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A-STABLE METHODS OF HIGH ORDER FOR VOLTERRA INTEGRAL EQUATIONS

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1. INTRODUCTION

Consider the linear Volterra integral equation of the second kind

$$y(t) = \int_0^t K(t, s)\, y(s)\, \mathrm{d}s + g(t), \quad 0 \leqq s \leqq t \leqq T < \infty, \tag{1.1}$$

where $g(t)$ and $K(t, s)$ are continuous on $0 \leqq s \leqq t \leqq T$.

It is well known that under this condition there is a unique continuous solution of (1.1) on the interval $[0, T]$. Although we will treat only the linear case, it is simple to adjust the present method for the numerical solution of integral equations of the first kind, nonlinear and integro-differential equations of Volterra type.

For efficient numerical solution one must often ask not only for high asymptotic accuracy but also for other requirements. One of these is Dahlquist's A-stability.

Recently, de Hoog and Weiss ([2] and [3]) have suggested, for the numerical solution of the Volterra integral equations of the first kind, methods which are A-stable (they call them numerically stable). The methods are block by block methods, i.e., many matrix inversions are needed.

The main result of the present paper is the proof of existence of A-stable high order methods for Volterra integral equations of the second kind. Our method connects the good features of block by block methods (A-stability plus high asymptotic accuracy, not attainable by step by step methods) with relatively easy numerical realization of step by step methods. At least we need no matrix inversion.

Our method for numerical solution of (1.1) which is denoted by $\mathfrak{M}$ is based on the following procedure.

Denote

$$z(t, u) = \int_0^u K(t, s)\, y(s)\, \mathrm{d}s, \quad 0 \leqq u \leqq t. \tag{1.2}$$

For every fixed $t \in [0, T]$, the equation (1.1) is equivalent to the problem

$$\frac{\partial}{\partial u} z(t, u) = K(t, u)\, y(u), \quad u \leqq t \tag{1.3}$$

$$z(t, 0) = 0 \tag{1.4}$$

and

$$z(t, t) = y(t) - g(t). \tag{1.5}$$

Then our method $\mathfrak{M}$ consists in two steps. First, we compute an approximate solution of the problem (1.3)–(1.4) by an O.I.M. (overimplicit multistep) method (on such method cf. [1]) then an approximate solution of (1.1) is computed by means of (1.5).

Since the method $\mathfrak{M}$ is essentially based on the O.I.M. methods, part 2 is a quotation of some basic concepts and results from [1]. In part 3, method $\mathfrak{M}$ is described in detail. Convergence theorem and order of $\mathfrak{M}$ is examined in part 4. Part 5 is devoted to the extension of Dahlquist's A-stability concept to integral equations and to the proof of existence of A-stable method $\mathfrak{M}$. Finally, a numerical example is given in part 6.

In the end, we introduce the notation needed. Let $t_q = qh$, $q = 0(1)N$, $t_N = T$ and let h positive be the stepsize. Let y be a real function defined on $[0, T]$. The approximation value of $y(t_q)$ computed by a numerical method is denoted by y_q. We always mention explicitly this numerical method if it is not clear from the context. Vector $\mathbf{e}$ is the k-dimensional vector $[1, \ldots, 1]^T$, $\mathbf{0}$ is the zero vector and $\mathbf{0}_{i,j}$ is the zero matrix of type $i \times j$ while $\mathbf{I}_j$ is the identity matrix of order j.

2. PRELIMINARIES

In this part we briefly quote some concepts and results from [1]. Consider the problem

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = f(t, y(t)), \quad t \in [0, T] \tag{2.1}$$

$$y(0) = y^0, \tag{2.2}$$

where the right hand term is continuous and satisfying the Lipschitz condition with respect to y in the strip $0 \leqq t \leqq T$, $-\infty < y < +\infty$. O.I.M. methods for numerical solution of the problem (2.1)–(2.2) differ from linear l-step methods in such a way that instead of computing the approximate solution at one point, from the known approximate solutions at l preceding points, we compute the approximate solution

at k successive points simultaneously. Hence one step of an O.I.M. method, given by matrices $\boldsymbol{B} = (b_{ij})_{i,j=1}^{k,l}$, $\boldsymbol{C} = (c_{ij})_{i,j=1}^{k}$ and $\boldsymbol{D} = (d_{ij})_{i,j=1}^{k,l}$, consists in computing k values $y_{n+1}, \ldots, y_{n+k}$ of the approximate solution at k successive points $t_{n+1}, \ldots$ $\ldots, t_{n+k}$, from known l values at the preceding points $t_{n-l+1}, \ldots, t_n$, by means of the system:

$$\begin{bmatrix} y_{n+1} \\ \vdots \\ y_{n+k} \end{bmatrix} + \boldsymbol{B} \begin{bmatrix} y_{n-l+1} \\ \vdots \\ y_n \end{bmatrix} - h\boldsymbol{C} \begin{bmatrix} f(t_{n+1}, y_{n+1}) \\ \vdots \\ f(t_{n+k}, y_{n+k}) \end{bmatrix} - h\boldsymbol{D} \begin{bmatrix} f(t_{n-l+1}, y_{n-l+1}) \\ \vdots \\ f(t_n, y_n) \end{bmatrix} = \boldsymbol{0}.$$

In the next step of the method we choose new l initial values from the values $y_{n-l+2}, \ldots, y_{n+k}$. Because this can be done in many different ways, we are bound to say how to proceed. So we introduce a parameter $s\,(1 \leqq s \leqq k)$ and new initial values are $y_{n-l+1+s}, \ldots, y_{n+s}$. If $s < k$ we forget the values $y_{n+s+1}, \ldots, y_{n+k}$ just computed and recompute them in the next step. Nevertheless, and this cannot lead to any misunderstanding, we shall always denote the value of approximate solution at the point t_i by the only symbol y_i. To the O.I.M. method just defined we shall refer as to the O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$.

The O.I.M. method for which $l = 1$, $\boldsymbol{B} = -\boldsymbol{e}$ will be called the selfstarting method.

Let y be $(p+1)$-times continuously differentiable (p positive integer) on $[0, T]$ and denote

$$\begin{bmatrix} y(t+h) \\ \vdots \\ y(t+kh) \end{bmatrix} + \boldsymbol{B} \begin{bmatrix} y(t-(l-1)h) \\ \vdots \\ y(t) \end{bmatrix} - h\boldsymbol{C} \begin{bmatrix} y'(t+h) \\ \vdots \\ y'(t+kh) \end{bmatrix} - h\boldsymbol{D} \begin{bmatrix} y'(t-(l-1)h) \\ \vdots \\ y'(t) \end{bmatrix} = \begin{bmatrix} L_1(y(t); h) \\ \vdots \\ L_k(y(t); h) \end{bmatrix}.$$

The vector $[L_1, \ldots, L_k]^T$ is called the local error of the method.

Definition. *O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ is said to be of order p if the local error of the method is of the order h^{p+1}. Moreover, if p is at least one the O.I.M. method is said to be consistent.*

For the sake of brevity we suppose to be $l \leqq s$. All the following theorems hold also for $l > s$. Definition of the method $\mathfrak{M}$ remains unchanged.

Let us define the matrix

$$\boldsymbol{R} = [\boldsymbol{0}_{l,s-l}, \boldsymbol{I}_l, \boldsymbol{0}_{l,k-s}]$$

and the matrix

$$\boldsymbol{E} = -\boldsymbol{R}\boldsymbol{B}\,.$$

Definition. *O.I.M. method* $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ *is said to be stable if there is a positive constant* c *such that*

$$\|\boldsymbol{E}^n\| \leqq c \quad \textit{for every integer } n\,.$$

Theorem 1. *A stable and consistent O.I.M. method is convergent. Moreover, if the method is of order* p *and the starting error defined by* $e_i = y_i - y(t_i)$ $i = 0(1)\, l-1$ *is of order at least* h^p, *then*

$$y_n = y(t_n) + O(h^p)$$

for every t_n *from* $[0, T]$ *fixed.*

3. DESCRIPTION OF THE METHOD $\mathfrak{M}$

Consider O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ which computes from values of the approximate solution at l preceding points the values at k successive points. Denote $z^p(t_q) = z(t_p, t_q)$ for $t_p \geqq t_q$ and by z_q^p value of the approximate solution of the problem (1.3)–(1.4) at the point t_q, obtained by the O.I.M. method under consideration. Let sufficiently exact starting values of the approximate solution of the integral equation at the points $t_0, \ldots, t_{k+l-2}$

$$y_0 = g(0)\,, \quad y_1, \ldots, y_{k+l-2} \tag{3.1}$$

and of the problem (1.3)–(1.4) at the points $t_0, \ldots, t_{l-1}$

$$z_0^p = 0\,, \quad z_1^p, \ldots, z_{l-1}^p\,, \tag{3.2}$$

where $t_p \in (0, T]$ is an arbitrary fixed point, be given and $p \geqq k + l - 1$.

