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ON GENERALIZED MEASURES OF RELATIVE INFORMATION AND INACCURACY

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1. INTRODUCTION

In their former papers, Sharma and Taneja [11] and Sharma and Gupta [9] characterized a generalized measure of type $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$ given by

$$(1.1)\qquad I^{(\alpha,\beta)}_{(\gamma,\delta)}(P;Q) = (2^{-\beta} - 2^{-\delta})^{-1}\sum_{i=1}^{n}(p_i^\alpha q_i^\beta - p_i^\gamma q_i^\delta)\,,\quad \alpha,\beta,\gamma,\delta > 0$$

$$\beta \neq \delta(\alpha \neq \gamma) \quad \text{whenever} \quad \alpha = \gamma(\beta = \delta)\,,$$

where $P = (p_1, \ldots, p_n)$, $p_i \geqq 0$, $\sum_{i=1}^{n} p_i = 1$ and $Q = (q_1, \ldots, q_n)$, $q_i > 0$, $\sum_{i=1}^{n} q_i \leqq 1$ are two discrete probability distributions of a discrete random variable.

The measure (1.1) under certain conditions (see [11]) gives Kullback's [5] relative information and Kerridge's [4] inaccuracy. These measures have many applications in statistics, economics etc..

The measure (1.1) can also be taken as

$$(1.2)\qquad I^{(\alpha,\beta)}_{(\gamma,\delta)}(P;Q) = (2^{\alpha-\beta} - 2^{\gamma-\delta})^{-1}\sum_{i=1}^{n}(p_i^\alpha q_i^{\beta-\alpha} - p_i^\gamma q_i^{\delta-\gamma})\,,$$

where $\alpha, \beta, \gamma, \delta > 0$, $\alpha \neq \gamma(\beta \neq \delta)$ whenever $\beta = \delta(\alpha = \gamma)$.

When $\gamma = \delta = 1$, the measure (1.2) reduces to

$$(1.3)\qquad I^{(\alpha,\beta)}_{(1,1)}(P;Q) = (2^{\alpha-\beta} - 1)^{-1}\left[\sum_{i=1}^{n} p_i^\alpha q_i^{\beta-\alpha} - 1\right],\quad \alpha \neq \beta\,,\quad \alpha > 0\,.$$

The measure (1.3) has been characterized by Sharma and Autar [7, 8] and reduces to Kullback's relative information and Kerridge's inaccuracy when $\beta = 1$, $\alpha \to 1$ and $\alpha = 1$, $\beta \to 1$, respectively.

The measure (1.2) is related with (1.3) as follows:

$$(1.4) \qquad I^{(\alpha,\beta)}_{(\gamma,\delta)}(P; Q) = \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} I^{(\alpha,\beta)}_{(1,1)}(P; Q) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} I^{(1,1)}_{(\gamma,\delta)}(P; Q),$$

where $A_{\alpha,\beta} = (2^{\alpha-\beta} - 1)$* and $A_{\gamma,\delta} = (2^{\gamma-\delta} - 1)$*.

It can be easily seen that the measure $I^{(\alpha,\beta)}_{(\gamma,\delta)}(P; Q)$ satisfies the following branching property:

$$(1.5) \qquad I^{(\alpha,\beta)}_{(\gamma,\delta)}(P; Q) - I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) =$$

$$= \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} (p_1 + p_2)^\alpha (q_1 + q_2)^{\beta-\alpha} I^{(\alpha,\beta)}_{(1,1)} \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} (p_1 + p_2)^\gamma (q_1 + q_2)^{\delta-\gamma} I^{(1,1)}_{(\gamma,\delta)} \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right), \quad A_{\alpha,\beta} \neq A_{\gamma,\delta}$$

where $p_1 + p_2 > 0$, $q_1 + q_2 > 0$.

When $\gamma = \delta = 1$ (or $\alpha = \beta = 1$), (1.5) reduces to a branching property studied by Sharma and Autar [7, 8].

In this communication, we characterize the measure (1.2) by taking the branching property (1.5) along with two other axioms. Some properties of this measure are also studied.

2. GENERALIZED MEASURE OF TYPE $\binom{\alpha,\beta}{\gamma,\delta}$

In this section, we shall characterize measure of type $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$ associated with a pair of probability distributions $P = (p_2, \ldots, p_n)$, $p_i \geqq 0, \sum_{i=1}^{n} p_i = 1$ and $Q = (q_1, \ldots, q_n)$, $q_i \geqq 0, \sum_{i=1}^{n} q_i = 1$ of a discrete random variable. We consider the following axioms:

(I) (*Symmetry*). $I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1, p_2, p_3; q_1, q_2, q_3)$ is a symmetric function of its variables provided the probabilities p_i and q_i $(i = 1, 2, 3)$ correspond to each other.

(II) (*Normality*). $I^{(\alpha,\beta)}_{(\gamma,\delta)}(1, 0; \frac{1}{2}, \frac{1}{2}) = 1$.

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*) Throughout the paper, we shall adopt the notation $A_{\alpha,\beta}$ for $(2^{\alpha-\beta} - 1)$ and $A_{\gamma,\delta}$ for $(2^{\gamma-\delta} - 1)$.

(III) (*Branching property*).

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(P; Q) - I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) =$$

$$= \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} (p_1 + p_2)^\alpha (q_1 + q_2)^{\beta-\alpha} I_{(1,1)}^{(\alpha,\beta)} \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2};\right.$$

$$\left.\frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} (p_1 + p_2)^\gamma (q_1 + q_2)^{\delta-\gamma} .$$

$$. I_{(\gamma,\delta)}^{(1,1)} \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}; \frac{q_1}{q_1 + q_2}, \frac{q_2}{q_1 + q_2}\right)$$

where $p_1 + p_2 > 0$, $q_1 + q_2 > 0$, $A_{\alpha,\beta} \neq A_{\gamma,\delta}$ and α, β, γ and δ are positive parameters different from unity.

