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TABLES FOR TWO NORMAL-SCORES RANK TESTS FOR THE TWO-SAMPLE SCALE PROBLEM

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Introduction. In this paper we present the tables of scores and of upper-tail and lower-tail critical values for the Capon test and for the Klotz test for cases when the pooled sample size $m + n$ lies within the bounds $6 \leqq m + n \leqq 20$ and the one-sided significance levels lie near 0·5%, 1%, 2·5%, 5%. These tests are optimal (in a sense to be specified later) for the two-sample scale problem whenever the basic distributions are normal.

We shall use throughout the terminology and the results from the book by Hájek-Šidák [2] without further explicit quotations.

Description of the tests. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two random samples with the densities f_1 and f_2, respectively. Suppose that the notation is chosen so that $m \leqq n$, and put $N = m + n$. We wish to test the hypothesis H_0 that f_1 and f_2 are identical (and otherwise they may be arbitrary) against the alternatives that they differ in scale, which may be expressed by $f_1(x) = \sigma^{-1} f((x - \mu)/\sigma), f_2(x) = f(x - \mu)$ where f is some density, μ is a common location parameter, σ is the parameter to be tested.

Let us denote by $R_1, \ldots, R_m$ the ranks of $X_1, \ldots, X_m$ in the pooled sample of all observations $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ arranged in order of their magnitude. The theoretical Capon test is based on the statistic $S' = \sum_{i=1}^{m} a'_{1N}(R_i)$, where $a'_{1N}(k)$ are the normal scores for the scale problem, i.e. $a'_{1N}(k) = \mathsf{E}[V_N^{(k)}]^2$ where $V_N^{(1)} < V_N^{(2)} < \ldots < V_N^{(N)}$ is an ordered random sample of size N from the standardized normal distribution. The test with the critical region $\{S' \geqq c_u\}$, c_u being some upper-tail critical value, is the locally most powerful rank test of H_0 against $\sigma > 1$ whenever f is the normal density; it is also asymptotically optimum for such a density. Of course, in practice we must round off the scores $a'_{1N}(k)$. Since, moreover, the test remains unchanged if we multiply S' by any positive constant, we work in subsequent tables, for the sake of simplicity, with the test statistic

$$S = \sum_{i=1}^{m} a_{1N}(R_i)$$

where $a_{1N}(k) = 100a'_{1N}(k)$ rounded off to integer values. If we are testing against the one-sided alternative $\sigma > 1$ (or $\sigma < 1$) expressing that the first density f_1 has a larger (smaller) variance than the second density f_2, we base the test on a one-sided critical region $\{S \geqq c_u\}$ (or $\{S \leqq c_l\}$) where $c_u(c_l)$ is some upper-tail (lower-tail, respectively) critical value. The test against $\sigma \neq 1$ uses a two-sided critical region $\{S \geqq c_u \text{ or } S \leqq c_l\}$.

The theoretical Klotz test is based similarly on the statistic $T' = \sum_{i=1}^{m} b'_{1N}(R_i)$, where $b'_{1N}(k)$ are approximate normal scores for the scale problem given by $b'_{1N}(k) = [\Phi^{-1}(k/(N+1))]^2$ where Φ^{-1} is the inverse of the standardized normal distribution function. This test is asymptotically optimum for the two-sample problem of scale described above with f being the normal density. For the sake of simplicity, we consider here the test statistic

$$T = \sum_{i=1}^{m} b_{1N}(R_i)$$

where $b_{1N}(k) = 100b'_{1N}(k)$ rounded off to integer values. Otherwise, everything is similar as in the preceding case of the Capon test.

Description of the tables. The scores $a_{1N}(k)$ for the Capon test are given in Table 1 for $6 \leqq N \leqq 20$. They have been obtained from Teichroew's tables [4] by multiplying the relevant numbers by 100 and rounding them off afterwards to integer values.

Table 2 concerns the Capon test and has four double-columns for $\alpha_1 = 0{\cdot}5\%$, $\alpha_2 = 1\%$, $\alpha_3 = 2{\cdot}5\%$, $\alpha_4 = 5\%$; in its i-th double-column, in upper line for each pair n, m it contains the upper-tail critical value c_{ui} of the statistic S and the corresponding exact probability $P\{S \geqq c_{ui}\}$ in per cents such that this probability is the closest possible to the value α_i $(i = 1, 2, 3, 4)$; similarly, in the lower line it contains the lower-tail critical value c_{li} of the statistic S and the exact probability $P\{S \leqq c_{li}\}$ in per cents such that this probability is the closest possible to α_i $(i = 1, 2, 3, 4)$. For example, precise rules and formulas for tabulating the upper-tail critical values c_{ui} are as follows: First, let there exist two consecutive possible values $c_{ui}^- < c_{ui}^+$ of the statistic S such that $P\{S \geqq c_{ui}^+\} < \alpha_i \leqq P\{S \geqq c_{ui}^-\}$; then, if $P\{S \geqq c_{ui}^-\} - \alpha_i \geqq \alpha_i - P\{S \geqq c_{ui}^+\}$ we tabulate c_{ui}^+ and $P\{S \geqq c_{ui}^+\}$; if, conversely, $P\{S \geqq c_{ui}^-\} - \alpha_i < \alpha_i - P\{S \geqq c_{ui}^+\}$ we tabulate c_{ui}^- and $P\{S \geqq c_{ui}^-\}$. Second, let there exist no c_{ui}^+ with $P\{S \geqq c_{ui}^+\} < \alpha_i$; then we take for c_{ui}^- the maximal possible value of S and tabulate this c_{ui}^- and $P\{S \geqq c_{ui}^-\}$ in the i-th double column corresponding to the largest α_i (among $\alpha_1, \alpha_2, \alpha_3, \alpha_4$) for which $\alpha_i \leqq P\{S \geqq c_{ui}^-\}$ is satisfied; the double-columns for smaller α_j's (if there are any) are then filled by dashes, the double-columns for larger α_j's (if there are any) are filled according to the first rule. Third, if this second case occurs for α_1, the value c_{u1}^- is preceded by a star indicating that the tabulated c_{u1}^- is the largest possible value (so that it can be distinguished from the first case). As for the lower-tail critical values c_{li} contained in the lower line for each n, m, the rules for their tabulation are analogous, and we hope we need not write them in detail.

