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FINITE ELEMENT ANALYSIS FOR UNILATERAL PROBLEMS WITH OBSTACLES ON THE BOUNDARY

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INTRODUCTION

In the present paper the finite element analysis for unilateral problems with obstacles on the boundary Γ of a polygonal domain $\Omega \subset R^2$ is given. Using the technique of [9], [7], the rate of convergence is proved provided the exact solution is smooth enough. We obtain the same results as in [2], [3], where the technique of [5], [6] was used for investigating the primary and the dual problems. In the present paper the case of nonhomogeneous obstacles as well as the case of two obstacles, "lower and upper", is studied. As the regularity hypotheses are not fulfilled in general, we prove — for one class of problems — the convergence of finite element approximations to the exact solution without any regulatity assumptions.

1. SETTING OF THE PROBLEM

Let $\Omega \subset R^m$ be a bounded domain. $H^k(\Omega)$ ($k \geqq 0$ integer) will denote the space of all functions, the derivatives of which up to the order k (in the sense of distributions) are square integrable in Ω. The norm of $u \in H^k(\Omega)$ (defined by the usual manner) will be denoted by $\|u\|_k$. For simplicity we write $H^0(\Omega) = L^2(\Omega)$ and the scalar product of $u, v \in L^2(\Omega)$ will be denoted by (u, v). A repeated Latin index implies always summation over the range $1, \ldots, n$.

Let $\Omega \subset R^2$ be a bounded domain with a Lipschitz boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \mathscr{R}$, where Γ_1, Γ_2 are open in Γ, $\Gamma_2 \neq \emptyset$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\text{mes}_1\, \mathscr{R} = 0$ (one-dimensional Lebesgue measure) and $\bar{\Gamma}_1 \cap \bar{\Gamma}_2 = \{A, B\}$ if Γ_1 is non-empty.[1])

[1]) After slight modifications of proofs of this paper, one can easily extend its results to the case of multiply connected domains.

Let us set

$$V = \{v \in H^1(Q) : v = 0 \quad \text{on} \quad \Gamma_1\},$$

$$K = \{v \in V : v \geqq \psi \qquad \text{on} \quad \Gamma_2\},$$

where ψ is a given function defined on Γ_2 such that $\psi(A) = \psi(B) = 0$.
Let us define

$$J(v) = \int_Q \left(a_{ij} \frac{\partial v}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0 v^2 \right) \mathrm{d}x - 2 \int_Q f v \,\mathrm{d}x ,$$

where

(1.1) $$f \in L^2(Q),$$

(2.1) $$a_{ij}, a_0 \in L^\infty(Q), \quad i, j = 1, 2,$$

(3.1) $$a_{ij}(x) = a_{ji}(x) \quad \text{a.e. in } Q, \quad i, j = 1, 2,$$

(4.1) $$a_0 \geqq 0 \quad \text{and either} \quad \Gamma_1 \neq \emptyset \quad \text{or} \quad \exists c = \text{const.} > 0,$$

$$a_0(x) \geqq c \quad \text{a.e. in } Q,$$

(5.1) $$\exists \alpha = \text{const.} > 0 : \forall \zeta \in R^2$$

$$a_{ij}\zeta_i\zeta_j \geqq \alpha \|\zeta\|^2$$

a.e. in Q.

We shall consider the following problem (P):

(P) $$\textit{find } u \in K : J(u) = \min_{v \in K} J(v).$$

Theorem 1.1. *If* (1.1)–(5.1) *hold, then there exists a unique solution of* (P), *which is characterized by*

(6.1) $$u \in K : a(u, v - u) \geqq (f, v - u) \quad \forall v \in K.$$

where

$$a(u, v) = \int_Q \left(a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} + a_0 u v \right) \mathrm{d}x .$$