From axiom (I), we have

(2.1)
$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(y_1, 1 - x_1 - y_1, x_1; y_2, 1 - x_2 - y_2, x_2) =$$
$$= I_{(\gamma,\delta)}^{(\alpha,\beta)}(x_1, 1 - y_1 - x_1, y_1; x_2, 1 - y_2 - x_2, y_2),$$

which together with axiom (III) gives

(2.2)
$$f(1 - x_1; 1 - x_2) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} (1 - x_1)^\alpha (1 - x_2)^{\beta-\alpha} g\left(\frac{y_1}{1 - x_1}; \frac{y_2}{1 - x_2}\right) +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} (1 - x_1)^\gamma (1 - x_2)^{\delta-\gamma} h\left(\frac{y_1}{1 - x_1}; \frac{y_2}{1 - x_2}\right) =$$

$$= f(1 - y_1; 1 - y_2) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} (1 - y_1)^\alpha (1 - y_2)^{\beta-\alpha} g\left(\frac{x_1}{1 - y_1}; \frac{x_2}{1 - y_2}\right) +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} (1 - y_1)^\gamma (1 - y_2)^{\delta-\gamma} h\left(\frac{x_1}{1 - y_1}; \frac{x_2}{1 - y_2}\right),$$

where

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(x_1, 1 - x_1; x_2, 1 - x_2) = f(x_1; x_2),$$

$$I_{(1,1)}^{(\alpha,\beta)}(x_1, 1 - x_1; x_2, 1 - x_2) = g(x_1; x_2),$$

and
$$I_{(\gamma,\delta)}^{(1,1)}(x_1, 1 - x_1; x_2, 1 - x_2) = h(x_1; x_2).$$

Setting $x_1 = x_2 = 1$, $y_1 = y_2 = 0$ in (2.2), we get

(2.3)
$$f(0; 0) = f(1; 1) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} g(1; 1) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} h(1; 1).$$

By axiom (III), we get

$$(2.4)\quad I^{(\alpha,\beta)}_{(\gamma,\delta)}(1,0,0;1,0,0) = I^{(\alpha,\beta)}_{(\gamma,\delta)}(1,0;1,0) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}} I^{(\alpha,\beta)}_{(1,1)}(1,0;1,0) +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}} I^{(1,1)}_{(\gamma,\delta)}(1,0;1,0) = f(1;1) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}} g(1;1) +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}} h(1;1),$$

and

$$(2.5)\quad I^{(\alpha,\beta)}_{(\gamma,\delta)}(0,1,0;0,1,0) = I^{(\alpha,\beta)}_{(\gamma,\delta)}(1,0;1,0) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}} I^{(\alpha,\beta)}_{(1,1)}(0,1;0,1) +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}} I^{(1,1)}_{(\gamma,\delta)}(0,1;0,1) =$$

$$= f(1;1) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}} g(0;0) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}} h(0;0).$$

Expressions (2.4) and (2.5) together with axiom (I) give

$$(2.6)\qquad A_{\alpha,\beta}[g(1;1) - g(0;0)] = A_{\gamma,\delta}[h(1;1) - h(0;0)].$$

Now for $\gamma = \delta = 1$, (2.3) and (2.6) give

$$(2.7)\qquad g(1;1) = g(0;0) = 0.$$

Again for $\alpha = \beta = 1$, (2.3) and (2.6) give

$$(2.8)\qquad h(1;1) = h(0;0) = 0.$$

Thus (2.3) together with (2.7) and (2.8) gives

$$(2.9)\qquad f(1;1) = f(0;0).$$

Further, if $x_1 = x_2 = 0$, $y_1 = 0$, $y_2 = \frac{1}{2}$ in (2.2) then by axiom (II), (2.7) and (2.8), we get

$$(2.9')\qquad f(1;1) = f(0;0) = 0.$$

Next, substituting $y_1 = 1 - x_1$, $y_2 = 1 - x_2$ in (2.2) and using (2.7) and (2.8), we get

$$(2.10)\qquad f(x_1;x_2) = f(1-x_1;1-x_2).$$

Thus, the functional equation (2.2) with (2.10) reduces to

$$(2.11)\quad f(x_1;x_2) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}} (1-x_1)^{\alpha}(1-x_2)^{\beta-\alpha} g\left(\frac{y_1}{1-x_1};\frac{y_2}{1-x_2}\right) +$$

$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}(1-x_1)^{\gamma}(1-x_2)^{\delta-\gamma}h\left(\frac{y_1}{1-x_1};\frac{y_2}{1-x_2}\right)=$$

$$=f(y_1;y_2)+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}(1-y_1)^{\alpha}(1-y_2)^{\beta-\alpha}g\left(\frac{x_1}{1-y_1};\frac{x_2}{1-y_2}\right)+$$

$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}(1-y_1)^{\gamma}(1-y_2)^{\delta-\gamma}h\left(\frac{x_1}{1-y_1};\frac{x_2}{1-y_2}\right),$$

for all $x_1, y_1, x_2, y_2 \in [0, 1)$ with $x_1 + x_2 \leqq 1$, $y_1 + y_2 \leqq 1$.

Now we shall first obtain a relation between f, g and h and then find the values of these functions in the following lemma:

Lemma. *The functions f, g and h given in* (2.11) *satisfy the relation*

$$f(x;y)=\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}g(x;y)+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}h(x;y), \tag{2.12}$$

where

$$g(x;y)=A_{\alpha,\beta}^{-1}[x^{\alpha}y^{\beta-\alpha}+(1-x)^{\alpha}(1-y)^{\beta-\alpha}-1], \quad A_{\alpha,\beta}\neq 0, \tag{2.13}$$

and

$$h(x;y)=A_{\gamma,\delta}^{-1}[x^{\gamma}y^{\delta-\gamma}+(1-x)^{\gamma}(1-y)^{\delta-\gamma}-1], \quad A_{\gamma,\delta}\neq 0, \tag{2.14}$$

for all $x, y \in [0, 1]$.

Proof. Setting $p_1 = 1 - x_1$, $q_1 = 1 - x_2$, $p_2 = y_1/(1 - x_1)$ and $q_2 = y_2/(1 - x_2)$ in (2.11), we get

$$f(p_1;q_1)+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}p_1^{\alpha}q_1^{\beta-\alpha}g(p_2;q_2)+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}p_1^{\gamma}q_1^{\delta-\gamma}h(p_2;q_2)= \tag{2.15}$$

$$=f(p_1p_2;q_1q_2)+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}(1-p_1p_2)^{\alpha}(1-q_1q_2)^{\beta-\alpha}g\left(\frac{1-p_1}{1-p_1p_2};\frac{1-q_1}{1-q_1q_2}\right)+$$

$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}(1-p_1p_2)^{\gamma}(1-q_1q_2)^{\delta-\gamma}h\left(\frac{1-p_1}{1-p_1p_2};\frac{1-q_1}{1-q_1q_2}\right),$$

for all $p_1, q_1 \in (0, 1]$, $p_2, q_2 \in [0, 1]$ with $p_1p_2 < 1$ and $q_1q_2 < 1$.

