}\,\varphi_j(\mathring{A}_i) = \frac{\partial^2}{\partial x\,\partial y}\,\varphi_j(\mathring{A}_i) = 0$$

$$i, j = 1, \ldots, M\,,$$

where $\mathring{A}_i$, $i = 1, \ldots, M$ are all the interior nodes of $\mathscr{R}_h$. Hence any $v \in V_h$ can be written in the form

$$(3.25) \qquad v(x) = \sum_{j=1}^{R} q_j\,\varphi_j(x)\,,$$

where $q_j = v(\mathring{A}_j)$, $j = 1, \ldots, M$. From the definition of $\mathscr{K}_h$ and (3.24), (3.25) it follows

$$v \in \mathscr{K}_h \Leftrightarrow \boldsymbol{q}^{\mathrm{T}} = (q_1, \ldots, q_R) \in \mathscr{K}_E\,,$$

where

$$\mathscr{K}_E = \{\boldsymbol{q} \in E_R : q_j \geqq \psi(\mathring{A}_j),\ \mathring{A}_j \in \mathscr{N}_h \cap Q,\ j = 1, \ldots, M\}\,.$$

Substituting (3.25) into $\mathscr{J}(v)$ we obtain

$$\mathscr{L}(\boldsymbol{q}) \equiv \mathscr{J}(v) = \boldsymbol{q}^{\mathrm{T}} A \boldsymbol{q} - 2\boldsymbol{f}^{\mathrm{T}}\boldsymbol{q}\,,$$

where $\boldsymbol{f}^{\mathrm{T}} = (f_1, \ldots, f_R)$, $A = (a_{ij})_{i,j=1}^{R}$, $f_j = \int_Q f\varphi_j \,\mathrm{d}x\,\mathrm{d}y$, $a_{ij} = a(\varphi_i; \varphi_j)$.

Problem $(\mathscr{P}_h)$ can be written in the following equivalent form:

find $\boldsymbol{q}^* \in \mathscr{K}_E$ *such that*

$$(\tilde{\mathscr{P}}_h) \qquad \mathscr{L}(\boldsymbol{q}^*) = \min_{\boldsymbol{q} \in \mathscr{K}_E} \mathscr{L}(\boldsymbol{q})\,.$$

It seems that one of the most effective numerical method for solving $(\tilde{\mathscr{P}}_h)$ is the modification of the well-known SOR method:

let $\boldsymbol{q}^0 \in \mathscr{K}_E$ *be given,*

$$q_i^{m+1/2} = -\frac{1}{a_{ii}}\Big(\sum_{j=1}^{i-1} a_{ij} q_j^{m+1} + \sum_{j=i+1}^{R} a_{ij} q_j^m - f_i\Big)$$

$$q_i^{m+1} = \max\{\psi(\mathring{A}_i), (1-\omega)\,q_i^m + \omega q_i^{m+1/2}\}\,, \quad i = 1, \ldots, M$$

$$q_i^{m+1} = (1-\omega)\,q_i^m + \omega q_i^{m+1/2}\,, \quad i = M+1, \ldots, R\,; \quad m = 1, 2, \ldots\,,$$

where $\omega \in (0, 2)$ is some selected weighting factor. For the proof of the convergence of this method see [7].

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Souhrn

O NUMERICKÉM ŘEŠENÍ JEDNÉ VARIAČNÍ NEROVNOSTI 4. ŘÁDU METODOU KONEČNÝCH PRVKŮ

JAROSLAV HASLINGER

V práci je řešen problém tenké vetknuté desky, jejíž průhyb je zespodu omezen dokonale tuhou překážkou. Užitím metody konečných prvků docházíme k úloze kvadratického programování: nalézt minimum kvadratického funkcionálu na konvexní podmnožině $\mathscr{K}_E \subset E_n$. Užívají se dva typy konečných prvků na obdélnících a to prvky bikubické a redukované Ari-Adiniovy prvky. Je dokázána konvergence metody a navržena konkrétní numerická metoda pro řešení úlohy v konečné dimensi. Výhodou tohoto přístupu je to, že jen nepatrná úprava umožňuje užít stávající algoritmy pro řešení klasického problému desky.

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