Table 2 (except the maximal and the minimal values, i.e. except those c preceded either by dashes or by a star) can be used in two somewhat different ways, and we shall illustrate the idea in the case of the upper-tail critical values: If we are willing to admit a slight exceeding of the significance level α_i, we can always use the critical region $\{S \geqq c_{ui}\}$ with the exact significance level $P\{S \geqq c_{ui}\}$ given directly in Table 2. However, if we wish to have a significance level not exceeding α_i but we find in the table $P\{S \geqq c_{ui}\} > \alpha_i$, we can base the test on the critical region $\{S > c_{ui}\}$ whose significance level is $<\alpha_i - (P\{S \geqq c_{ui}\} - \alpha_i)$.

Naturally, the two-sided critical region $\{S \geqq c_{ui}$ or $S \leqq c_{li}\}$, applied in testing against $\sigma \neq 1$, has the significance level $P\{S \geqq c_{ui}\} + P\{S \leqq c_{li}\}$.

Passing further to the Klotz test, the scores $b_{1N}(k)$ for it are given in Table 3. We have computed them anew, and checked by means of Klotz's table in [3]. (It was found that Klotz's scores $b_{1,14}(4)$, $b_{1,15}(4)$, $b_{1,18}(7)$, $b_{1,20}(1)$, $b_{1,20}(3)$ are wrong.)

Table 4 contains the upper-tail and the lower-tail critical values c_{ui} and c_{li} of the statistic T and the corresponding exact probabilities $P\{T \geqq c_{ui}\}$ and $P\{T \leqq c_{li}\}$. The table is arranged in the same manner as the Table 2, so that all remarks made above for Table 2 and the statistic S continue to hold also here.

Tables 2 and 4 have been computed by direct combinatorial procedures, so that the probabilities in these tables are exact (except, of course, their natural rounding off). The author is deeply indebted to M. Nosál for programming necessary computations, and to J. Hájek for suggesting the way of tabulation used here.

Concerning the history, the Capon test was derived by Capon [1] but it seems that no tables for it were available before the present ones. The Klotz test was introduced by Klotz [3], who published also the upper-tail and the lower-tail critical values for $8 \leqq N \leqq 20$ and the corresponding exact significance levels near 0·5%, 1%, 2·5%, 5%, 10%, using the scores $b'_{1N}(k)$ rounded off to four decimal places. Our point of view is more practical than that of Klotz; we think that a precision to two decimal places in the scores of a non-parametric test is quite sufficient for practical purposes.

Example 1. Let the X sample be 98; 112; 121; 127; 146, and the Y sample 4; 51; 80; 97; 115; 132; 150; 171; 208; 259, so that $m = 5$, $n = 10$, $N = 15$. We would like to test H_0 against $\sigma < 1$ (i.e. against the one-sided alternative that the first density has a smaller variance than the second one) by means of the Capon test at a significance level not exceeding 1%. We must use a critical region $\{S \leqq c_l\}$, and in Table 2 we find $c_{l2} = 133$ with the exact level $P\{S \leqq 133\} = 1{\cdot}03\%$. However, since we do not want to exceed 1%, we base the test on the critical region $\{S < 133\}$ and its level is then $<1\% - (1{\cdot}03\% - 1\%) = 0{\cdot}97\%$. The actual value of S in this example is $S = 121$ so that we can reject H_0 at a significance level $<0{\cdot}97\%$.

Example 2. Let the X sample be 14; 79; 143; 208, and the Y sample 51; 85; 94; 112; 138, so that $m = 4$, $n = 5$, $N = 9$. We want to test H_0 against the two-sided alternative $\sigma \neq 1$ by means of the Klotz test at some significance level between 5% and 10%. The test will be based on a two-sided critical region $\{T \geqq c_u \text{ or } T \leqq c_l\}$, and it is reasonable to make both probabilities $P\{T \geqq c_u\}$ and $P\{T \leqq c_l\}$ at least roughly equal, which means for our case to make them lie roughly between 2·5% and 5%. Looking into Table 4, we find two possibilities: to choose either the critical region $\{T \geqq c_{u4} \text{ or } T \leqq c_{l3}\} = \{T \geqq 427 \text{ or } T \leqq 62\}$ with the significance level $3{\cdot}97\% + 3{\cdot}17\% = 7{\cdot}14\%$, or the critical region $\{T \geqq c_{u4} \text{ or } T \leqq c_{l4}\} = \{T \geqq 427 \text{ or } T \leqq 83\}$ with the significance level $3{\cdot}97\% + 5{\cdot}56\% = 9{\cdot}53\%$. (Of course, we must choose the critical region in advance, before the experiment.) In the present example we get $T = 427$ so that, in both cases of critical regions, we reject H_0 at the corresponding level.