We shall prove that for arbitrary p's and q's as above, the function F defined by

$$F(p_1,p_2;q_1,q_2)=f(p_1;q_1)+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}[p_1^{\alpha}q_1^{\beta-\alpha}+ \tag{2.16}$$

$$+(1-p_1)^{\alpha}(1-q_1)^{\beta-\alpha}]g(p_2;q_2)+$$

$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}\left[p_1^\gamma q_1^{\delta-\gamma}+(1-p_1)^\gamma(1-q_1)^{\delta-\gamma}\right]h(p_2;q_2),$$

is symmetric, i.e.

$$F(p_1,p_2;q_1,q_2)=F(p_2,p_1;q_2,q_1). \tag{2.17}$$

Now (2.16) together with (2.15) gives

$$F(p_1,p_2;\ q_1,q_2)=f(p_1p_2;q_1q_2)+\frac{A_{\alpha,\beta}}{A_{\alpha,\beta}-A_{\gamma,\delta}}(1-p_1p_2)^\alpha\cdot \tag{2.18}$$
$$\cdot(1-q_1q_2)^{\beta-\alpha}G(p_1,p_2;\ q_1,q_2)+$$
$$+\frac{A_{\gamma,\delta}}{A_{\gamma,\delta}-A_{\alpha,\beta}}(1-p_1p_2)^\gamma(1-q_1q_2)^{\delta-\gamma}H(p_1,p_2;\ q_1,q_2),$$

where

$$G(p_1,p_2;\ q_1,q_2)=g\left(\frac{1-p_1}{1-p_1p_2};\frac{1-q_1}{1-q_1q_2}\right)+ \tag{2.19}$$
$$+\left(\frac{1-p_1}{1-p_1p_2}\right)^\alpha\left(\frac{1-q_1}{1-q_1q_2}\right)^{\beta-\alpha}g(p_2;q_2)$$

and

$$H(p_1,p_2;\ q_1,q_2)=h\left(\frac{1-p_1}{1-p_1p_2};\frac{1-q_1}{1-q_1q_2}\right)+ \tag{2.20}$$
$$+\left(\frac{1-p_1}{1-p_1p_2}\right)^\gamma\left(\frac{1-q_1}{1-q_1q_2}\right)^{\delta-\gamma}h(p_2;q_2).$$

Now to prove that F is symmetric, we have to show that the functions G and H are symmetric. To this aim, first let $\gamma=\delta=1$ in (2.15). Then we get

$$g(p_1;q_1)+p_1^\alpha q_1^{\beta-\alpha}g(p_2;q_2)= \tag{2.21}$$
$$=g(p_1p_2;q_1q_2)+(1-p_1p_2)^\alpha(1-q_1q_2)^{\beta-\alpha}$$
$$g\left(\frac{1-p_1}{1-p_1p_2};\frac{1-q_1}{1-q_1q_2}\right),$$

for all $p_1,q_1\in(0,1]$, $p_2,q_2\in[0,1]$ with $p_1p_2<1$ and $q_1q_2<1$.

Next set $p_1^*=(1-p_1)/(1-p_1p_2)$ and $q_1^*=(1-q_1)/(1-q_1q_2)$ in (2.19), we get

$$G(p_1,p_2;\ q_1,q_2)=g(p_1^*;q_1^*)+p_1^{*\alpha}q_1^{*\beta-\alpha}g(p_2;q_2)=$$
$$=g(p_1^*p_2;q_1^*q_2)+(1-p_1^*p_2)^\alpha(1-q_1^*q_2)^{\beta-\alpha}g\left(\frac{1-p_1^*}{1-p_1^*p_2};\frac{1-q_1^*}{1-q_1^*q_2}\right)$$

(from (2.21))

$$= g(1 - p_1^*p_2; 1 - q_1^*q_2) + (1 - p_1^*p_2)^\alpha (1 - q_1^*q_2)^{\beta-\alpha} g\left(\frac{1 - p_1^*}{1 - p_1^*p_2}; \frac{1 - q_1^*}{1 - q_1^*q_2}\right)$$

(from (2.10))

$$= g\left(\frac{1 - p_2}{1 - p_1p_2}; \frac{1 - q_2}{1 - q_1q_2}\right) + \left(\frac{1 - p_2}{1 - p_1p_2}\right)^\alpha \left(\frac{1 - q_2}{1 - q_1q_2}\right)^{\beta-\alpha} g(p_1; q_1) =$$

$$= G(p_2, p_1; q_2, q_1).$$

Similarly, for $\alpha = \beta = 1$, we can easily show that

$$H(p_1, p_2; q_1, q_2) = H(p_2, p_1; q_2, q_1).$$

This proves (2.17).

Now putting $p_2 = q_2 = 1$ in (2.17), we get

$$0 = F(p_1, 1; q_1, 1) - F(1, p_1; 1, q_1) = \tag{2.22}$$

$$= f(p_1; q_1) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} [p_1^\alpha q_1^{\beta-\alpha} + (1 - p_1)^\alpha (1 - q_1)^{\beta-\alpha}] g(1; 1) +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} [p_1^\gamma q_1^{\delta-\gamma} + (1 - p_1)^\gamma (1 - q_1)^{\delta-\gamma}] h(1; 1) - f(1; 1) -$$

$$- \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} g(p_1; q_1) - \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} h(p_1; q_1).$$

Expression (2.22) together with (2.7), (2.8) and (2.9) gives (2.12).

Again taking $p_2 = 0$, $q_2 = \frac{1}{2}$ in (2.17), from (2.16), (2.17) and axiom (II), we get for $A_{\alpha,\beta} \neq A_{\gamma,\delta}$,

$$A_{\alpha,\beta}\{g(p_1; q_1) + [p_1^\alpha q_1^{\beta-\alpha} + (1 - p_1)^\alpha (1 - q_1)^{\beta-\alpha} - 1] - 2^{\alpha-\beta} g(p_1; q_1)\} \tag{2.23}$$

$$= A_{\gamma,\delta}\{h(p_1; q_1) + [p_1^\gamma q_1^{\delta-\gamma} + (1 - p_1)^\gamma (1 - q_1)^{\delta-\gamma} - 1] - 2^{\gamma-\delta} h(p_1; q_1)\} =$$

$$= C\,(\text{say}),$$

where C is any arbitrary constant.