Later we shall return to the problem of determining the values (3.1)–(3.2).

Suppose that the values $y_1, \ldots, y_{p-1}$ are available. Method $\mathfrak{M}$ consists in the following two steps:

(a) We apply the O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ to the problem (1.3)–(1.4):

$$\begin{bmatrix} z_{n+1}^p \\ \vdots \\ z_{n+k}^p \end{bmatrix} + \boldsymbol{B} \begin{bmatrix} z_{n-l+1}^p \\ \vdots \\ z_n^n \end{bmatrix} - h\boldsymbol{C} \begin{bmatrix} K(t_p, t_{n+1})\, y_{n+1} \\ \vdots \\ K(t_p, t_{n+k})\, y_{n+k} \end{bmatrix} - h\boldsymbol{D} \begin{bmatrix} K(t_p, t_{n-l+1})\, y_{n-l+1} \\ \vdots \\ K(t_p, t_n)\, y_n \end{bmatrix} = \boldsymbol{0} \tag{3.3}$$

for $n + k < p$.

Let us denote by r the least integer for which $(r+1)s + k > p$ and $r_1 = p - (rs + k)$. We calculate the values z_q^p for $q \leqq rs + k$ from the system (3.3).

(b) Let us carry out one step of the O.I.M. method (3.3), starting from the point $n = rs + r_1$. The last equation of the system (3.3) is then

$$z_p^p = -\sum_{j=1}^{l} b_{kj} z_{p-k-l+j}^p + h\sum_{j=1}^{k} c_{kj} K(t_p, t_{p-k+j})\, y_{p-k+j} + \\ + h\sum_{j=1}^{l} d_{kj} K(t_p, t_{p-k-l+j})\, y_{p-k-l+j}\,. \tag{3.4}$$

Using (1.4), the equation (3.4) yields for y_p

$$y_p = (g(t_p) - \sum_{j=1}^{l} b_{kj} z_{p-k-l+j}^p + h\sum_{j=1}^{k-1} c_{kj} K(t_p, t_{p-k+j})\, y_{p-k+j} + \\ + h\sum_{j=1}^{l} d_{kj} K(t_p, t_{p-k-l+j})\, y_{p-k-l+j})/(1 - hc_{kk} K(t_p, t_p)) \tag{3.5}$$

for $h < 1/(|c_{kk}| \max_{[0,T]} |K(t,t)|)$ and $p \geqq k + l - 1$.

Applying successively **(a)**–**(b)** for $p = k + l - 1(1)N$ we can compute y_p using (3.5). Thus we have established the method $\mathfrak{M}$.

Remark. To point out which of O.I.M. methods is used for numerical solution of the problem (1.3)–(1.4), we shall sometimes refer to the method $\mathfrak{M}$ as to the method generated by O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$.

Definition. *Method $\mathfrak{M}$ is said to be of order m if the O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ generating it is of order m.*

Let us come back to the problem of determinig the starting values of approximate solution (3.1)–(3.2). Let $\{-\mathbf{e}, \boldsymbol{F}, \boldsymbol{G}, w\}$ be a selfstarting O.I.M. method which computes from one value the values of approximate solution at m successive points. For this method it is sufficient to know only the first m values (3.1). Let us denote the method for numerical solution of (1.1) generated by the selfstarting method $\{-\mathbf{e}, \boldsymbol{F}, \boldsymbol{G}, w\}$ by $\mathfrak{M}_1$. As for the starting values (3.2), they are reduced to $z_0^p = 0$. Thus we can compute, using this method $\mathfrak{M}_1$, the required values of the starting approximate solutions $y_m, \ldots, y_{k+l-2}$. System (3.3) for O.I.M. method $\{-\mathbf{e}, \boldsymbol{F}, \boldsymbol{G}, w\}$ yields also the values $z_1^p, \ldots, z_{l-1}^p$. In the course of calculations, the following case can occur: The least integer n_0 such that $n_0 + k > k + l - 2$ is reached and we have not yet calculated all z_j^p, $j \leqq l - 1$. Then the missing values z_j^p are computed from the system

$$\begin{bmatrix} z_{l-m}^p \\ \vdots \\ z_{l-1}^p \end{bmatrix} - \mathbf{e} z_{l-m-1}^p - h\boldsymbol{F} \begin{bmatrix} K(t_p, t_{l-m})\, y_{l-m} \\ \vdots \\ K(t_p, t_{l-1})\, y_{l-1} \end{bmatrix} - \\ - h\boldsymbol{G} K(t_p, t_{l-m-1})\, y_{l-m-1} = \boldsymbol{0}\,.$$

The relationship between m, the order of the selfstarting method and the order of the O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ will be discussed in the next part.