2. APPROXIMATION OF (P)

Let $Q \subset R^2$ be a polygonal domain, $\bar{\Gamma}_2 = \bigcup_{j=1}^{m} A_j A_{j+1}$, $A_1 = A$, $A_{m+1} = B$ (if $\Gamma_1 \neq \emptyset$). Let $\{\mathcal{T}_h\}$, $h \in (0, 1)$ be a system of *regular triangulations* of $\bar{Q}$, satisfying the usual requirements concerning the mutual position of triangles T_i and such that A, B are vertices of $\mathscr{T}_h$ for every $h \in (0, 1)$. We define

$$V_h = \{v \in C(\bar{Q}) \cap V : v|_{T_i} \textit{ is linear in } T_i, \ \forall T_i \in \mathscr{T}_h\},$$

$$K_h = \{v \in V_h : v(a_i) \geqq \psi(a_i), \textit{ where } a_i \textit{ are vertices of } \mathscr{T}_h \textit{ on } \Gamma_2, i = 1, \dots, n\}.$$

In general, $K_h \not\subset K$.

We define the problem $(\mathrm{P_h})$ *(an approximation of* (P)*) in the following manner:*

$$(\mathrm{P}_h) \quad \textit{find} \quad u_h \in K_h : J(u_h) = \min_{v \in K_h} J(v).$$

Theorem 1.2. *There exists a unique solution of* $(\mathrm{P_h})$, *which is characterized by*

$$u_h \in K_h : a(u_h, v_h - u_h) \geqq (f, v_h - u_h) \quad \forall v_h \in K_h. \tag{1.2}$$

For the proof see [1]. Our aim is to estimate the rate of convergence of $\|u - u_h\|_1$.

Lemma 1.2. *It holds*

$$\|u - u_h\|_1^2 \leqq c\{(f, u - v_h) + (f, u_h - v) + a(u_h - u, v_h - u) + \\ + a(u, v - u_h) + a(u, v_h - u)\} \quad \forall v \in K, v_h \in K_h, \tag{2.2}$$

where c *is an absolute constant.*

Proof. Let $v \in K$, $v_h \in K_h$. Then

$$\alpha\|u - u_h\|_1^2 \leqq a(u - u_h, u - u_h) = a(u, u) + a(u_h, u_h) - a(u, u_h) - \\ - a(u_h, u) \leqq a(u, v) + (f, u - v) + a(u_h, v_h) + (f, u_h - v_h) - \\ - a(u, u_h) - a(u_h, u) = (f, u - v_h) + (f, u_h - v) + a(u, v - u_h) + \\ + a(u_h - u, v_h - u) + a(u, v_h - u).$$

Consequence. *If* $K_h \subset K$, $h \in (0, 1)$, *then substituting* $v = u_h$ *in* (2.2) *we obtain for all* $v_h \in K_h$:

$$\|u - u_h\|_1^2 \leqq c\{(f, u - v_h) + a(u_h - u, v_h - u) + a(u, v_h - u)\}. \tag{3.2}$$

Theorem 2.2. *Let* $\psi = 0$ *on* Γ_2, *let the solution* u *satisfy* $u \in K \cap H^2(Q)$ *and* $u|_{\Gamma_2} \in H^2(A_j A_{j+1})$ $\forall j = 1, \dots, m$. *Then*

$$\|u - u_h\|_1 = O(h).$$

Proof. The Green formula yields

$$a(u, v_h - u) = \int_Q \left(- \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) (v_h - u) + a_0 \, u(v_h - u) \right) \mathrm{d}x + $$
$$+ \int_{\Gamma_2} \frac{\partial u}{\partial n_A} (v_h - u) \, \mathrm{d}s \,,$$

where $\partial u / \partial n_A = a_{ij}(\partial u / \partial x_j) n_i$, n_i being the components of the unit outward normal to Γ. It is easy to verify that

$$(4.2) \qquad - \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + a_0 u = f \quad \text{a.e. in } Q \,.$$

This together with (3.2) implies

$$(5.2) \qquad \|u - u_h\|_1^2 \leqq c \left\{ a(u_h - u, v_h - u) + \int_{\Gamma_2} \frac{\partial u}{\partial n_A} (v_h - u) \, \mathrm{d}s \right\} \leqq$$
$$\leqq c\{ \|u_h - u\|_1 \|v_h - u\|_1 + \|v_h - u\|_{L_2(\Gamma_2)} \} \leqq$$
$$\leqq c\varepsilon \|u - u_h\|_1^2 + \frac{c}{\varepsilon} \|u - v_h\|_1^2 + c\|v_h - u\|_{L_2(\Gamma_2)} \,,$$

where $\varepsilon > 0$ is arbitrary.