Now putting $p_1 = q_1 = 1$ in (2.23) and using (2.7), we get $C = 0$. Thus (2.13) and (2.14) follow.

Now (2.12) together with (2.13) and (2.14) gives

$$f(p; q) = \tag{2.24}$$

$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} [p^\alpha q^{\beta-\alpha} + (1 - p)^\alpha (1 - q)^{\beta-\alpha} - p^\gamma q^{\delta-\gamma} + (1 - p)^\gamma (1 - q)^{\delta-\gamma}],$$

$$A_{\alpha,\beta} \neq A_{\gamma,\delta}$$

which is an *information function of type* $\begin{pmatrix}\alpha, & \beta\\ \gamma, & \delta\end{pmatrix}$.

Again from the branching property (i.e., axiom (III)) we can write

$$(2.25)\qquad I_{(\gamma,\delta)}^{(\alpha,\beta)}(P;Q) = \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \sum_{i=2}^{n} s_i^{\alpha} t_i^{\beta-\alpha}\, g(p_i/s_i;\, q_i/t_i) +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \sum_{i=2}^{n} s_i^{\gamma} t_i^{\delta-\gamma}\, h(p_i/s_i;\, q_i/t_i)\,,$$

where $s_i = p_1 + \ldots + p_i$; $t_i = q_1 + \ldots + q_i$ $(i = 2, 3, \ldots, n)$.

Now (2.25) together with (2.13) and (2.14) gives

$$(2.26)\qquad I_{(\gamma,\delta)}^{(\alpha,\beta)}(P;Q) = (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} (p_i^{\alpha} q_i^{\beta-\alpha} - p_i^{\gamma} q_i^{\delta-\gamma})\,, \quad A_{\alpha,\beta} \neq A_{\gamma,\delta}\,,$$

which is an *information measure of type* $\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}$.

Thus, we have proved:

Theorem 2.1. *The measure determined by axioms* (I)–(III), *associated with a pair of probability distributions* $P = (p_1, \ldots, p_n)$, $p_i \geqq 0$, $\sum_{i=1}^{n} p_i = 1$ *and* $Q = (q_1, \ldots, q_n)$, $q_i \geqq 0$, $\sum_{i=1}^{n} q_i = 1$ *is given by* (2.26).

2.1. PARTICULAR CASES

Setting $\gamma = \delta = 1$ in (2.26), we get

$$(2.27)\qquad I^{(\alpha,\beta)}(P;Q) = (2^{\alpha-\beta} - 1)^{-1} \left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{\beta-\alpha} - 1\right].$$

Case I. (*Kullback's relative information*): The measure (2.27) together with the condition

$$(2.28)\qquad I^{(\alpha,\beta)}(p, 1-p;\, p, 1-p) = 0\,, \quad 0 < p < 1$$

gives $\beta = 1$.

Thus under the condition (2.28), the measure (2.27) reduces to

$$(2.29)\qquad I^{\alpha}(P;Q) = (2^{\alpha-1} - 1)^{-1} \left[\sum_{i=1}^{n} p_i^{\alpha} q_i^{1-\alpha} - 1\right], \quad \alpha \neq 1\,,$$

which is the relative information of type α studied by many authors ([6], [10], [12], [13]).

Also

$$\lim_{\alpha \to 1} I^{\alpha}(P; Q) = \sum_{i=1}^{n} p_i \log_2 (p_i/q_i), \tag{2.30}$$

which is Kullback's [5] relative information. This measure has also been characterized by Hobson [3], Campbell [1] and Sharma and Taneja [10].

Case II. (*Kerridge's Inaccuracy*): The measure (2.26) together with the condition

$$I^{(\alpha,\beta)}(p_1, p_2, p_3; q_1, q_2, q_2) = I^{(\alpha,\beta)}(p_1, p_2 + p_3; q_1, q_2), \tag{2.31}$$

gives $\alpha = 1$.

Thus under the condition (2.31), the measure (2.27) reduces to

$$I^{\beta}(P; Q) = (2^{1-\beta} - 1)^{-1} \left[\sum_{i=1}^{n} p_i q_i^{\beta-1} - 1\right], \quad \beta \neq 1, \tag{2.32}$$

which is the inaccuracy measure of type β studied by many authors ([10], [12], [13]).

It may also be noted that

$$\lim_{\beta \to 1} I^{\beta}(P; Q) = -\sum_{i=1}^{n} p_i \log_2 q_i, \tag{2.33}$$

which is Kerridge's [4] inaccuracy measure.

3. PROPERTIES OF THE MEASURE $I_{(\gamma,\delta)}^{(\alpha,\beta)}(P; Q)$

The measure of information $I_{(\gamma,\delta)}^{(\alpha,\beta)}(P; Q)$, $P, Q \in \Delta_n$, where $\Delta_n = \{P = (p_1, \ldots, p_n), p_i \geqq 0, \sum_{i=1}^{n} p_i = 1\}$ has the following properties:

Theorem 3.1. (i) (*Symmetry*): *$I_{(\gamma,\delta)}^{(\alpha,\beta)}(P; Q)$ is a symmetric function of its arguments provided the probabilities p_i and q_i $(i = 1, 2, \ldots, n)$ correspond to each other i.e.,*

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_{n-1}, p_n; q_1, \ldots, q_{n-1}, q_n) = I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_n, p_1, \ldots, p_{n-1}; q_n, q_1, \ldots, q_{n-1}).$$

(ii) (*Expansibility*): $I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_n, 0; q_1, \ldots, q_n, 0) = I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_n; q_1, \ldots, q_n)$.