Remark on asymptotic normality. The statistic S' has, under H_0, the mean value $\mathsf{E}S' = m$, the variance

$$\operatorname{var} S' = \frac{mn}{N(N-1)} \sum_{k=1}^{N} [a'_{1N}(k)]^2 - \frac{mn}{N-1},$$

and the standardized variable $(S' - \mathsf{E}S')(\operatorname{var} S')^{-1/2}$ has asymptotically the standardized normal distribution whenever $m \to \infty$, $n \to \infty$ in an arbitrary way. The statistic T' has, under H_0,

$$\mathsf{E}T' = \frac{m}{N} \sum_{k=1}^{N} b'_{1N}(k),$$

$$\operatorname{var} T' = \frac{mn}{N(N-1)} \sum_{k=1}^{N} [b'_{1N}(k)]^2 - \frac{n}{m(N-1)} (\mathsf{E}T')^2,$$

and an analogous assertion on its asymptotic normality is true.

Table 5 shows the quality of the normal approximation for our modification S of the Capon statistic for all pairs m, n with $m + n = N = 20$. This table is arranged in a similar way as Table 2, but the first numbers in each double-column are the approximate levels in per cents, the second numbers are the exact levels; i.e. in the upper line we tabulate $1 - \Phi((10^{-2}c_{ui} - \mathsf{E}S')/(\operatorname{var} S')^{1/2})$ in per cents (Φ being the standardized normal distribution function, c_{ui} being the critical value from Table 2), and $P\{S \geqq c_{ui}\}$ in per cents taken from Table 2; in the lower line we tabulate similarly $\Phi((10^{-2}c_{li} - \mathsf{E}S')/(\operatorname{var} S')^{1/2})$ and $P\{S \leqq c_{li}\}$.

Remark on ties. If the samples contain some equal observations, so that their ranks are not well defined, we can either randomize the ordering of these observations by means of a supplementary experiment, or use the following method of average scores: Suppose that we consider a group of τ equal observations, and that they would have the ranks (say) $r + 1, \ldots, r + \tau$ if they were distinct but if they had otherwise the same position among the other observations; then we assign to each of these observations the average score $\tau^{-1} \sum_{i=1}^{\tau} a_{1N}(r + i)$ in the case of S, and analogously for T. If the samples do not contain many equal observations, our Tables 2 and 4 may still serve as approximations.

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Souhrn

TABULKY PRO DVA POŘADOVÉ TESTY S NORMÁLNÍMI SKÓRY PRO DVOUVÝBĚROVÝ PROBLÉM ŠKÁLY

ZBYNĚK ŠIDÁK

Publikují se tabulky skórů a horních i dolních kritických hodnot pro Caponův test a pro Klotzův test pro případy, kdy rozsah $m + n$ dvou spojených výběrů leží v mezích $6 \leqq m + n \leqq 20$ a jednostranné hladiny významnosti leží blízko 0,5%, 1%, 2,5%, 5%. Tyto testy jsou optimální pro dvouvýběrový problém škály, jsou-li základní rozložení normální.

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Table 1. Scores $a_{1N}(k)$ for the Capon statistic S

k \ N	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	202	222	240	256	271	285	298	310	321	331	341	351	360	368	376
2	69	83	97	109	122	133	144	155	164	174	182	191	199	207	214
3	29	34	42	51	61	70	79	87	96	104	112	119	126	133	140
4	29	21	21	25	30	36	43	50	57	63	70	77	83	89	95
5	69	34	21	17	17	19	23	27	32	38	43	49	54	60	65
6	202	83	42	25	17	14	14	15	18	22	26	30	34	39	44
7		222	97	51	30	19	14	12	12	13	15	18	21	25	28
8			240	109	61	36	23	15	12	10	10	11	13	15	18
9				256	122	70	43	27	18	13	10	9	9	10	11
10					271	133	79	50	32	22	15	11	9	8	8
11						285	144	87	57	38	26	18	13	10	8
12							298	155	96	63	43	30	21	15	11
13								310	164	104	70	49	34	25	18
14									321	174	112	77	54	39	28
15										331	182	119	83	60	44
16											341	191	126	89	65
17												351	199	133	95
18													360	207	140
19														368	214
20															376

Table 2. Upper-tail critical values c_{ui} and lower-tail critical values c_{li} of the Capon statistic S, and significance levels $P\{S \geqq c_{ui}\}$ and $P\{S \leqq c_{li}\}$ in per cents