Let us set $v_h = R_h u$, where $R_h u \in V_h$ is defined by the relation

$$R_h u = \Pi_{T_i} u \quad \text{on} \quad T_i \,, \quad \forall T_i \in \mathcal{T}_h \,.$$

$\Pi_{T_i} u$ denotes the linear Lagrange interpolate of u on T_i. As $v_h \in K_h$, the assertion of the theorem follows from the well-known properties of Lagrange interpolation, (5.2) and the regularity assumptions.

Let us consider the general case $K_h \not\subset K$.

Theorem 3.2. *Let* $\psi \in H^2(A_j A_{j+1}) \cap H^1(\Gamma_2)$, $u \in K \cap H^2(Q)$, $u|_{\Gamma_2} \in H^2(A_j A_{j+1})$, $j = 1, \ldots, m$. *Then*

$$\|u - u_h\|_1 = O(h) \,.$$

Proof follows from the Green formula, (2.2) and (4.2):

$$(6.2) \qquad \|u - u_h\|_1^2 \leqq c \left\{ a(u_h - u, v_h - u) + \int_{\Gamma_2} \frac{\partial u}{\partial n_A} (v - u_h) \, \mathrm{d}s + \right.$$
$$\left. + \int_{\Gamma_2} \frac{\partial u}{\partial n_A} (v_h - u) \, \mathrm{d}s \right\} \quad \forall v \in K \,, \quad \forall v_h \in K_h \,.$$

The first and the third member on the right hand side of (6.2) can be estimated in the same manner as in Theorem 2.2 (we substitute $v_h = R_h u$). Let us consider the second member. We define

$$\bar{v} = \sup(u_h, \psi) \quad \text{on} \quad \Gamma_2 ,$$

$$\bar{v} = 0 \qquad\qquad \text{on} \quad \Gamma - \Gamma_2 .$$

Then $\bar{v} \in H^1(\Gamma)$, $\bar{v} \geqq \psi$ on Γ_2 and there exists a function $v \in H^1(Q)$ such that $v = \bar{v}$ on Γ. Hence $v \in K$ and

$$u_h - \bar{v} = \begin{cases} 0 & \text{if} \quad u_h \geqq \psi \\ u_h - \psi & \text{if} \quad u_h < \psi . \end{cases}$$

As $u_h(a_i) \geqq \psi(a_i)$, $i = 1, \ldots, n$, it is $u_h \geqq r_h\psi$ on Γ_2, where $r_h\psi|_{a_i a_{i+1}}$ is the linear Lagrange interpolate of ψ on $a_i a_{i+1}$. Now

$$\int_{\Gamma_2} (\bar{v} - u_h)^2 \,\mathrm{d}s = \int_{\Gamma_2 +} (\psi - u_h)^2 \,\mathrm{d}s ,$$

where

$$\Gamma_2^+ = \{x \in \Gamma_2 : u_h(x) < \psi(x)\} .$$

Since $0 < (\psi - u_h)(x) \leqq (\psi - r_h\psi)(x)$ on Γ_2^+, we have

$$\int_{\Gamma_2} (\bar{v} - u_h)^2 \,\mathrm{d}s = \int_{\Gamma_2 +} (\psi - u_h)^2 \,\mathrm{d}s \leqq \int_{\Gamma_2 +} (\psi - r_h\psi)^2 \,\mathrm{d}s = O(h^4) .$$

Remark. *Supposing* $u \in H^2(Q)$, *we can prove that* $\|u - u_h\|_1 = O(h^{3/4})$. See also [8].

