Aplikace matematiky

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Aplikace matematiky, Vol. 17 (1972), No. 6, 409–421

Persistent URL: `http://dml.cz/dmlcz/103434`

ROOTS OF THE CIRCULAR CYLINDRICAL SHELL CHARACTERISTIC EQUATION

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(Received April 14, 1971)

For the solution of the closed isotropic elastic circular cylindrical shell by means of linear bending theory in cylindrical coordinates, the decomposition into the simple Fourier series with respect to the circumferential angle can be used for the separation of independent variables. In this case, for each member of the series it is necessary to find the general solution of an ordinary homogeneous linear differential equation with constant coefficients

$$\left[a_8 \frac{\mathrm{d}^8}{\mathrm{d}x^8} + a_6 \frac{\mathrm{d}^6}{\mathrm{d}x^6} + a_4 \frac{\mathrm{d}^4}{\mathrm{d}x^4} + a_2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + a_0\right] W_n = 0 \tag{1}$$

where W_n denotes the source-function and a_i $(i = 8, 6, 4, 2, 0)$ are real coefficients dependent on the parameters n, H, ν; n is the ordinal number of the member of the series, ν is Poisson's ratio of the shell material, S is the shell thickness, r is the middle surface radius and H is the dimensionless quantity

$$H = \frac{S^2}{12r^2}. \tag{2}$$

From the point of view of technical problems, the following restrictions may be introduced:

$$H \in (0; 0{\cdot}001)\,, \quad \nu \in \langle 0; 0{\cdot}4\rangle\,, \quad n \geqq 2 \tag{3}$$

(The cases $n = 0$ and 1 are usually solved by simpler methods.)

The above mentioned bending theory is known in some versions differing by their authors' approach to the geometry of a shell deformation. Each of them yields other coefficients a_i. With the exception of some of them (e.g. Lurje, Galerkin) coefficients a_i may be expressed as polynomials in n, H, ν. The representatives of three important streams are introduced in Tab. I., for a more detailed account see [1] and [2]. The Wlasow's version coincides with that by Flügge, but subsequently it neglects some little members. The Goldenweiser's version which is presented here without neglecting little members, is different from that by Wlasow because of the different opinion of the change of circumferential curvature. In this point it coincides with Love-Timoshenko's version. The simplest Donnell's version which is also mentioned here is known as the technical theory.

The characteristic equation to (1) is

$$a_8 m^8 + a_6 m^6 + a_4 m^4 + a_2 m^2 + a_0 = 0 . \tag{4}$$

Wlasow [3] proves that in consequence of the law of conservation of energy the

Tab. I. Coefficients of Equation (4) divided by H, according to the most important authors.

$\frac{a_i}{H}$	Flügge	Goldenweiser	Donnell
$\frac{a_8}{H}$	$1 + 2H - 3H^2$	$1 + 4H$	1
$\frac{a_6}{H}$	$-4n^2\left[1 + \frac{H}{8}(11 - 3\nu) + \frac{9H^2}{8}(1 - \nu)\right] + 2\nu(1 + 3H)$	$-4n^2[1 + H]$	$-4n^2$
$\frac{a_4}{H}$	$6n^4\left[1 + \frac{H}{2}(2 - \nu) - \frac{H^2}{6}\nu^2\right] - 6n^2\left[1 + \frac{H}{2}(2 - \nu + \nu^2)\right] + \frac{1 - \nu^2}{H} + 4 - 3\nu^2 + 3H$	$6n^4\left[1 + \frac{H}{6}(1 - \nu^2)\right] - 2n^2(4 - \nu^2) + \frac{1 - \nu^2}{H} + 4(1 - \nu^2)$	$6n^4 + \frac{1 - \nu^2}{H}$
$\frac{a_2}{H}$	$-4n^2(n^2 - 1)^2 - (n^6/2)H(7 - 3\nu) - \frac{3}{2}n^6H^2(1 - \nu) - 2n^2\nu(n^2 - 1) + n^4H(7 - 5\nu) + 3n^4H^2(1 - \nu) - \frac{7}{2}n^2H(1 - \nu) - \frac{3}{2}n^2H^2(1 - \nu)$	$-4n^2(n^2 - 1)^2$	$-4n^6$
$\frac{a_0}{H}$	$n^4(n^2 - 1)^2(1 + H)$	$n^4(n^2 - 1)^2$	n^8