(iii) $\left(\text{\textit{Recursive of type}} \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}\right)$: *For $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n) \in \Delta_n$, we have*

(3.1)

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1, \ldots, p_n; q_1, \ldots, q_n) - I_{(\gamma,\delta)}^{(\alpha,\beta)}(p_1 + p_2, p_3, \ldots, p_n; q_1 + q_2, q_3, \ldots, q_n) =$$

$$= \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} (p_1 + p_2)^{\alpha}(q_1 + q_2)^{\beta-\alpha} I^{(\alpha,\beta)}_{(1,1)}\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2};\right.$$

$$\left.\frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2}\right) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} (p_1 + p_2)^{\gamma}(q_1 + q_2)^{\delta-\gamma}.$$

$$\cdot I^{(1,1)}_{(\gamma,\delta)}\left(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}; \frac{q_1}{q_1+q_2}, \frac{q_2}{q_1+q_2}\right), \quad A_{\alpha,\beta} \neq A_{\gamma,\delta}$$

(iv) $\left(\textit{Generalized recursive of type}\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}\right)$: *For* $n \geqq N + 1$ *where* $N \geqq 2$ *and* $(p_1, \ldots, p_n) \in \Delta_n$, $(q_1, \ldots, q_n) \in \Delta_n$, *we have*

(3.2)
$$I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1, \ldots, p_n; q_1, \ldots, q_n) -$$
$$- I^{(\alpha,\beta)}_{(\gamma,\delta)}\left(\sum_{i=1}^{N} p_i, p_{N+1}, \ldots, p_n; \sum_{i=1}^{N} q_i, q_{N+1}, \ldots, q_n\right) =$$
$$= \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \left(\sum_{i=1}^{N} p_i\right)^{\alpha} \left(\sum_{i=1}^{N} q_i\right)^{\beta-\alpha} I^{(\alpha,\beta)}_{(1,1)}\left(p_1/\sum_{i=1}^{N} p_i, \ldots, p_N/\sum_{i=1}^{N} p_i;\right.$$
$$\left. q_1/\sum_{i=1}^{N} q_i, \ldots, q_N/\sum_{i=1}^{N} q_i\right) +$$
$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \left(\sum_{i=1}^{N} p_i\right)^{\gamma} \left(\sum_{i=1}^{N} q_i\right)^{\delta-\gamma} I^{(1,1)}_{(\gamma,\delta)}\left(p_1/\sum_{i=1}^{N} p_i, \ldots, p_N/\sum_{i=1}^{N} p_i;\right.$$
$$\left. q_1/\sum_{i=1}^{N} q_i, \ldots, q_N/\sum_{i=1}^{N} q_i\right).$$

(v) $\left(\textit{Strongly-additive of type}\begin{pmatrix}\alpha, \beta\\ \gamma, \delta\end{pmatrix}\right)$:

(3.3)
$$I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1p_{11}, \ldots, p_1p_{1n}, \ldots, p_mp_{m1}, \ldots, p_mp_{mn};$$
$$q_1q_{11}, \ldots, q_1q_{1n}, \ldots, q_mq_{m1}, \ldots, q_mq_{mn}) +$$
$$= I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1, \ldots, p_m; q_1, \ldots, q_m) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \sum_{j=1}^{m} p_j^{\alpha} q_j^{\beta-\alpha} I^{(\alpha,\beta)}_{(1,1)}(p_{j1}, \ldots, p_{jn};$$
$$q_{j1}, \ldots, q_{jn}) +$$
$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \sum_{j=1}^{m} p_j^{\gamma} q_j^{\delta-\gamma} I^{(1,1)}_{(\gamma,\delta)}(p_{j1}, \ldots, p_{jn}; q_{j1}, \ldots, q_{jn})$$

for all $(p_1, \ldots, p_m)$ *and* $(q_1, \ldots, q_m) \in \Delta_m$, $(p_{j1}, \ldots, p_{jn}) \in \Delta_n$ *and* $(q_{j1}, \ldots, q_{jn}) \in \Delta_n$, $(j = 1, 2, \ldots, m)$.

Proof. Properties (i) and (ii) are obvious and can be verified easily. Property (iii) is axiom (III) considered in Section 2. We prove (iv) and (v) by direct computation.

(iv)
$$I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1, \ldots, p_n; q_1, \ldots, q_n) -$$
$$- I^{(\alpha,\beta)}_{(\gamma,\delta)}(\sum_{i=1}^{N} p_i, p_{N+1}, \ldots, p_n; \sum_{i=1}^{N} q_i, q_{N+1}, \ldots, q_n) =$$
$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} [\sum_{i=1}^{n} (p_i^\alpha q_i^{\beta-\alpha} - p_i^\gamma q_i^{\delta-\gamma}) -$$
$$- \{(\sum_{i=1}^{N} p_i)^\alpha (\sum_{i=1}^{N} q_i)^{\beta-\alpha} + \sum_{i=N+1}^{n} p_i^\alpha q_i^{\beta-\alpha} - (\sum_{i=1}^{N} p_i)^\gamma (\sum_{i=1}^{N} q_i)^{\delta-\gamma} - \sum_{i=N+1}^{n} p_i^\gamma q_i^{\delta-\gamma}\}] =$$
$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} [\sum_{i=1}^{N} p_i^\alpha q_i^{\beta-\alpha} - (\sum_{i=1}^{N} p_i)^\alpha (\sum_{i=1}^{N} q_i)^{\beta-\alpha} - \sum_{i=1}^{N} p_i^\gamma q_i^{\delta-\gamma} +$$
$$+ (\sum_{i=1}^{N} p_i)^\gamma (\sum_{i=1}^{N} q_i)^{\delta-\gamma}] =$$
$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} (\sum_{i=1}^{N} p_i)^\alpha (\sum_{i=1}^{N} q_i)^{\beta-\alpha} [\sum_{i=1}^{N} (p_i/\sum_{i=1}^{N} p_i)^\alpha (q_i/\sum_{i=1}^{N} q_i)^{\beta-\alpha} - 1] -$$
$$- (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} (\sum_{i=1}^{N} p_i)^\gamma (\sum_{i=1}^{N} q_i)^{\delta-\gamma} [\sum_{i=1}^{N} (p_i/\sum_{i=1}^{N} p_i)^\gamma (q_i/\sum_{i=1}^{N} q_i)^{\delta-\gamma} - 1] =$$
$$= \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} (\sum_{i=1}^{N} p_i)^\alpha (\sum_{i=1}^{N} q_i)^{\beta-\alpha} I^{(\alpha,\beta)}_{(1,1)}(p_1/\sum_{i=1}^{N} p_i, \ldots, p_N/\sum_{i=1}^{N} p_i;$$
$$q_1/\sum_{i=1}^{N} q_i, \ldots, q_N/\sum_{i=1}^{N} q_i) +$$
$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} (\sum_{i=1}^{N} p_i)^\gamma (\sum_{i=1}^{N} q_i)^{\delta-\gamma} I^{(1,1)}_{(\gamma,\delta)}(p_1/\sum_{i=1}^{N} p_i, \ldots, p_N/\sum_{i=1}^{N} p_i;$$
$$q_1/\sum_{i=1}^{N} q_i, \ldots, q_N/\sum_{i=1}^{N} q_i).$$