n	m	0·5%		1·0%		2·5%		5·0%	
5	1	—	—	—	—	—	—	202	33·33
		—	—	—	—	—	—	29	33·33
4	2	—	—	—	—	—	—	404	6·67
		—	—	—	—	—	—	58	6·67
3	3	—	—	—	—	—	—	473	10·00
		—	—	—	—	—	—	127	10·00
6	1	—	—	—	—	—	—	222	28·57
		—	—	—	—	—	—	21	14·29
5	2	—	—	—	—	444	4·76	444	4·76
		—	—	—	—	—	—	55	9·52
4	3	—	—	—	—	—	—	527	5·71
		—	—	—	—	89	2·86	89	2·86
7	1	—	—	—	—	—	—	240	25·00
		—	—	—	—	—	—	21	25·00
6	2	—	—	—	—	480	3·57	480	3·57
		—	—	—	—	42	3·57	42	3·57
5	3	—	—	—	—	577	3·57	577	3·57
		—	—	—	—	84	3·57	84	3·57
4	4	—	—	674	1·43	674	1·43	619	7·14
		—	—	126	1·43	126	1·43	181	7·14
8	1	—	—	—	—	—	—	256	22·22
		—	—	—	—	—	—	17	11·11
7	2	—	—	—	—	512	2·78	512	2·78
		—	—	—	—	—	—	42	5·56
6	3	—	—	621	2·38	621	2·38	563	4·76
		—	—	67	1·19	67	1·19	93	5·95
5	4	* 730	0·79	730	0·79	672	3·97	672	3·97
		—	—	118	1·59	144	3·17	176	5·56
9	1	—	—	—	—	—	—	271	20·00
		—	—	—	—	—	—	17	20·00
8	2	—	—	542	2·22	542	2·22	542	2·22
		—	—	34	2·22	34	2·22	34	2·22
7	3	—	—	664	1·67	603	3·33	572	5·00
		—	—	64	1·67	77	3·33	95	5·00
6	4	786	0·48	786	0·48	725	2·38	694	4·29
		94	0·48	94	0·48	125	2·38	156	4·76
5	5	* 847	0·79	847	0·79	803	2·38	755	6·35
		* 155	0·79	155	0·79	199	2·38	247	6·35

Table 2 (continued)

n	m	0·5%		1·0%		2·5%		5·0%	
10	1	—	—	—	—	—	—	285	18·18
		—	—	—	—	—	—	14	9·09
9	2	—	—	570	1·82	570	1·82	570	1·82
		—	—	—	—	33	3·64	38	5·45
8	3	—	—	703	1·21	640	2·42	589	4·85
		* 52	0.61	52	0·61	69	3·03	86	4·85
7	4	836	0·30	773	1·52	739	2·73	710	4·85
		* 88	0·61	105	1·21	122	2·12	139	4·55
6	5	906	0·43	872	0·87	843	1·95	792	5·41
		124	0·22	158	1·08	192	2·60	226	4·98
11	1	—	—	—	—	—	—	298	16·67
		—	—	—	—	—	—	14	16·67
10	2	—	--	596	1·52	596	1·52	442	7·58
		—	—	28	1·52	28	1·52	37	7·58
9	3	* 740	0·91	740	0·91	639	2·73	586	5·45
		* 51	0·91	51	0·91	71	2·73	80	6·36
8	4	884	0·20	819	1·01	763	2·63	698	5·25
		74	0·20	94	1·01	123	2·83	139	4·65
7	5	927	0·51	907	0·76	862	2·27	826	4·55
		137	0·51	153	1·01	173	2·02	211	4·80
6	6	1006	0·54	986	0·97	941	2·81	905	4·76
		196	0·54	216	0·97	261	2·81	297	4·76
12	1	—	—	—	—	—	—	310	15·38
		—	—	—	—	—	—	12	7·69
11	2	—	—	620	1·28	620	1·28	465	6·41
		—	—	—	—	27	2·56	30	3·85
10	3	* 775	0·70	775	0·70	647	2·80	620	4·55
		42	0·35	42	0·35	57	2·45	77	4·90
9	4	862	0·70	825	1·26	790	2·52	719	4·90
		81	0·56	92	0·98	107	2·66	139	4·48
8	5	957	0·47	942	0·85	877	2·72	829	5·05
		119	0·39	142	0·93	171	2·41	203	4·90
7	6	1044	0·52	1029	0·93	969	2·39	927	5·01
		184	0·52	206	0·99	243	2·39	286	5·01
13	1	—	—	—	—	—	—	321	14·29
		—	—	—	—	—	—	12	14·29
12	2	—	—	642	1·10	642	1·10	485	5·49
		—	—	24	1·10	24	1·10	30	5·49
11	3	* 806	0·55	738	1·10	660	2·75	581	6·04
		* 42	0·55	48	1·10	56	1·65	76	4·95

Table 2 (continued)

n	*m*	0·5%		1·0%		2·5%		5·0%	
10	4	902	0·50	863	0·90	795	2·60	717	5·09
		74	0·50	88	1·00	113	2·70	133	5·09
9	5	988	0·50	959	1·00	895	2·70	842	5·00
		117	0·40	137	1·20	170	2·60	201	5·09
8	6	1084	0·47	1045	1·00	991	2·70	946	5·13
		174	0·47	202	1·03	233	2·53	276	4·96
7	7	1155	0·52	1135	0·99	1087	2·39	1034	4·90
		245	0·52	265	0·99	313	2·39	366	4·90
14	1	—	—	—	—	—	—	331	13·33
		—	—	—	—	—	—	10	6·67
13	2	* 662	0·95	662	0·95	662	0·95	505	4·76
		—	—	23	1·90	26	2·86	32	4·76
12	3	836	0·44	766	0·88	679	2·64	609	5·05
		36	0·22	45	1·10	57	2·20	70	4·40
11	4	940	0·37	870	1·03	804	2·34	122	5·13
		74	0·51	83	1·10	102	2·20	727	4·91
10	5	1020	0·43	978	0·97	912	2·63	851	5·00
		112	0·40	133	1·03	158	2·36	196	5·09
9	6	1111	0·48	1070	0·94	1016	2·58	959	5·11
		168	0·50	193	1·02	228	2·42	270	5·03
8	7	1190	0·57	1162	1·01	1105	2·46	1054	5·22
		234	0·47	263	1·09	308	2·50	357	5·05
15	1	—	—	—	—	—	—	341	12·50
		—	—	—	—	—	—	10	12·50
14	2	* 682	0·83	682	0·83	523	4·17	523	4·17
		* 20	0·83	20	0·83	25	4·17	30	5·00
13	3	864	0·36	752	1·07	697	2·50	593	5·71
		35	0·36	46	1·07	51	2·50	68	5·36
12	4	934	0·49	890	0·99	809	2·58	735	5·05
		66	0·49	78	0·99	99	2·58	122	5·11
11	5	1046	0·50	1002	0·92	929	2·47	860	4·99
		109	0·55	126	0·96	159	2·52	190	5·04
10	6	1131	0·54	1098	0·95	1030	2·49	970	5·09
		163	0·54	185	1·00	221	2·52	264	4·91
9	7	1227	0·51	1185	1·03	1126	2·53	1072	4·93
		227	0·51	249	1·01	300	2·48	345	5·05
8	8	1300	0·47	1270	0·98	1217	2·48	1167	5·08
		298	0·47	328	0·98	381	2·48	431	5·08
16	1	—	—	—	—	—	—	351	11·76
		—	—	—	—	—	—	9	5·88