equation (4) in region (3) should have only complex roots. Each of the authors – Wlasow in [3], Flügge in [5] and Goldenweiser in [4]-affirms about his own equation that it has only complex roots in region (3). This property is obvious for Donnell's equation because its left hand side is the sum of a positive number and the fourth power of a binomial. On the other hand, from this point of view Wlasow criticises Goldenweiser's version. If we neglect H compared with 1 in Goldenweiser's equation (4), then for certain n's greater than n_0,

$$n_0 = \sqrt{\left(\frac{1 - \nu^2 - 3H}{2H(1 - \nu^2)}\right)}, \tag{5}$$

it has four real roots, as it is proved in [1]. However, the complete (i.e. without neglecting H compared with 1) Goldenweiser's equation has not yet been analysed.

Theorem. *Characteristic equation (4) with Goldenweiser's coefficients from Tab. I. has only complex roots in region (3). They are of the following form:*

$$\begin{array}{llll} \alpha_n + i\beta_n\,, & \alpha_n - i\beta_n\,, & -\alpha_n + i\beta_n\,, & -\alpha_n - i\beta_n \\ \gamma_n + i\delta_n\,, & \gamma_n - i\delta_n\,, & -\gamma_n + i\delta_n\,, & -\gamma_n - i\delta_n \end{array} \tag{6}$$

where α_n, β_n, γ_n, δ_n are real positive numbers.

Proof. By the transformation

$$z = m^2 \tag{7}$$

the characteristic equation becomes an algebraic one of the fourth degree for z. As in the whole region (3) the successive coefficients of the equation have always the opposite signs, the equation for z cannot have negative roots and that is why the equation (4) cannot have pure imaginary roots. Zero cannot be a root, which follows from the form of a_0. Therefore a real root of the equation for z yields a real root of the equation (4). It is sufficient to analyse the equation for z. From each complex root of the equation for z we obtain two roots of the equation (4) differing by the sign only. Hence the form (6) follows.

We shall use the results from [6], p. 60: the character of the roots of an equation of the fourth degree is above all decided by the discriminant of its resolvent; if it is positive, the equation has different roots, either all real or all complex; if the discriminant is equal to zero, the equation has multiple roots. The region (3) is simply connected. Although the quantity n is discrete, we extend it to the continuous quantity in the proof. Then the equation (4) has continuous coefficients a_i with respect to n, H, ν in the region (3). If we prove the discriminant to be positive in the whole region (3), then multiple roots cannot exist and hence the roots will be either all real or all complex in the whole region (3): under the above mentioned suppositions it is impossible for equation (4) to have only real roots in a subregion of region (3) and only complex roots in another subregion because in this case the multiple roots should have to exist on the boundary between the subregions. However, the roots of the resolvent depend continuously on the coefficients of the 8-th degree equation in region (3) (see e.g. trigonometric solution), and hence at least two of its roots should be equal to zero on the above mentioned boundary between subregions. Hence we obtain the existence of multiple roots of the 4-th degree equation, which is in contradiction to the statement that the discriminant is positive in the whole region (3).

The positive sign of the discriminant in the whole region (3) and the existence of only complex roots at one point of the region (3) form a sufficient condition for equation (4) to have only complex roots in the whole region (3). By a direct calculation we obtain complex roots e.g. for $v = 0{\cdot}3$, $n = 2$, $H = 1/1200$.