(v)
$$I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1, \ldots, p_m; q_1, \ldots, q_m) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \sum_{j=1}^{m} p_j^\alpha q_j^{\beta-\alpha} I^{(\alpha,\beta)}_{(1,1)}(p_{j1}, \ldots, p_{jn};$$
$$q_{j1}, \ldots, q_{jn}) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \sum_{j=1}^{m} p_j^\gamma q_j^{\delta-\gamma} I^{(1,1)}_{(\gamma,\delta)}(p_{j1}, \ldots, p_{jn}; q_{j1}, \ldots, q_{jn}) =$$
$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} [\sum_{j=1}^{m} (p_j^\alpha q_j^{\beta-\alpha} - p_j^\gamma q_j^{\delta-\gamma}) + \sum_{j=1}^{m} p_j^\alpha q_j^{\beta-\alpha} (\sum_{i=1}^{n} p_{ji}^\alpha q_j^{\beta-\alpha} - 1) -$$
$$- \sum_{j=1}^{m} p_j^\gamma q_j^{\delta-\gamma} (\sum_{i=1}^{n} p_{ji}^\gamma q_{ji}^{\delta-\gamma} - 1)] =$$
$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} [\sum_{i=1}^{n} \sum_{j=1}^{m} (p_j p_{ji})^\alpha (q_j q_{ji})^{\beta-\alpha} - (p_j p_{ji})^\gamma (q_j q_{ji})^{\delta-\gamma}] =$$
$$= I^{(\alpha,\beta)}_{(\gamma,\delta)}(p_1 p_{11}, \ldots, p_1 p_{1n}, \ldots, p_m p_{m1}, \ldots, p_m p_{mn};$$
$$q_1 q_{11}, \ldots, q_1 q_{1n}, \ldots, q_m q_{m1}, \ldots, q_m q_{mn}).$$

Theorem 3.2. *Let $P_1 = (p_{11}, p_{12}, \ldots, p_{1n}) \in \Delta_n$ and $P_2 = (p_{21}, p_{22}, \ldots, p_{2m}) \in \Delta_m$ with a similar notation for Q_1 and Q_2. If $P_1^* P_2 = (p_{11}p_{21}, \ldots, p_{11}p_{2m}, \ldots \ldots, p_{1n}p_{21}, \ldots, p_{1n}p_{2m})$, then*

(i) (*Generalized additivity*):

(3.3′)

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(P_1^* P_2;\, Q_1^* Q_2) = G_{(\gamma,\delta)}^{(\alpha,\beta)}(P_1;\, Q_1)\, I_{(\gamma,\delta)}^{(\alpha,\beta)}(P_2;\, Q_2) + G_{(\gamma,\delta)}^{(\alpha,\beta)}(P_2;\, Q_2)\, I_{(\gamma,\delta)}^{(\alpha,\beta)}(P_1;\, Q_1)\,,$$

where

$$G_{(\gamma,\delta)}^{(\alpha,\beta)}(P;\, Q) = \tfrac{1}{2}\sum_{i=1}^{n} (p_i^{\alpha} q_i^{\beta-\alpha} + p_i^{\gamma} q_i^{\delta-\gamma})\,, \quad \alpha, \beta, \gamma, \delta > 0\,. \tag{3.4}$$

(ii) (*Sub-additivity*): *For $\alpha, \gamma \geqq 1$, $\beta - \alpha \geqq 1$, $\delta - \gamma \geqq 1$, we have*

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(P_1^* P_2;\, Q_1^* Q_2) \leqq I_{(\gamma,\delta)}^{(\alpha,\beta)}(P_1;\, Q_1) + I_{(\gamma,\delta)}^{(\alpha,\beta)}(P_2;\, Q_2)\,. \tag{3.5}$$

Proof. (i) R.H.S.

$$= \tfrac{1}{2}\sum_{i=1}^{n} (p_{1i}^{\alpha} q_{1i}^{\beta-\alpha} + p_{1i}^{\gamma} q_{1i}^{\delta-\gamma}) (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{j=1}^{m} (p_{2j}^{\alpha} q_{2j}^{\beta-\alpha} - p_{2j}^{\gamma} q_{2j}^{\delta-\gamma}) +$$

$$+ \tfrac{1}{2}\sum_{j=1}^{m} (p_{2j}^{\alpha} q_{2j}^{\beta-\alpha} + p_{2j}^{\gamma} q_{2j}^{\delta-\gamma}) (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} (p_{1i}^{\alpha} q_{1i}^{\beta-\alpha} - p_{1i}^{\gamma} q_{1i}^{\delta-\gamma}) =$$

$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} \sum_{j=1}^{m} [(p_{1i}p_{2j})^{\alpha} (q_{1i}q_{2j})^{\beta-\alpha} - (p_{1i}p_{2j})^{\gamma} (q_{1i}q_{2j})^{\delta-\gamma}] =$$

$$= I_{(\gamma,\delta)}^{(\alpha,\beta)}(P_1^* P_2;\, Q_1^* Q_2) = \text{L.H.S.}$$

(ii) Now for $\alpha, \gamma \geqq 1$ with $\beta - \alpha \geqq 1$, $\delta - \gamma \geqq 1$ it follows from (3.4) that

$$G_{(\gamma,\delta)}^{(\alpha,\beta)}(P_1;\, Q_1) = \tfrac{1}{2}\sum_{i=1}^{n} (p_{1i}^{\alpha} q_{1i}^{\beta-\alpha} + p_{1i}^{\gamma} q_{1i}^{\delta-\gamma}) \leqq 1\,,$$

which together with (3.3) proves (3.5).

Theorem 3.3. *For $(p_1, \ldots, p_n) \in \Delta_n$, $(q_{i1}, \ldots, q_{mi})$ and $(q_1, \ldots, q_m) \in \Delta_m$ $(i = 1, 2, \ldots, n)$, we have*

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}\Big(\sum_{i=1}^{n} p_i q_{1i}, \ldots, \sum_{i=1}^{n} p_i q_{im};\, q_1, \ldots, q_m\Big) \geqq \sum_{i=1}^{n} p_i\, I_{(\gamma,\delta)}^{(\alpha,\beta)}(q_{1i}, \ldots, q_{mi};\, q_1, \ldots\, q_m). \tag{3.6}$$

for all α, β, γ and δ such that either $\beta > \alpha > 1$, $0 < \delta < \gamma < 1$ or $\delta > \gamma > 1$, $0 < \beta < \alpha < 1$.