Now let us consider the case of "two obstacles" on Γ_2. Let

$$K = \{v \in V : \varphi \leqq v \leqq \psi \quad \text{on} \quad \Gamma_2\} ,$$

where φ, ψ are functions defined on Γ_2, $\varphi(x) \leqq \psi(x)$ for each $x \in \Gamma_2$, $\varphi(A) = \varphi(B) = \psi(A) = \psi(B) = 0$ if $\Gamma_1 \neq \emptyset$. *We define the problem* $(\tilde{\mathrm{P}})$:

$(\tilde{\mathrm{P}})$ *find* $u \in K : J(u) = \min_{v \in K} J(v) .$

Let $K_h = \{v \in C(\bar{Q}) \cap V, v|_{T_i}$ *is linear on* $T_i \in \mathcal{T}_h$, $\varphi(a_i) \leqq v(a_i) \leqq \psi(a_i)$, $i = 1, \ldots, n\}$.
Let us recall that $a_1, \ldots, a_n$ are the vertices of $\mathcal{T}_h$ on Γ_2.

We define the problem $(\tilde{P}_h)$:

$(\tilde{P}_h)$ *find* $u_h \in K_h : J(u_h) = \min_{v \in K_h} J(v) .$

Theorem 4.2. *Let the solution u satisfy $u \in K \cap H^2(Q)$, $u|_{\Gamma_2} \in H^2(A_j A_{j+1})$; let $\varphi, \psi \in H^2(A_j A_{j+1}) \cap H^1(\Gamma_2)$, $j = 1, \ldots, m$. Then*

$$\|u - u_h\|_1 = O(h) .$$

Proof. We use (6.2). It is sufficient to estimate

$$\int_{\Gamma_2} \frac{\partial u}{\partial n_A} (v - u_h) \, ds .$$

Let us set

$$\bar{v} = \begin{cases} \max(\min(u_h, \psi), \varphi) & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma - \Gamma_2 . \end{cases}$$

It is easy to see that $\varphi \leqq \bar{v} \leqq \psi$ on Γ_2, $\bar{v} \in H^1(\Gamma)$ so that there exists a function $v \in H^1(Q) : v = \bar{v}$ on Γ. Hence $v \in K$ and

$$u_h - \bar{v} = \begin{cases} u_h - \psi & \text{if } u_h > \psi \\ 0 & \text{if } \varphi \leqq u_h \leqq \psi \\ u_h - \varphi & \text{if } u_h < \varphi . \end{cases}$$

We can write

$$\int_{\Gamma_2} (u_h - \bar{v})^2 \, ds = \int_{\Gamma_2 +} (u_h - \psi)^2 \, ds + \int_{\Gamma_2 -} (u_h - \varphi)^2 \, ds ,$$

where $\Gamma_2^+ = \{x \in \Gamma_2 : u_h(x) > \psi(x)\}$, $\Gamma_2^- = \{x \in \Gamma_2 : u_h(x) < \varphi(x)\}$. The inequality $u_h(a_i) \leqq \psi(a_i)$ implies $u_h \leqq r_h \psi$ on Γ_2 and similarly $u_h \geqq r_h \varphi$ on Γ_2.
Hence

$$0 < u_h(x) - \psi(x) \leqq r_h \psi(x) - \psi(x) \quad \text{on} \quad \Gamma_2^+ ,$$
$$0 < \varphi(x) - u_h(x) \leqq \varphi(x) - r_h \varphi(x) \quad \text{on} \quad \Gamma_2^-$$

and

$$\int_{\Gamma_2 +} (u_h - \psi)^2 \, ds \leqq \int_{\Gamma_2} (r_h \psi - \psi)^2 \, ds = O(h^4) ,$$
$$\int_{\Gamma_2 -} (u_h - \varphi)^2 \, ds \leqq \int_{\Gamma_2} (\varphi - r_h \varphi)^2 \, ds = O(h^4) .$$

APPENDIX

In the above analysis we needed very strong regularity assumptions concerning the solution u in order to be able to prove the rate of convergence. Unfortunately there are no reasons to expect such a great smoothness in general case. In this Appendix we prove the convergence of u_h to u for a particular problem without any regularity assumptions. We shall consider the problem (P) with

$$K = \{v \in H^1(Q) : v \geqq \psi \text{ on } \Gamma\} ,$$

where (A 1) ψ *is the trace of a function* $\Psi \in H^{1+\varepsilon}(Q)$ ($\varepsilon > 0$). First we prove two auxiliary lemmas.