For the reduced equation of the fourth degree

$$\eta^4 + C\eta^2 + D\eta + E = 0$$

the discriminant is introduced in [6] in the form

$$(8) \qquad \Delta = 16C^4E - 4C^3D^2 - 128C^2E^2 + 144CD^2E + 256E^3 - 27D^4 .$$

We shall write the equation for z in a more convenient form

$$(9) \qquad b_4z^4 - 4b_3z^3 + 6b_2z^2 - 4b_1z + b_0 = 0 ,$$

where

$$(10) \qquad b_4 = a_8 , \quad -4b_3 = a_6 , \quad 6b_2 = a_4 , \quad -4b_1 = a_2 , \quad b_0 = a_0 .$$

Equation (9) can be transformed to the reduced form by means of the substitution

$$b_4z - b_3 = \eta .$$

Calculating C, D, E by means of b_i and substituting into (8) we get the discriminant of equation (9), which is equal to the following expression:

$$(8a) \qquad \Delta = 64b_4^6[b_4^3b_0^3 - 3b_4^2(4b_3b_1b_0^2 + 6b_2^2b_0^2 - 18b_2b_1^2b_0 + 9b_1^4) + \\ + 3b_4(18b_3^2b_2b_0^2 - 2b_3^2b_1^2b_0 - 60b_3b_2^2b_1b_0 + 36b_3b_2b_1^3 + \\ + 27b_2^4b_0 - 18b_2^3b_1^2) - 27b_3^4b_0^2 + b_3^3(108b_2b_1b_0 - 64b_1^3) + \\ + b_3^2(36b_2^2b_1^2 - 54b_2^3b_0)] .$$

The sum of numerical coefficients on the right hand side of (8a) is zero which is a consequence of multiple roots in the case of the equality of coefficients b_i. The change of both the signs of b_1, b_3 simultaneously maintains this property.

The conclusion of the proof consists in substituting b_i into (8a) from Tab. I. by means of (10) and in proving that the form obtained is always positive in region (3). We shall substitute only coefficients of the complete Goldenweiser's equation. By means of

$$G = Hn^2 , \quad Z = Hv^2 , \quad \mu = 1 - v^2$$

we can write the result in the following form

$$\text{(8b)} \qquad \Delta = 64(1 + 4H)^6\, G^2 n^2 (n^2 - 1)^2 \,.$$

$$.\{\sum_{j=0}^{4} n^{2-2j} G^j (G - 1)^{8-2j} f_j + 54\nu^4 G n^{-2}[(G - 1)^2 + 4Gn^{-2}]^2\}\,,$$

where f_j are polynomials in n^{-2}, Z, ν^2, μ:

$$\text{(8c)} \qquad f_0 = \frac{\mu^4 - 4Z\mu^3}{16n^4}\,,$$

$$f_1 = \mu^3 - \frac{\mu^3 + 54\nu^4}{n^2} + \frac{\mu^4}{n^4} - 4{\cdot}5Z(\mu^2 + 6\nu^2) -$$

$$- 4{\cdot}5Z\,\frac{\mu^2 + 12\nu^2}{n^2} - 4Z\,\frac{\mu^3 + 6{\cdot}75\nu^2}{n^4}\,,$$

$$f_2 = 12\mu^3 - 81\nu^4 - \frac{12\mu^3 + 488\nu^4}{n^2} + \frac{6\mu^4 + 27\nu^4}{n^4} - 54Z(1 + \nu^4) +$$

$$+ 54Z\,\frac{\mu^2 + 7\nu^2}{n^2} - 24Z\,\frac{\mu^3 + 9\nu^2}{n^4}\,,$$

$$f_3 = 48\mu^3 - 648\nu^4 - \frac{48\mu^3 + 1296\nu^4}{n^2} + \frac{16\mu^4 + 216\nu^4}{n^4} -$$

$$- 216Z(\mu^2 - 2\nu^2) + 216Z\,\frac{1 + \nu^4}{n^2} - 16Z\,\frac{4\mu^3 + 27\nu^2}{n^4}\,,$$

$$f_4 = 64\mu^3 - 1296\nu^4 - \frac{64\mu^3 + 864\nu^4}{n^2} + \frac{16\mu^4 + 27\nu^4}{n^4} -$$

$$- 288Z(\mu^2 - 6\nu^2) + 288Z\,\frac{\mu^2 - 3\nu^2}{n^2} - 64Z\,\frac{\mu^3}{n^4}\,.$$