Proof. We have

$$I_{(\gamma,\delta)}^{(\alpha,\beta)}\Big(\sum_{i=1}^{n} p_i q_{1i}, \ldots, \sum_{i=1}^{n} p_i q_{mi};\, q_1, \ldots, q_m\Big) =$$

$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{j=1}^{m} [(\sum_{i=1}^{n} p_i q_{ji})^\alpha q_j^{\beta-\alpha} - (\sum_{i=1}^{n} p_i q_{ji})^\gamma q_j^{\delta-\gamma}] \geqq$$

$$\geqq (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{j=1}^{m} [\sum_{i=1}^{n} p_i q_{ji}^\alpha q_j^{\beta-\alpha} - \sum_{i=1}^{n} p_i q_{ji}^\gamma q_j^{\delta-\gamma}]$$

for $\beta > \alpha > 1$ and $0 < \delta < \gamma < 1$ (see Gallager [2], p. 523)

$$= (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} \sum_{i=1}^{n} p_i [\sum_{j=1}^{m} q_{ji}^\alpha q_j^{\beta-\alpha} - \sum_{j=1}^{m} q_{ji}^\gamma q_j^{\delta-\gamma}] =$$

$$= \sum_{i=1}^{n} p_i I_{(\gamma,\delta)}^{(\alpha,\beta)}(q_{1i}, \ldots, q_{mi}; q_1, \ldots, q_m) \quad \text{for} \quad \beta > \alpha > 1 \quad \text{and} \quad 0 < \delta < \gamma < 1 .$$

By symmetry in α, γ and β, δ, the above result is also true for $0 < \beta < \alpha < 1$ and $\delta > \gamma > 1$.

Theorem 3.4. (*Inversion Theorem*): *If we define the functions ϕ, ϕ_1 and ϕ_2 as*

(3.7) $$I_{(\gamma,\delta)}^{(\alpha,\beta)}(1/n, \ldots, 1/n; 1/s, \ldots, 1/s) = \phi(n; s) ,$$

(3.8) $$I_{(1,1)}^{(\alpha,\beta)}(1/n, \ldots, 1/n; 1/s, \ldots, 1/s) =$$

$$= \phi_1(n; s) = (1/n)^\alpha (1/s)^{\beta-\alpha} \sum_{j=2}^{n} (j)^\beta g(1/j; 1/j) ,$$

and

(3.9) $$I_{(\gamma,\delta)}^{(1,1)}(1/n, \ldots, 1/n; 1/s, \ldots, 1/s) =$$

$$= \phi_2(n; s) = (1/n)^\gamma (1/s)^{\delta-\gamma} \sum_{j=2}^{n} (j)^\delta h(1/j; 1/j) ,$$

then for all rationals m/n, r/s, where $1 \leqq m \leqq n$, $1 \leqq r \leqq s$, the function f defined by

(3.10) $$f\left(\frac{m}{n}, \frac{r}{s}\right) = \phi(n; s) - \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \left[\left(\frac{m}{n}\right)^\alpha \left(\frac{r}{s}\right)^{\beta-\alpha} \phi_1(m; r) + \right.$$

$$\left. + \left(1 - \frac{m}{n}\right)^\alpha \left(1 - \frac{r}{s}\right)^{\beta-\alpha} \phi_1(n-m; s-r)\right] -$$

$$- \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \left[\left(\frac{m}{n}\right)^\gamma \left(\frac{r}{s}\right)^{\delta-\gamma} \phi_2(m; r) + \left(1 - \frac{m}{n}\right)^\gamma \left(1 - \frac{r}{s}\right)^{\delta-\gamma} \phi_2(n-m; s-r)\right] .$$

Proof. Let $p_1 = m/n$, $q_1 = r/s$ be any two rational numbers lying in $(0, 1)$.

Next, putting in (3.3) $m = 2$, $p_1 = m/n, q_1 = r/s, p_2 = 1 - p_1 = 1 - m/n$, $q_2 = 1 - q_1 = 1 - r/s, 1 \leqq m \leqq n, 1 \leqq r \leqq s$, and

$$p_{1k} = \begin{cases} 1/m & \text{if } k = 1, 2, \ldots, m \\ 0 & \text{if } k = m+1, m+2, \ldots, n, \end{cases}$$

$$p_{2k} = \begin{cases} 1/(n-m) & \text{if } k = 1, 2, \ldots, n-m \\ 0 & \text{if } k = n-m+1, n-m+2, \ldots, n, \end{cases}$$

$$q_{1k} = \begin{cases} 1/r & \text{if } k = 1, 2, \ldots, m \\ 0 & \text{if } k = m+1, m+2, \ldots, n, \end{cases}$$

$$q_{2k} = \begin{cases} 1/(s-r) & \text{if } k = 1, 2, \ldots, n-m \\ 0 & \text{if } k = n-m+1, n-m+2, \ldots, n, \end{cases}$$

we have

(3.11)
$$I_{(\gamma,\delta)}^{(\alpha,\beta)}(\underbrace{1/n, \ldots, 1/n}_{m}, \underbrace{0, \ldots, 0}_{n-m}, \underbrace{1/n, \ldots, 1/n}_{n-m}, \underbrace{0, \ldots, 0}_{m}; \underbrace{1/s, \ldots, 1/s}_{m}, \underbrace{0, \ldots, 0}_{n-m}, \underbrace{1/s, \ldots, 1/s}_{n-m}, \underbrace{0, 0 \ldots, 0}_{m})$$

$$= I_{(\gamma,\delta)}^{(\alpha,\beta)}(m/n, 1 - m/n; r/s, 1 - r/s) +$$

$$+ \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \Big\{ (n/m)^{\alpha} (r/s)^{\beta-\alpha} I_{(1,1)}^{(\alpha,\beta)}(\underbrace{1/m. \ldots, 1/m}_{m}; \underbrace{0, \ldots, 0}_{n-m}; \underbrace{1/r, \ldots, 1/r}_{m}, \underbrace{0, \ldots, 0}_{n-m}) +$$