Finally, the values $y_1, \ldots, y_{m-1}$ could be computed using only the value y_0 which is given explicitly, by the methods described in [2].

4. CONVERGENCE AND ORDER OF $\mathfrak{M}$

Definition. *The method $\mathfrak{M}$ for numerical solution of* (1.1) *is said to be convergent when*

$$\lim_{p\to\infty ph=t} y_p = y(t)$$

t is a fixed point from $[0,T]$.

Let a method $\mathfrak{M}$ generated by a consistent and stable O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ of order v be given. Denote

$$e_q^p = z_q^p - z_p(t_q)\,.$$

Using (1.4) we have $e_p^p = y_p - y(t_p)$.

Careful examination of the definition of the method $\mathfrak{M}$ results in the conclusion that the convergence of our method, i.e., the convergence of the approximate solutions $y_p = z_p^p - g(t_p)$, is equivalent to the convergence of the approximate solutions z_q^p of the problem (1.3)–(1.4), obtained by means of the O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$.

Theorem 2. *A method $\mathfrak{M}$ generated by a stable and consistent O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ is convergent. Moreover, if the order of $\mathfrak{M}$ is v and starting values are given such that e_i^p, $i = (1)\,l-1$ and e_i^i, $i = 1(1)\,k+l-2$ are of order at least h^v then e_q^p is of the order h^v.*

The proof of the theorem repeats almost literally the proof of the convergence theorem for O.I.M. methods (cf. [1], Theorem 3.1), so we omit it.

Remark. Let the starting values (3.2) be computed by means of the method $\mathfrak{M}_1$ generated by a selstarting O.I.M. method $\{-\boldsymbol{e}, \boldsymbol{F}, \boldsymbol{G}, w\}$ of order m. Then $e_q^p = O(h^{m+1})$, $q = 0(1)\,l-1$, though the convergence theorem for O.I.M. methods guarantee only the order m. The increase of the exponent by one is caused by the finiteness of the steps of the method $\{-\boldsymbol{e}, \boldsymbol{F}, \boldsymbol{G}, w\}$ during the calculation of the values (3.2). Thus the order of the "starting" method $\mathfrak{M}_1$ can be less by one then the order of the method $\mathfrak{M}$ preserving the original order of error.

5. A-STABILITY OF METHOD $\mathfrak{M}$

Our definition of A-stable methods for numerical solution of (1.1) is a direct analogy of A-stability (in the sense of Dahlquist) of methods for numerical solution of Cauchy problems.

Definition. *A method* $\mathfrak{M}$ *generated by an O.I.M. method* $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$ *is said to be A-stable if when applied to the equation*

$$y(t) - \alpha \int_0^t y(s)\,\mathrm{d}s = 0\,, \tag{5.1}$$

$t \in [0, T]$, $Re\,\alpha < 0$, *solutions* y_n *of the corresponding difference equations converge to zero as* n *tends to infinity.*

Remark. One can define the A-stability concept for the methods of the type $\mathfrak{M}$ for numerical solutions of Volterra integral equations of the first kind as in the definition, replacing the equation (5.1) by

$$0 = \alpha \int_0^t y(s)\,\mathrm{d}s\,, \tag{5.2}$$

$t \in [0, T]$, $Re\,\alpha < 0$.

Let us apply the A-stability concept just defined to the methods of de Hoog and Weiss. Their method is a block by block method and applied to the equation (5.2) could be written in the form

$$\boldsymbol{A}\boldsymbol{Y}_i = \boldsymbol{B}\boldsymbol{Y}_{i-1}\,.$$

$\boldsymbol{Y}_i$ are vectors of values of approximate solutions in n successive points calculated simultaneously and $\boldsymbol{A}$ regular and $\boldsymbol{B}$ are matrices which define the method. Then A-stability is equivalent to the condition that eigenvalues of $\boldsymbol{A}^{-1}\boldsymbol{B}$ are in the open unit disc. This phenomenon is called by de Hoog and Weiss numerical stability.