The arrangement of this expression was chosen like this so that the polynomials (8c) in region (3) might be positive for $n > 4$, which can be easily proved. There is no doubt as to $n \in \langle 2; 4 \rangle$, but to make a mathematical proof it would be necessary to return from G back to H and to arrange the expression according to the powers of H

$$\text{(8d)} \qquad \Delta = 4(1 - \nu^2)^4\, H^2 n^4 (n^2 - 1)^2 \left[1 + 16Hn^2\left(\frac{n^2}{1 - \nu^2} - \frac{1}{2}\right) + \ldots\right].$$

Here the members with higher powers of H are neglected with respect to little n's. For ν approaching the value 0·5 which is out of the region (3) the expression (8b) is not always positive (plastic state).

The expression (8b) found by a computer practically never serves for substituting because of a big span of powers.

The highest power of n in expression (8b) is 6 if $G \neq 1$ but it is 0 if $G = 1$. Hence it follows that the algorithm for the calculation of the discriminant must be carefully chosen with respect to numerical stability in the case of a limited number of decimal places of mantissa in the subregion

$$(3a) \qquad G \doteq 1\,, \quad \text{i.e.} \quad n \doteq 3{\cdot}464\,\frac{r}{S}$$

which we call the first instability region. In region (3) it is always $n \geqq 32$. Lower powers of n can be neglected in comparison with higher ones for sufficiently big n's, which are also in subregion (3a). Goldenweiser's characteristic equation approaches that of Donnell, the complex roots of which have relatively big real parts α_n, γ_n in comparison with the imaginary parts β_n, δ_n in this subregion; this property is more and more apparent with growing n:

$$m = \pm\, n \pm \tfrac{1}{2}\sqrt[4]{[-(1-\nu^2)/H]} = \pm\, n \pm \tfrac{1}{2}\sqrt[4]{[-(1-\nu^2)]}\,.\,\sqrt{\left[3{\cdot}464\,\frac{r}{S}\right]}\,,$$

which we obtain solving Donnell's equation and neglecting small members.

If we return from G back to H then the highest power of n is 24 when $H \neq 0$ but only 8 when $H \to 0$ as it can be seen from (8b) and (8d). Hence another requirement concerning the calculation of discriminant arises, viz., the numerical stability in the subregion which we call the second instability region:

$$(3b) \qquad H \doteq 0\,, \quad n \text{ little.}$$

In this case the characteristic equation approaches the form (where Hn^4 was neglected in comparison with 1)

$$(11) \qquad [m^4 + (1-\nu^2)/H]\,[m^4 + a_0/(1-\nu^2)] = 0$$

in which the modul of the so called "big" root $\alpha_n + i\beta_n$ (derived from the first bracket) is much bigger than the modul of the so called "little" root $\gamma_n + i\delta_n$, see [4]. In the region (3b) it holds

$$(12) \qquad \sqrt{(\alpha_n^2 + \beta_n^2)} \gg \sqrt{(\gamma_n^2 + \delta_n^2)}\,,$$

$$\alpha_n - \beta_n \ll \frac{2\alpha_n\beta_n}{\alpha_n + \beta_n}\,, \quad \gamma_n - \delta_n \ll \frac{2\gamma_n\delta_n}{\gamma_n + \delta_n}\,.$$

The author of this paper carried out an analogous proof for Flügge's equation, too. The result analogous to (8b) cannot be presented here since it would need too much space.

Although another expression for equation (4) is obtained by other authors, the properties of roots are the same.

We shall introduce the relations between the roots and the coefficients of equation (4):

$$\alpha_n^2 - \beta_n^2 + \gamma_n^2 - \delta_n^2 = -\frac{a_6}{2a_8} \tag{13}$$

$$(\alpha_n^2 + \beta_n^2)^2 + (\gamma_n^2 + \delta_n^2)^2 + 4(\alpha_n^2 - \beta_n^2)(\gamma_n^2 - \delta_n^2) = \frac{a_4}{a_8}$$

$$(\alpha_n^2 - \beta_n^2)(\gamma_n^2 + \delta_n^2)^2 + (\gamma_n^2 - \delta_n^2)(\alpha_n^2 + \beta_n^2)^2 = -\frac{a_2}{2a_8}$$

$$(\alpha_n^2 + \beta_n^2)^2 \cdot (\gamma_n^2 + \delta_n^2)^2 = \frac{a_0}{a_8}.$$