$$+ (1 - m/n)^{\alpha} (1 - r/s)^{\beta-\alpha} I_{(1,1)}^{(\alpha,\beta)} \Big(\underbrace{\frac{1}{n-m}, \ldots, \frac{1}{n-m}}_{n-m}, \underbrace{0, \ldots, 0}_{m}; \underbrace{\frac{1}{s-r}, \ldots, \frac{1}{s-r}}_{n-m}, \underbrace{0, \ldots, 0}_{m} \Big) \Big\} +$$

$$+ \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \Big\{ (m/n)^{\gamma} (r/s)^{\delta-\gamma} I_{(\gamma,\delta)}^{(1,1)} (\underbrace{1/m, \ldots, 1/m}_{m}, \underbrace{0, \ldots, 0}_{n-m}; \underbrace{1/r, \ldots, 1/r}_{m}, \underbrace{0, \ldots, 0}_{n-m}) +$$

$$+ (1 - m/n)^{\gamma} (1 - r/s)^{\delta - \gamma} I^{(1,1)}_{(\gamma,\delta)} \Big(\underbrace{\frac{1}{n-m}, \ldots, \frac{1}{n-m}}_{n-m}, \underbrace{0, \ldots, 0}_{m}; \underbrace{\frac{1}{s-r}, \ldots, \frac{1}{s-r}}_{n-m}, \underbrace{0, \ldots, 0}_{m} \Big).$$

As $I^{(\alpha,\beta)}_{(\gamma,\delta)} : \Delta_n \times \Delta_n \to R (n = 2, 3, \ldots)$ are symmetric and expansible (Theorem 3.1) and

$$f(p; q) = I^{(\alpha,\beta)}_{(\gamma,\delta)}(p, 1 - p; q, 1 - q) . \tag{3.12}$$

Now (3.11) together with (3.7), (3.8), (3.9) and (3.12) gives the desired result (3.10).

Corollary. *If the functions ϕ, ϕ_1 and ϕ_2 satisfy the relation*

$$\phi(n; s) = \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} \phi_1(n; s) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} \phi_2(n; s) \tag{3.13}$$

with

$$\phi_1(1; 2) = \phi_2(1; 2) = 1; \quad \phi_1(1; 1) = \phi_2(1; 1) = 0 , \tag{3.14}$$

then

$$\phi(n; s) = (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} (n^{1-\alpha} s^{\alpha-\beta} - n^{1-\gamma} s^{\gamma-\delta}) \tag{3.15}$$

and

$$f(m/n; r/s) = (A_{\alpha,\beta} - A_{\gamma,\delta})^{-1} [(m/n)^{\alpha} (r/s)^{\beta-\alpha} + (1 - m/n)^{\alpha} (1 - r/s)^{\beta-\alpha} - (m/n)^{\gamma} (r/s)^{\delta-\gamma} - (1 - m/n)^{\gamma} (1 - r/s)^{\delta-\gamma}] . \tag{3.16}$$

Proof. Setting $p_j = 1/m$, $p_{jk} = 1/n$, $q_j = 1/r$, $q_{jk} = 1/s$ $(j = 1, 2, \ldots, m; k = 1, 2, \ldots, n)$, $1 \leqq m \leqq r$, $1 \leqq n \leqq s$ in (3.3), we get

$$\phi(mn; rs) = \phi(m; r) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} m^{1-\alpha} r^{\alpha-\beta} \phi_1 (n; s) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} m^{1-\gamma} r^{\gamma-\delta} \phi_2(n; s) \tag{3.17}$$

$$\phi(mn; rs) = \phi(n; s) + \frac{A_{\alpha,\beta}}{A_{\alpha,\beta} - A_{\gamma,\delta}} n^{1-\alpha} s^{\alpha-\beta} \phi_1(m; r) + \frac{A_{\gamma,\delta}}{A_{\gamma,\delta} - A_{\alpha,\beta}} n^{1-\gamma} s^{\gamma-\delta} \phi_2(m; r) . \tag{3.18}$$

Now (3.17) and (3.18) together with (3.13) give

$$(3.19) \qquad A_{\alpha,\beta}\{(1 - n^{1-\alpha}s^{\alpha-\beta})\,\phi_1(m;r) + (n^{1-\alpha}r^{\alpha-\beta} - 1)\,\phi_1(n;s)\} =$$
$$= A_{\gamma,\delta}\{(1 - n^{1-\gamma}s^{\gamma-\delta})\,\phi_2(m;r) + (m^{1-\gamma}r^{\gamma-\delta} - 1)\,\phi_2(n;s)\,.$$

Taking $n = 1$, $s = 2$ in (3.19) and using (3.14), we get

$$(3.20) \qquad A_{\alpha,\beta}\{(1 - 2^{\alpha-\beta})\,\phi_1(m;r) + (m^{1-\alpha}r^{\alpha-\beta} - 1)\} =$$
$$= A_{\gamma,\delta}\{(1 - 2^{\gamma-\delta})\,\phi_2(m;r) + (m^{1-\gamma}r^{\gamma-\delta} - 1)\} = K\,(\text{say}),$$

where K is an arbitrary constant.

Next, taking $m = r = 1$ in (3.20) and using (3.14), we obtain $K = 0$.

Thus, we have

$$(3.21) \qquad \phi_1(m;r) = A_{\alpha,\beta}^{-1}(m^{1-\alpha}r^{\alpha-\beta} - 1)\,, \quad A_{\alpha,\beta} \neq 0$$

and

$$(3.22) \qquad \phi_2(m;r) = A_{\gamma,\delta}^{-1}(m^{1-\gamma}r^{\gamma-\delta} - 1)\,, \quad A_{\gamma,\delta} \neq 0\,.$$

Thus (3.13) together with (3.21) and (3.22) gives (3.15). Finally, (3.16) follows from (3.10), (3.13), (3.21) and (3.22).

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Souhrn

O ZOBECNĚNÝCH MÍRÁCH RELATIVNÍ INFORMACE A NEPŘESNOSTI

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Kullbackova relativní informace a Kerridgeova nepřesnost jsou dvě informačněteoretické míry pro dvojice pravděpodobnostních distribucí diskrétních náhodných veličin. V článku se studuje zobecněná míra, která speciálně zahrnuje parametrické zobecnění relativní informace a nepřesnosti. Jsou rovněž odvozeny některé důležité vlastnosti této zobecněné míry a věta o inversi.

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