Consider a method $\mathfrak{M}$ generated by an A-stable O.I.M. method $\{\boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, s\}$. Such methods of arbitrary high order of asymptotic accuracy exist (cf. [1], Part 4). The problem (1.3)–(1.4) for equation (5.1) is of the form

$$z(t, u) = \alpha \int_0^u y(s)\,\mathrm{d}s\,, \quad u \leqq t$$

$$\frac{\partial}{\partial u} z(t, u) = \alpha\, y(u)\,. \tag{5.3}$$

The equation (5.1) implies

$$y(u) = \int_0^u \alpha\, y(s)\,\mathrm{d}s = z(t, u)\,. \tag{5.4}$$

Substitution of (5.4) into (5.3) yields

$$\frac{\partial}{\partial u} z(t, u) = \alpha\, z(t, u)\,. \tag{5.5}$$

The equation (5.5) is solved by an A-stable O.I.M. method $\{\mathbf{B}, \mathbf{C}, \mathbf{D}, s\}$. This means:

$$z_q^p \to 0 \quad \text{as} \quad p \to \infty\,, \quad q \leqq p\,.$$

Particularly,

$$z_p^p \to 0 \quad \text{as} \quad p \to \infty\,. \tag{5.6}$$

Thus (1.4) and (5.6) imply

Theorem 3. *Let an A-stable O.I.M. method* $\{\mathbf{B}, \mathbf{C}, \mathbf{D}, s\}$ *of order* m *be given. Then the method* $\mathfrak{M}$ *generated by this O.I.M. method is an A-stable method of order* m.

6. NUMERICAL EXAMPLE

Careful examination of the definition of the method $\mathfrak{M}$ displays one of the advantages of our method. Namely, in the course of calculation of approximate solutions y_p we do not "solve" any system of equations, "solve" in the sense of inverting a matrix of the system in any form. This is necessary in the methods suggested by de Hoog and Weiss. The order of the matrix that they must invert equals to the order of the asymptotic accuracy, i.e., grown if higher order of accuracy is needed. On the other hand, we need just operations of addition and matrix multiplication. Moreover, if the method $\mathfrak{M}$ is generated by a selfstarting O.I.M. method, no matrix multiplication is needed. Also the kernel $K(t, s)$ of the integral equation need not be defined for $t < s$ as it is required for the first method in [2].

To illustrate our method $\mathfrak{M}$ we consider the method generated by an A-stable selfstarting O.I.M. method $\{-\mathbf{e}, \mathbf{F}, \mathbf{G}, w\}$ with $s = 2$, $k = 2$ and

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is of order two, applied to the equation:

$$y(t) = \int_0^t (1/2 + t^4 - s^4)\, y(s)\, \mathrm{d}s - t(1/2 + 4t^4/5) + 1\,,$$

where $0 \leqq s \leqq t \leqq T = 1.40$.
The exact solution is

$$y(t) = 1\,.$$

In the table below the errors for $h = \cdot 14$ and $h = \cdot 028$ are tabulated. Among other, it points out one interesting feature of our method. There is no "explosion"

in the error for $h = \cdot 14$ and $t > 1 \cdot 2$ as it is the case in the first of de Hoog and Weiss methods, i.e., global error of the method is still estimated by *const.* h^2 for these h and t.

t	·28	·42	·56	·70	·84	·98	1·12	1·26	1·40
$h = \cdot 14$	—·0004	·0211	·0087	·0268	·0098	—·0116	·0093	—·0231	·0098
$h = \cdot 028$	—·00000	·00015	·00009	·00017	·00006	—·00018	·00008	—·00021	·00007

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Souhrn

A-STABILNÍ METODY VYSOKÉHO ŘÁDU PŘESNOSTI PRO INTEGRÁLNÍ ROVNICE VOLTERROVA TYPU

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V článku je ukázána možnost užití O.I.M. metod na řešení Volterrových rovnic. V třídě těchto metod existují A-stabilní metody libovolně vysokého řádu asymptotické přesnosti. V části 5 je dokázáno, že tyto metody generují metody na řešení Volterrových rovnic, jež jsou také A-stabilní a libovolně vysokého řádu asymptotické přesnosti. Početní výhodou námi definovaných metod je, že tyto, při numerické realizaci, nikde nevyžadují inverzi matice v jakékoliv formě.

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