CALCULATION OF ROOTS

Mechanical application of usual algorithms does not yield any sufficiently accurate results in the whole region (3), which is caused by the properties of the roots described in the preceding section. Owing to unsuitable calculating methods we sometimes even get real roots from equation (4) by any author. For further analysis of the shell the roots should be known with the highest possible accuracy because especially in contact-problems the decisive role is played by the shear forces which are obtained by means of multiple differentiation of the source-function.

Now let us introduce three algorithms for the solution of equation (4). The algorithm based on the classical method of solving the algebraic equation of the 4-th degree by means of its cubic resolvent is applicable in the largest extent from region (3). Equation (4) is first reduced so that the module of the big roots should be near to 1. Algorithm is made for complex roots only. If it is used for another problem, then in the case of other roots than complex ones the return from the procedure comes through label L due to the negative sign of the discriminant of the resolvent (4 roots real) or due to the positive sign of all the roots of the resolvent (8 roots real). Solution of these cases see e.g. [1]. After finding the big roots, numbers γ_n, δ_n are found in a less usual way from coefficients a_0, a_2 according to (13) because of the above mentioned reason. The procedure is written in ALGOL 60 and it is described in Appendix. It uses arrays $A[1:5]$, $K[1:8]$ and the shell parameter

$$B = b^2 = \frac{r}{S}\sqrt{[3(1 - \nu^2)]} = \sqrt{\frac{1 - \nu^2}{4H}}. \tag{14}$$

The recommended function ARCSIN is applied. Procedure RADICES does not depend on the approach of the authors of the equation (4). Coefficients a_i must be calculated numerically before, without neglecting little members, with the highest possible accuracy. The squares of the roots, which are partial results of the algorithm, must be also included into the result, because their later calculation from the roots may cause a big inaccuracy in the second instability region.

With respect to the above mentioned instability regions it would be certainly advantageous to apply the "long real" ("double precission"). TRIPLEX-ALGOL would be advantageous for the estimation of the inaccuracy. However, these devices are not usually in the software of little computers. Procedure RADICES is practically used with a simple length of a number in author's programme for the calculation of closed circular cylindrical shells on the computer MINSK 22 with 29-bits mantissa.

In the second instability region the discriminant is the difference of two nearly identical numbers, so that in some cases all bits of mantissa are lost and test $L\,1$ stops incorrectly the calculation by return through L. However, if it is proved that even in this case all roots are complex, it is no mistake to change the algorithm in line $L\,1$ into the form

$$L\,1 : \textbf{if } DE \leqq 1 \textbf{ then } DE := DE\,1\;;$$

where $DE\,1$ denotes the nearest expressible number in computer, bigger than one. Such a modified algorithm gives very accurate results for $H \in \langle 10^{-7};\ 10^{-3} \rangle$, $n \in \langle 2;\ 120 \rangle$. If coefficients are calculated back from the roots, then the new coefficients coincide with the original ones in all ciphers except a_6, which is sometimes less accurate in the instability regions. The roots of equation (4) by the three authors by means of procedure RADICES are calculated in Tab. II. with $\nu = 0{\cdot}3$.

As another possibility of the calculation, an iterative algorithm by Newton's method applied to a complex function was tested. The starting value for $\alpha_n + i\beta_n$ is the big root of Donnell's equation

(15)
$$m_0 = \frac{b}{2}\left\{1 + \sqrt{\left[\frac{2n^2}{b^2} + \sqrt{\left(\frac{4n^4}{b^4} + 1\right)}\right]}\right\} + \frac{bi}{2}\left\{1 + \sqrt{\left[-\frac{2n^2}{b^2} + \sqrt{\left(\frac{4n^4}{b^4} + 1\right)}\right]}\right\},$$

where b is taken from (14). Let us denote the polynomial on the left hand side of equation (4) by $P(m)$ and $P'(m) = \mathrm{d}P/\mathrm{d}m$. Then

$$m_{j+1} = m_j + \Delta m_j\,, \tag{16}$$

where the complex increment is

$$\Delta m_j = -\frac{P(m_j)}{P'(m_j)}\,. \tag{17}$$

This course can be repeated till we reach the required accuracy for $|P(m_{j+1}) - P(m_j)|$ with respect to the number of bits of mantissa. Except number m_j it is recommended to keep also m_j^2 during the iterative procedure, or more precisely, to add Re (Δm_j) –