Table 2 (continued)

n	*m*	0·5%		1·0%		2·5%		5·0%	
15	2	* 702	0·74	702	0·74	542	3·68	542	3·68
		—	—	20	1·47	22	2·21	27	3·68
14	3	821	0·59	779	0·88	711	2·50	619	4·85
		38	0·74	40	1·03	52	2·35	69	5·15
13	4	942	0·55	911	0·92	828	2·48	743	5·08
		68	0·63	77	1·05	99	2·65	122	5·04
12	5	1061	0·50	1021	0·99	947	2·46	863	5·06
		107	0·50	124	1·02	154	2·54	185	5·22
11	6	1161	0·51	1119	1·03	1051	2·48	982	4·99
		156	0·52	180	0·98	215	2·48	257	5·03
10	7	1257	0·50	1219	1·02	1150	2·45	1090	4·94
		216	0·50	243	0·99	292	2·53	335	5·01
9	8	1340	0·53	1305	1·00	1246	2·48	1186	5·05
		284	0·52	318	0·99	369	2·49	423	4·99
17	1	—	—	—	—	—	—	360	11·11
		—	—	—	—	—	—	9	11·11
16	2	* 720	0·65	720	0·65	559	3·27	486	5·88
		* 18	0·65	18	0·65	22	3·27	26	3·92
15	3	846	0·49	774	0·98	729	2·21	613	5·15
		35	0·49	39	0·74	52	2·45	64	4·90
14	4	972	0·46	929	0·98	824	2·48	763	5·00
		64	0·46	73	0·88	94	2·39	118	5·16
13	5	1079	0·49	1027	1·00	953	2·54	875	5·04
		103	0·49	119	1·05	147	2·45	180	5·04
12	6	1182	0·54	1140	1·00	1068	2·46	994	4·96
		152	0·54	172	0·96	212	2·49	252	4·99
11	7	1279	0·49	1236	1·00	1170	2·51	1101	5·03
		210	0·51	235	1·05	280	2·49	329	4·98
10	8	1371	0·48	1330	0·99	1266	2·52	1203	5·00
		273	0·50	304	0·98	360	2·52	412	4·98
9	9	1449	0·50	1415	1·02	1356	2·47	1297	4·99
		349	0·50	383	1·02	442	2·47	501	4·99
18	1	—	—	—	—	—	—	368	10·53
		—	—	—	—	—	—	8	5·26
17	2	* 736	0·58	736	0·58	575	2·92	501	5·26
		—	—	18	1·17	23	2·92	25	5·26
16	3	869	0·41	782	1·03	708	2·79	634	4·64
		33	0·52	40	1·03	50	2·89	64	5·26
15	4	982	0·46	929	1·03	835	2·58	766	4·98
		60	0·49	72	0·98	89	2·40	117	4·93

Table 2 (continued)

n	m	0·5%		1·0%		2·5%		5·0%	
14	5	1092	0·51	1047	0·97	968	2·58	881	5·05
		99	0·51	116	0·95	147	2·63	178	5·17
13	6	1204	0·52	1154	1·00	1078	2·51	1001	4·99
		149	0·50	170	0·99	207	2·53	246	5·00
12	7	1309	0·48	1260	0·99	1186	2·50	1112	5·02
		202	0·48	231	1·00	275	2·47	324	5·01
11	8	1398	0·48	1354	1·00	1283	2·50	1215	5·09
		269	0·50	299	0·99	351	2·51	404	4·98
10	9	1482	0·50	1446	1·01	1378	2·54	1313	4·99
		338	0·50	374	0·98	433	2·51	492	5·00
19	1	—	—	—	—	—	—	376	10·00
		—	—	—	—	—	—	8	10·00
18	2	* 752	0·53	752	0·53	590	2·63	516	4·74
		* 16	0·53	16	0·53	19	2·63	26	5·26
17	3	847	0·53	796	1·05	730	2·46	634	4·74
		34	0·53	37	1·23	47	2·63	63	5·09
16	4	987	0·54	942	0·97	835	2·50	768	4·97
		58	0·47	68	0·97	91	2·66	112	5·04
15	5	1117	0·50	1054	1·01	972	2·52	888	5·02
		96	0·46	110	0·98	142	2·46	174	5·01
14	6	1221	0·48	1171	0·99	1088	2·49	1010	5·01
		141	0·49	164	1·01	203	2·46	241	4·98
13	7	1328	0·50	1277	1·00	1198	2·51	1122	4·98
		196	0·50	223	1·01	269	2·58	316	4·98
12	8	1424	0·51	1377	1·01	1301	2·50	1228	5·02
		259	0·50	289	1·01	343	2·50	398	4·97
11	9	1514	0·50	1469	1·01	1398	2·49	1328	5·01
		327	0·50	362	0·99	423	2·49	483	5·00
10	10	1596	0·50	1555	0·99	1489	2·51	1424	4·99
		402	0·50	443	0·99	509	2·51	574	4·99