Lemma A1. *Let* $\tilde{K} = \{v \in H^1(Q) : v \geqq 0$ *on* $\Gamma\}$. *Then* $\tilde{K}$ *is the closure in* $H^1(\bar{Q})$ *of the set*

$$\mathscr{E}_+(\bar{Q}) = \{v \in \mathscr{E}(\bar{Q}) : v \geqq 0 \text{ on } \Gamma\}\,.^1)$$

Proof. Let $u \in \tilde{K}$ be arbitrary. Then $u|_\Gamma \in H^{1/2}(\Gamma)$ (for the definition see [4]), $u|_\Gamma \geqq 0$ and there exists a function $U \in H^1(Q)$ such that $U = u$ on Γ, $U \geqq 0$ in Q. For the construction of U see [4], p. 100. We can write

$$u = U + Z\,,$$

where $Z \in H^1_0(Q)$. The density of $D(Q)$ in $H^1_0(Q)^2)$ implies that there exist $Z_h \in D(Q)$ such that $Z_h \to Z$ in $H^1(Q)$. The regularization U_h of U are also non-negative in Q and if U is suitably extended on a domain G, $\bar{Q} \subset G$ then $U_h \to U$ in $H^1(Q)$. Setting $u_h = U_h + Z_h \in \mathscr{E}_+(\bar{Q})$ we have $u_h \to u$ in $H^1(Q)$.

Lemma A2. *Let* φ *be a continuous function defined on* $\langle a, b\rangle$ $(-\infty < a < b < \infty)$, $D_n : a = x_0^n < x_1^n < \ldots < x_n^n = b$ *a division of* $\langle a, b\rangle$, $v(D_n) = \max\limits_{i=1,\ldots,n} |x_i^n - x_{i-1}^n| \to$ $\to 0$ *for* $n \to \infty$. *Let* $\{\psi_n\}_{n=1}^\infty$ *be a sequence of piecewise linear functions with nodes at* x_n^i *such that* $\psi_n(x_i^n) \geqq \varphi(x_n^i)\ \forall i = 0, \ldots, n;\ n = 1, 2, \ldots$. *Let* $\psi_n \to \psi$ *a.e. in* $\langle a, b\rangle$. *Then* $\psi \geqq \varphi$ *a.e. in* $\langle a, b\rangle$.

Proof. Let $M \subseteq \langle a, b\rangle$, $\mu(M) = b - a$ (one-dimensional Lebesgue measure) and such that $\forall x \in M : \psi_n(x) \to \psi(x)$. We shall show that for all $x \in M$, $\psi(x) \geqq$ $\geqq \varphi(x)$. Let there exist $x \in M$ such that $\psi(\bar{x}) < \varphi(\bar{x})$. Since φ is continuous there exists $\delta > 0$ such that

$$\text{(A2)} \qquad \varphi(x) > \psi(\bar{x}) + 1/2(\varphi(\bar{x}) - \psi(\bar{x})) \quad \text{for} \quad \forall x \in (\bar{x} - \delta, \bar{x} + \delta)\,, \quad \delta > 0\,.$$

On the other hand, there exists n_0 such that

$$\text{(A3)} \qquad \begin{aligned} &\psi_n(\bar{x}) < \psi(\bar{x}) + 1/2(\varphi(\bar{x}) - \psi(\bar{x}))\,, \\ &v(D_n) < \delta \end{aligned}$$

for $\forall n \geqq n_0$. We distinguish two cases:

---

[1]) $\mathscr{E}(\bar{Q}) = \bigcap\limits_{k=1}^{\infty} C^k(\bar{Q})$ ($k \geqq 0$ integer), where $C^k(\bar{Q})$ denotes the space of continuous functions, the derivatives of which up to the order k are also continuous and continuously extensible onto $\bar{Q}$.

[2]) $H^1_0(Q)$ is the subspace of $H^1(Q)$ of functions, the traces of which are equal to zero on Γ. $D(Q) \subset \mathscr{E}(\bar{Q})$ is the space of functions with compact support in Q.