Tab. II. Roots of Equation (4)

n	$\frac{r}{S}$		by Means of Procedure RADICES		
			Flügge	Goldenweiser	Donnell
2	1000	α_n	40·695534	40·697363	40·697391
		β_n	40·600826	40·598976	40·599005
		γ_n	0·042651353	0·042649427	0·049262158
		δ_n	0·042570117	0·042572059	0·049143044
		$\alpha_n^2 - \beta_n^2$	7·6994001	7·9984297	7·99843069
		$\gamma_n^2 - \delta_n^2$	$0{\cdot}692302 . 10^{-5}$	$0{\cdot}659336 . 10^{-5}$	$0{\cdot}11721492 . 10^{-4}$
40	1000	α_n	61·556824	61·556535	61·559335
		β_n	30·336285	30·330874	30·341412
		γ_n	20·904378	20·904959	20·911197
		δ_n	10·307549	10·308405	10·306726
		$\alpha_n^2 - \beta_n^2$	2868·9524	2869·2452	2868·9505
		$\gamma_n^2 - \delta_n^2$	330·74745	330·75408	331·04953
2	10	α_n	4·5686288	4·5768969	4·5993876
		β_n	3·6388997	3·6096019	3·6415651
		γ_n	0·45590630	0·45491653	0·53457377
		δ_n	0·37916436	0·38163031	0·42324878
		$\alpha_n^2 - \beta_n^2$	7·6307777	7·9187592	7·8933703
		$\gamma_n^2 - \delta_n^2$	0·064084939	0·061307361	0·106629596
40	10	α_n	42·036994	40·982815	42·032499
		β_n	2·1190439	1·9612243	2·1356262
		γ_n	37·932186	38·945494	37·967768
		δ_n	2·1607581	0·68520762	1·9291042
		$\alpha_n^2 - \beta_n^2$	1762·6185	1675·7447	1762·1701
		$\gamma_n^2 - \delta_n^2$	1434·1819	1516·2820	1437·8299

$-\operatorname{Im}(\Delta m_j)$ to $(\alpha_{nj} - \beta_{nj})$ in each step. The solution can be carried out in a reduced form as in the previous algorithm. The iterative algorithm is slower than the previous one. The root $\gamma_n + i\delta_n$ can be found for little n's by the iterative course as well, if we start from the approximate value

$$m_0 = \frac{1 + i}{2b} \sqrt[4]{\frac{a_0}{H}} . \tag{18}$$

However, the simple calculation by means of a_2, a_0, a_8, α_n, β_n, $\alpha_n - \beta_n$ according to the last two equations (13) explicitly formed, is better.

As the third possibility of the calculation of roots an asymptotic series was tested. This method is applicable only in the second instability region $2n < b$. It is possible to derive the following expressions for the roots $\alpha_n + i\beta_n$, see [1]:

from Flügge's equation

$$(19)\quad b\left[1 + i + \frac{1-i}{b^2}\,\frac{2n^2 - \nu}{4} + \frac{1+i}{b^4}\,\frac{8n^4 + 4\nu n^2 - 12n^2 + 4 - 3\nu^2}{32} - \ldots\right],$$

from Goldenweiser's equation

$$(20)\qquad b\left[1 + i + \frac{1-i}{b^2}\,\frac{n^2}{2} + \frac{1+i}{b^4}\,\frac{2n^2 - 4 + \nu^2}{8}\,n^2 - \ldots\right],$$

from Donnell's equation

$$(21)\qquad b\left[1 + i + \frac{1-i}{b^2}\,\frac{n^2}{2} + \frac{1+i}{b^4}\,\frac{n^4}{4} - \ldots\right].$$