Table 3. Scores $b_{1N}(k)$ for the Klotz statistic T

k \ N	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	114	132	149	164	178	191	203	215	225	235	245	254	262	271	278
2	32	46	58	71	83	94	104	114	123	132	141	149	157	164	171
3	3	10	19	28	37	46	54	63	71	79	86	94	101	107	114
4	3	0	2	6	12	19	25	32	39	46	52	58	65	71	77
5	32	10	2	0	1	4	9	13	19	24	29	35	40	46	51
6	114	46	19	6	1	0	1	3	6	10	14	19	23	28	32
7		132	58	28	12	4	1	0	1	2	5	8	11	15	19
8			149	71	37	19	9	3	1	0	1	2	4	6	9
9				164	83	46	25	13	6	2	1	0	0	2	3
10					178	94	54	32	19	10	5	2	0	0	0
11						191	104	63	39	24	14	8	4	2	0
12							203	114	71	46	29	19	11	6	3
13								215	123	79	52	35	23	15	9
14									225	132	86	58	40	28	19
15										235	141	94	65	46	32
16											245	149	101	71	51
17												254	157	107	77
18													262	164	114
19														271	171
20															278

Table 4. Upper-tail critical values c_{ui} and lower-tail critical values c_{li} of the Klotz statistic T, and significance levels $P\{T \geqq c_{ui}\}$ and $P\{T \leqq c_{li}\}$ in per cents

n	m	0·5%		1·0%		2·5%		5·0%	
5	1	—	—	—	—	—	—	114	33·33
		—	—	—	—	—	—	3	33·33
4	2	—	—	—	—	—	—	228	6·67
		—	—	—	—	—	—	6	6·67
3	3	—	—	—	—	—	—	260	10·00
		—	—	—	—	—	—	38	10·00
6	1	—	—	—	—	—	—	132	28·57
		—	—	—	—	—	—	0	14·29
5	2	—	—	—	—	264	4·76	264	4·76
		—	—	—	—	—	—	10	9·52
4	3	—	—	—	—	—	—	310	5·71
		—	—	—	—	20	2·86	20	2·86
7	1	—	—	—	—	—	—	149	25·00
		—	—	—	—	—	—	2	25·00
6	2	—	—	—	—	298	3·57	298	3·57
		—	—	—	—	4	3·57	4	3·57
5	3	—	—	—	—	356	3·57	356	3·57
		—	—	—	—	23	3·57	23	3·57
4	4	—	—	414	1·43	414	1·43	375	7·14
		—	—	42	1·43	42	1·43	81	7·14
8	1	—	—	—	—	—	—	164	22·22
		—	—	—	—	—	—	0	11·11
7	2	—	—	—	—	328	2·78	328	2·78
		—	—	—	—	—	—	6	5·56
6	3	—	—	399	2·38	399	2·38	356	4·76
		—	—	12	1·19	12	1·19	34	5·95
5	4	* 470	0·79	470	0·79	427	3·97	427	3·97
		—	—	40	1·59	62	3·17	83	5·56
9	1	—	—	—	—	—	—	178	20·00
		—	—	—	—	—	—	1	20·00
8	2	—	—	356	2·22	356	2·22	356	2·22
		—	—	2	2·22	2	2·22	2	2·22
7	3	—	—	439	1·67	393	3·33	368	5·00
		—	—	14	1·67	25	3·33	39	5·00
6	4	522	0·48	522	0·48	476	2·38	451	4·29
		26	0·48	26	0·48	51	2·38	76	4·76
5	5	* 559	0·79	559	0·79	523	2·38	488	6·35
		* 63	0·79	63	0·79	99	2·38	134	6·35

Table 4 (continued)