1. Let there exist $n \geqq n_0$ such that $\bar{x} \in (x^n_{i_0}, x^n_{i_0+1})$ for some $i_0 \in \{0, 1, \ldots, n\}$. The restriction of ψ_n on $x^n_{i_0}x^n_{i_0+1}$ is linear and $\psi_n(x^n_{i_0}) \geqq \varphi(x^n_{i_0})$, $\psi_n(\bar{x}) < \varphi(\bar{x})$ and (A2), (A3) yield $\varphi(x^n_{i_0}) > \psi_n(\bar{x})$. Hence $\psi_n(x^n_{i_0+1}) < \psi_n(\bar{x}) < \psi(\bar{x}) + 1/2(\varphi(\bar{x}) - \psi(\bar{x})) < \varphi(x)$ $\forall x \in (\bar{x} - \delta, \bar{x} + \delta)$. In particular, for $x = x^n_{i_0+1}$ we obtain

$$\psi_n(x^n_{i_0+1}) < \varphi(x^n_{i_0+1}),$$

i.e. a contradiction with the assumptions of Lemma A2.

2. Let $\bar{x} = x^n_{i_0}$ for all $n \geqq n_0$. Then $\psi_n(\bar{x}) \geqq \varphi(\bar{x}) \Rightarrow \psi(\bar{x}) \geqq \varphi(\bar{x})$ which is a contradiction.

Now we can prove the main result of this Appendix.

Theorem A1. *Let (A1) be satisfied. Then* $\|u - u_h\|_1 \to 0$ *for* $h \to 0+$.

Proof. according to [1], p. 142, it is sufficient to prove

(i) $\forall v \in K \quad \exists v_h \in K_h : \|v - v_h\|_1 \to 0 \quad \text{for} \quad h \to 0+$,
(ii) $v_h \in K_h$, $v_h \rightharpoonup v$ (weakly) in $H^1(Q) \Rightarrow v \in K$.

(ad i) Let $v \in K$ be arbitrary. Then $v - \Psi \in H^1(Q)$ and $v - \Psi = v - \psi \geqq 0$ on Γ. Lemma A1 implies that there exist $\varphi_H \in \mathscr{E}(\bar{Q})$: $\varphi_H \to v - \Psi$ in $H^1(Q)$. As $H^{1+\varepsilon}(Q) \supset \subset C(\bar{Q})$, we can construct $R_h\Psi$ in the same manner as in Theorem 2.2 and

$$\|\Psi - R_h\Psi\|_1 \to 0, \quad h \to 0+.$$

The same we do for φ_H:

$$\|\varphi_H - R_h\varphi_H\|_1 \to 0, \quad h \to 0+.$$

Setting $v_h = R_h\Psi + R_h\varphi_H$ we have $v_h \in K_h$ for each $h > 0$ and

$$\|v - v_h\|_1 \leqq \|(v - \Psi) - \varphi_H + \varphi_H - R_h\varphi_H\|_1 + \|\Psi - R_h\Psi\|_1 \to 0$$

for $h, H \to 0+$.

(ad ii) Let $v_h \rightharpoonup v$ in $H^1(Q)$, $v_h \in K_h$. Then $v_h \to v$ in $L^2(\Gamma)$[1]) and we can extract a subsequence $\{v_{h_n}\}$ such that $v_{h_n} \to v$ a.e. in Γ. Using Lemma A2 we prove $v \geqq \psi$ a.e. in Γ, i.e. $v \in K$.

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Souhrn

ANALÝZA JEDNOSTRANNÝCH ÚLOH S PŘEKÁŽKAMI NA HRANICI METODOU KONEČNÝCH PRVKŮ

JAROSLAV HASLINGER

Práce se zabývá aplikací metody konečných prvků pro řešení (i) jednostranných úloh s obecně nehomogenní překážkou na hranici Γ (ii) dvoustranných úloh, kdy je na Γ zadána „horní a dolní" překážka. Jsou-li řešení uvedených problémů dostatečně hladká, potom je dokázáno, že konvergence přibližných řešení k přesnému je řádu $0(h)$. Přitom používáme po částech lineárních konečných prvků. Protože oblast, na níž problémy řešíme je polygonální, není obecně zaručena tak vysoká hladkost řešení. Proto v závěru práce je proveden důkaz konvergence (bez odhadu její rychlosti) pro problémy typu (i), aniž bychom předpokládali dodatečnou hladkost.

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