It is interesting that the third member of the series is already affected by members of equation (4) that are usually neglected. Analogously the root $\gamma_n + i\delta_n$ in the same region can be obtained, e.g. from Goldenweiser's version:

$$(22)\qquad \frac{n}{2b}\sqrt{(n^2 - 1)}\,.\left[1 + i + \frac{1-i}{b^2}\,\frac{n^2 - 1}{2} - \right.$$
$$\left. - \frac{1+i}{b^4}\,\frac{4n^4 - (6 - \nu^2)\,n^2 + 3 - 2\nu^2}{8} + \ldots\right].$$

In the region $n \ll b$ we get more accurate values of roots by means of the series by adding two or three members than by means of procedure RADICES, as the decimal orders of the members of the series decrease rapidly and the results are not influenced by the inaccuracy of the difference of the numbers as the case would be in the procedure RADICES in the second instability region. In the first example of Tab. II. we get

$m = 40{\cdot}695536 \quad + i\,.\,40{\cdot}600822$ by means of (19),
$m = 40{\cdot}697371 \quad + i\,.\,40{\cdot}598966$ by means of (20),
$m = \ 0{\cdot}042649427 + i\,.\ 0{\cdot}042572059$ by means of (22).

We can obtain another version of asymptotic series, if we start from the value m_0 according to (15), which satisfies Donnell's equation. E.g. for Goldenweiser's version the series is

$$(23)\qquad \alpha_n + i\beta_n = m_0 + \frac{1}{b}\left[-\frac{4 - \nu^2}{8}\,n^2\,\frac{1+i}{b^2} + \right.$$
$$\left. + \frac{(8 - \nu^2)\,n^2 - 8 + 6\nu^2}{16}\,n^2\,\frac{1-i}{b^4} - \ldots\right],$$

which in the same example yields $m = 40{\cdot}697363 + i\,.\,40{\cdot}598976$. However, for n approaching b the series are divergent.

The biggest part of region (3) is covered by the classical algebraic solution. Its accuracy fails only in both subregions of the numerical instability at the limited number of decimal places of mantissa. The method of the asymptotic series is suitable only in the second subregion (3b). The iterative algorithm is applicable in the same subregion and in other parts of region (3). It is possible to derive other versions of series or of the initial values of iterative algorithm suitable for the first subregion (3a), but this subregion occures practically very seldom.

Appendix. Procedure for Calculation of Roots of Characteristic Equation by Means of Classical Algebraic Method.

procedure RADICES (A) onedimensional array for the successive coefficients a_8, a_6, a_4, a_2, a_0 (B) parameter (N) ordinal number of the member of the series (K) onedimensional array for the successive results α_n, β_n, γ_n, δ_n, $\alpha_n^2 - \beta_n^2$, $2\alpha_n\beta_n$, $\gamma_n^2 - \delta_n^2$, $2\gamma_n\delta_n$ (L) label for output in the case of other roots than complex;

value A, B; **array** A, K; **real** B, N; **label** L;

begin real M, P, Q, $K1$, $K2$, D, E, DE, MD, FI, $T1$, $T2$, $T3$, AL, $B1$, $D1$, $E1$, BD, $BD2$, CE, $CE2$, $M1$, $M2$; **integer** I;
$M1 := SQRT((N \times N/B) \uparrow 2 + 0{\cdot}25) + 0{\cdot}5$;
$M := (SQRT(M1) + M1) \times B$; **comment** approximate value of the square of the module of the big root;
$P := M \times A[1]$;