n	m	0·5%		1·0%		2·5%		5·0%	
10	1	—	—	—	—	—	—	191	18·18
		—	—	—	—	—	—	0	9·09
9	2	—	—	382	1·82	382	1·82	382	1·82
		—	—	—	—	4	3·64	8	5·45
8	3	—	—	476	1·21	428	2·42	386	4·85
		* 8	0·61	8	0·61	23	3·03	38	4·85
7	4	570	0·30	522	1·52	495	2·73	474	4·85
		* 27	0·61	42	1·21	54	2·12	69	4·55
6	5	616	0·43	589	0·87	568	1·95	526	5·41
		46	0·22	73	1·08	100	2·60	130	4·98
11	1	—	—	—	—	—	—	203	16·67
		—	—	—	—	—	—	1	16·67
10	2	—	—	406	1·52	406	1·52	307	7·58
		—	—	2	1·52	2	1·52	10	7·58
9	3	* 510	0·91	510	0·91	431	2·73	407	5·45
		* 11	0·91	11	0·91	27	2·73	35	6·36
8	4	614	0·20	564	1·01	519	2·63	469	5·25
		20	0·20	36	1·01	60	2·83	73	4·65
7	5	639	0·51	618	1·01	589	2·27	560	4·55
		61	0·51	74	1·01	90	2·02	124	5·05
6	6	693	0·54	677	0·97	643	2·38	614	4·76
		99	0·54	115	0·97	149	2·38	178	4·76
12	1	—	—	—	—	—	—	215	15·38
		—	—	—	—	—	—	0	7·69
11	2	—	—	430	1·28	430	1·28	329	6·41
		—	—	—	—	3	2·56	6	3·85
10	3	* 544	0·70	544	0·70	462	2·10	430	4·55
		6	0·35	6	0·35	19	2·45	35	4·90
9	4	607	0·70	576	1·26	547	2·52	496	5·03
		29	0·56	38	0·98	51	2·66	77	4·48
8	5	671	0·47	658	0·85	610	2·72	570	5·05
		51	0·39	70	0·93	95	2·41	121	4·90
7	6	734	0·52	721	0·93	673	2·51	642	5·01
		96	0·52	114	0·99	145	2·39	177	5·01
13	1	—	—	—	—	—	—	225	14·29
		—	—	—	—	—	—	1	14·29
12	2	—	—	450	1·10	450	1·10	348	5·49
		—	—	2	1·10	2	1·10	7	5·49
11	3	* 573	0·55	521	1·10	469	2·75	419	6·04
		* 8	0·55	13	1·10	21	1·65	39	4·95

Table 4 (continued)

n	*m*	0·5%		1·0%		2·5%		5·0%	
10	4	644	0·50	612	0·90	560	2·60	508	5·09
		27	0·50	40	1·00	60	2·70	78	5·09
9	5	702	0·50	683	1·00	645	2·30	593	5·00
		53	0·40	71	1·20	98	2·50	123	5·09
8	6	773	0·47	741	1·00	702	2·66	664	5·13
		92	0·47	117	1·00	142	2·53	175	4·96
7	7	825	0·52	807	0·99	769	2·39	727	4·90
		143	0·52	161	0·99	199	2·39	241	4·90
14	1	—	—	—	—	—	—	235	13·33
		—	—	—	—	—	—	0	6·67
13	2	* 470	0·95	470	0·95	470	0·95	367	4·76
		—	—	2	1·90	4	2·86	10	4·76
12	3	602	0·44	549	0·88	480	2·64	446	5·05
		4	0·22	12	1·10	22	2·20	34	4·40
11	4	681	0·37	626	1·03	573	2·64	523	4·91
		28	0·51	36	1·10	52	2·20	74	4·91
10	5	734	0·43	694	1·03	652	2·43	605	5·13
		52	0·40	70	1·03	93	2·50	123	4·96
9	6	806	0·48	770	1·02	729	2·58	683	4·96
		92	0·50	114	1·02	144	2·42	177	5·03
8	7	861	0·57	837	1·01	794	2·49	753	5·13
		141	0·53	163	1·09	199	2·50	238	5·02
15	1	—	—	—	—	—	—	245	12·50
		—	—	—	—	—	—	1	12·50
14	2	* 490	0·83	490	0·83	386	4·17	386	4·17
		* 2	0·83	2	0·83	6	4·17	10	5·00
13	3	631	0·36	542	1·07	495	2·50	438	5·71
		7	0·36	16	1·07	20	2·50	35	5·36
12	4	683	0·49	645	0·99	581	2·58	532	5·05
		25	0·49	36	0·99	53	2·58	73	5·11
11	5	769	0·50	731	0·92	667	2·52	627	4·99
		54	0·55	69	0·96	97	2·52	122	5·04
10	6	830	0·49	800	0·95	751	2·49	702	5·04
		92	0·49	111	1·00	140	2·52	177	5·01
9	7	901	0·51	869	1·01	820	2·53	776	5·02
		139	0·51	158	1·01	197	2·53	235	5·02
8	8	954	0·47	930	0·98	889	2·54	845	4·96
		192	0·47	216	0·98	257	2·54	301	4·96
16	1	—	—	—	—	—	—	254	11·76
		—	—	—	—	—	—	0	5·88

Table 4 (continued)