for $I := 2$ **step** 1 **until** 5 **do**

begin $A[I] := A[I]/P$; $P := P \times M$
end reduction $m^2 = M\xi$, $A[1]\,\xi^4 + A[2]\,\xi^3 + A[3]\,\xi^2 + A[4]\,\xi + A[5] = 0$;
$K1 := A[2]/4; P := K1 \times K1; K2 := 2 \times P - A[3]/3$;
$E := ((-3 \times P + A[3]) \times K1 - A[4]) \times K1 + A[5]$;
$D := (4 \times P - A[3]) \times 2 \times K1 + A[4]$;
comment reduction $\xi = \eta - K1, \eta^4 - (K2/3)\,\eta^2 + D\eta + E = 0$,
resolvent $t^3 + (K2/3)\,t^2 - 4Et - \frac{4}{3}E\,K2 - D^2 = 0$;
$Q := K2 \uparrow 2; P := -E/0{\cdot}75 - Q; Q := (-4 \times E + Q) \times K2 - 0{\cdot}5 \times D \times D$;
$DE := -(P/Q) \uparrow 2 \times P$;
$L1$: **if** $DE \leqq 1$ **then go to** L; **comment** test for the separation of the case with 4 real roots;
$MD := SQRT(-4 \times P)$; $FI := (1{\cdot}570796327 - ARCSIN\ (2/DE - 1))/6$;
if $Q < 0$ **then** $FI := 1{\cdot}047197551 - FI$;
$T1 := 2 \times K2 + MD \times COS(FI)$;
$T2 := 2 \times K2 + MD \times COS(FI + 2{\cdot}094395102)$;
$T3 := 2 \times K2 + MD \times COS(FI + 4{\cdot}188790204)$;

```
comment roots of the resolvent, at least one of them is positive;
if T1 < 0 then begin if T2 < 0 then begin P := T3; T3 := T1 end
                                    else begin P := T2; T2 := T1 end
              end else P := T1;
AL:= SQRT(P);
L2 : if T2 > 0 then go to L;
comment test for the separation of the case with 8 real roots;
B1 := SQRT(-T2) + SQRT(-T3); D1 := AL - 2 × K1;
BD := B1 ↑ 2 + D1 ↑ 2; BD2 := SQRT(BD);
M1 := 0·25 × M; M2 := 0·5 × M;
CE := 16 × A[5]/BD; E1 := (-D1 × CE - 4 × A[4])/BD;
CE2 := SQRT(CE);
K[1] := SQRT(M1 × (D1 - BD2));
K[2] := SQRT(M1 × (-D1 + BD2));
K[3] := SQRT(M1 × (E1 + CE2));
K[4] := SQRT(M1 × (-E1 + CE2));
K[5] := M2 × D1; K[6] := M2 × B1;
K[7] := M2 × E1; K[8] := 2 × K[3] × K[4]
end
```

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Souhrn

KOŘENY CHARAKTERISTICKÉ ROVNICE ROTAČNĚ VÁLCOVÉ SKOŘEPINY

MILAN GERYK

Stav napjatosti a deformace uzavřené rotačně válcové skořepiny lze řešit rozkladem do Fourierovy řady podle obvodové proměnné. Pro n-tý člen řady dostane se obyčejná diferenciální rovnice tvaru (1), jejíž koeficienty a_i se liší podle přístupu jejich autorů, viz tab. I. Její charakteristická rovnice (4) v oblasti (3) parametrů n, H, ν má mít dle zákona o zachování energie pouze kořeny komplexní. Zanedbá-li se H proti jedné u Golděnvejzerovy verse, rovnice má pro určitá n větší než n_0 též kořeny reálné.

V článku se dokazuje věta, že úplná Golděnvejzerova rovnice dle tab. I., tedy bez zanedbání H proti jedné, má v oblasti (3) pouze komplexní kořeny. Zároveň jsou ukázány vlastnosti kořenů a nalezeny dvě oblasti numerické nestability při řešení s omezeným počtem míst mantisy.

Vzhledem k použití na kontaktní úlohy vyžaduje se znalost kořenů se značnou přesností. Pro jejich výpočet je uvedena jednak procedura RADICES v algolu založená na klasickém řešení algebraické rovnice 4. stupně, jednak iterační postup pomocí Newtonovy metody pro komplexní přírůstek, jednak asymptotický rozvoj — tento je vhodný v druhé oblasti numerické nestability $n \ll b$. V tab. II. jsou numerické výsledky procedury RADICES, která se osvědčila v autorově programu pro výpočet rotačně válcové skořepiny na počítači MINSK 22 s 29-bitovou mantisou.

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