n	*m*	0·5%		1·0%		2·5%		5·0%	
15	2	* 508	0·74	508	0·74	403	3·68	403	3·68
		—	—	2	1·47	4	2·21	8	3·68
14	3	602	0·59	566	0·88	508	2·50	442	5·15
		10	0·74	12	1·03	23	2·35	37	5·00
13	4	696	0·42	665	0·92	602	2·48	552	5·04
		29	0·63	37	1·05	56	2·65	75	5·04
12	5	786	0·50	751	1·02	694	2·52	640	4·95
		56	0·50	66	0·95	95	2·54	122	5·22
11	6	860	0·51	827	1·00	772	2·51	719	5·05
		90	0·48	110	0·97	141	2·55	177	5·05
10	7	934	0·50	902	1·02	847	2·45	797	5·01
		133	0·50	158	1·00	198	2·47	234	4·99
9	8	996	0·48	966	1·00	919	2·49	870	5·01
		184	0·49	214	0·99	256	2·49	298	4·99
17	1	—	—	—	—	—	—	262	11·11
		—	—	—	—	—	—	0	11·11
16	2	* 524	0·65	524	0·65	419	3·27	363	5·88
		* 0	0·65	0	0·65	4	3·27	8	3·92
15	3	625	0·49	576	0·98	524	2·21	459	5·39
		8	0·49	11	0·74	23	2·45	34	4·90
14	4	721	0·46	685	0·98	616	2·45	560	5·13
		26	0·46	34	0·88	54	2·39	74	5·16
13	5	805	0·49	769	1·00	704	2·54	648	5·04
		54	0·49	67	1·03	92	2·54	120	5·14
12	6	887	0·51	849	1·00	791	2·46	736	4·99
		88	0·46	106	0·96	141	2·49	174	4·96
11	7	961	0·49	924	0·99	869	2·50	815	5·01
		131	0·51	153	0·95	192	2·49	233	5·03
10	8	1028	0·49	995	0·98	943	2·48	891	5·08
		181	0·50	207	0·99	250	2·48	296	4·99
9	9	1090	0·50	1061	0·96	1012	2·50	962	4·99
		236	0·50	265	0·96	314	2·50	364	4·99
18	1	—	—	—	—	—	—	271	10·53
		—	—	—	—	—	—	0	5·26
17	2	* 542	0·58	542	0·58	435	2·92	378	5·26
		—	—	2	1·17	6	2·92	8	5·26
16	3	649	0·41	588	1·03	542	2·79	481	4·64
		8	0·52	14	1·03	23	2·89	36	5·47
15	4	734	0·46	706	1·03	619	2·48	570	5·21
		25	0·49	36	1·03	51	2·40	76	4·98

Table 4 (continued)

n	*m*	0·5%		1·0%		2·5%		5·0%	
14	5	823	0·52	783	0·99	723	2·48	660	5·01
		53	0·51	68	0·95	94	2·49	122	5·13
13	6	912	0·50	874	1·00	811	2·49	751	5·01
		88	0·49	108	0·99	140	2·50	174	5·00
12	7	991	0·50	949	1·00	891	2·50	835	4·97
		130	0·50	154	0·97	193	2·51	233	5·03
11	8	1061	0·50	1025	1·01	965	2·46	912	5·00
		179	0·51	206	1·01	250	2·50	294	5·01
10	9	1125	0·49	1094	0·99	1039	2·51	986	5·01
		234	0·50	264	0·99	313	2·52	362	5·02
19	1	—	—	—	—	—	—	278	10·00
		—	—	—	—	—	—	0	10·00
18	2	* 556	0·53	556	0·53	449	2·63	392	4·74
		* 0	0·53	0	0·53	3	2·63	9	5·26
17	3	633	0·53	588	1·05	556	2·46	481	4·74
		9	0·53	12	1·23	22	2·63	35	5·09
16	4	746	0·54	710	0·97	639	2·56	578	4·99
		25	0·47	34	0·97	53	2·33	73	5·12
15	5	844	0·52	803	0·99	737	2·52	673	4·94
		53	0·52	63	0·98	92	2·57	120	4·94
14	6	927	0·51	890	1·00	823	2·51	763	5·01
		85	0·50	105	1·01	139	2·47	172	5·04
13	7	1012	0·50	970	1·00	907	2·51	849	4·99
		127	0·51	149	1·00	190	2·45	230	4·97
12	8	1087	0·50	1049	0·99	986	2·51	930	4·99
		174	0·49	200	1·00	247	2·51	293	4·99
11	9	1158	0·50	1121	0·99	1061	2·51	1005	5·00
		228	0·50	259	1·01	309	2·50	359	5·00
10	10	1221	0·50	1188	1·00	1133	2·51	1079	5·00
		287	0·50	320	1·00	375	2·51	429	5·00

Table 5. Approximate and exact significance levels of the Capon test for $N = 20$

n	m	0·5%		1·0%		2·5%		5·0%	
19	1	—	—	—	—	—	—	0·65	10·00
		—	—	—	—	—	—	20·36	10·00
18	2	0·02	0·53	—	—	0·54	2·63	1·93	4·74
		7·13	0·53	—	—	11·82	2·63	12·74	5·26
17	3	0·13	0·53	0·32	1·05	0·90	2·46	3·32	4·74
		7·19	0·53	7·41	1·23	8·21	2·63	9·63	5·09
16	4	0·20	0·54	0·39	0·97	1·64	2·50	3·55	4·97
		4·67	0·47	5·17	0·97	6·48	2·66	7·88	5·04
15	5	0·26	0·50	0·60	1·01	1·62	2·52	3·93	5·02
		3·35	0·46	3·85	0·98	5·23	2·46	6·97	5·01
14	6	0·39	0·48	0·72	0·99	1·83	2·49	3·95	5·01
		2·47	0·49	2·54	1·01	4·45	2·46	6·20	4·98
13	7	0·49	0·50	0·88	1·00	2·02	2·51	4·12	4·98
		1·90	0·50	2·48	1·01	3·80	2·58	5·69	4·98
12	8	0·62	0·51	1·04	1·01	2·23	2·50	4·32	5·02
		1·51	0·50	2·03	1·01	3·35	2·50	5·36	4·97
11	9	0·77	0·50	1·24	1·01	2·47	2·49	4·56	5·01
		1·19	0·50	1·69	0·99	2·99	2·49	5·00	5·00
10	10	0·96	0·50	1·47	0·99	2·74	2·51	4·80	4·99
		0·94	0·50	1·44	0·99	2·69	2·51	4·72	4·99