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ON A METHOD OF D. MARSAL FOR EQUATIONS WITH POSITIVE OPERATORS[1])

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In [3] D. MARSAL proposes a new iteration method for an approximate solution of linear systems

$$\xi_j = \sum_{k=1}^{N} a_{jk}\xi_k + f_j,\, a_{jk} \geqq 0\,, \quad j, k = 1, \ldots, N\,, \quad N \leqq +\infty\,,$$

and proves the convergence assuming that the above system has a unique non-negative solution. In this note we reformulate Marsal's result in an abstract form and obtain a method for an approximate solution of linear equations of the type

$$x = Tx + f \tag{1}$$

where f is a given element of a Riesz space X partially ordered by a cone K which is invariant with respect to T. This abstract approach has not only purely mathematical value, it has new applications even for systems of linear equations with either finite or countable many unknowns in those cases when the corresponding matrices have some negative elements but leave invariant some cone K.

Let X be a real or complex Banach space partially ordered by a reproducing closed and normal cone K ([1]); $x \in X$, $x = x_1 - x_2$, $x_1, x_2 \in K$, and $\|x + y\| \geqq \delta\|x\|$ with some $\delta > 0$ independent of $x, y \in K$. We write $x \leqq y$ (or $y \geqq x$) whenever $(y - x) \in K$. It will be assumed that X is a Riesz space [4], i.e. for every $x, y \in X$ there exist $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$, where the supremum and infinum are to be taken with respect to the order induced by the cone K. In X one defines an absolute value of an element $x \in X$ by setting $|x| = x^+ + x^-$, where $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Let X' be the dual space to X and $[X]$ the space of continuous linear operators mapping X into itself; X' and $[X]$ are Banach spaces. By K' we denote the adjoint cone, i.e. $K' = \{x' \in X' : \langle x, x'\rangle \geqq 0 \text{ for } x \in K\}$ where $\langle x, x'\rangle$

[1]) The paper was completed during the author's visit at Case Western Reserve University in Cleveland, Ohio as Visiting Professor of Mathematics.

denotes $x'(x)$. An operator $T \in [X]$ is called positive (more precisely K-positive) if $TK \subset K$. Let $\sigma(T)$ denote the spectrum of T and $r(T)$ the spectral radius of T; $r(T) = \sup \{|\lambda|; \lambda \in \sigma(T)\}$. By T' we denote the dual transformation to $T : \langle Tx, x' \rangle = \langle x, T'x' \rangle$, $x \in X$, $x' \in X'$.

Let us consider (1) with a positive operator $T \in [X]$ and let f be a given element in X. Let $h_p \in K$ and $z'_p \in K'$, $p = 0, 1, \ldots$, be such elements that

$$Tx \geqq \langle x, z'_p \rangle h_p \tag{2}$$
$$T'x' \geqq \langle h_p, x' \rangle z'_p$$

holds for all $x \in K$ and $x' \in K'$, where $y' \geqq x'$ means $\langle x, x' \rangle \leqq \langle x, y' \rangle$ for all $x \in K$

Further, let

$$\langle h_p, z'_p \rangle < 1, \; p = 0, 1, \ldots \tag{3}$$

Define

$$g_p = v_p + \frac{\langle v_p, z'_p \rangle}{\langle h_p, z'_p \rangle} h_p, \tag{4}$$

where

$$v_p = v_{p-1} - u_{p-1} + Tu_{p-1}, \quad v_0 = f$$

and

$$u_p = g_p^+ .$$

Furthermore, let

$$w_p = \sum_{j=0}^{p} u_j, \quad p = 0, 1, \ldots$$

Theorem. *Let $T \in [X]$ be a positive operator, let (1) have a unique solution $x \in K$ for a fixed $f \in X$. Let I-T be invertible. Let $u_0 \in K$ be a lower bound for x*

$$u_0 \leqq x . \tag{5}$$

Then there exist bounds w and v

$$w = \lim_{p \to \infty} w_p, \tag{6}$$
$$v = \lim_{p \to \infty} v_p$$

and we have

$$w \leqq x \tag{7}$$

and

$$-v \in K. \tag{8}$$

The relation

$$w = x \tag{9}$$

is equivalent to $v = 0$.

If (10) *and* (11) *hold simultaneously, where*

$$v_s \in K \quad \textit{for some} \quad s \geqq 1\,; \tag{10}$$

$$\langle Tx, x' \rangle + \frac{\langle x, z'_p \rangle}{1 - \langle h_p, z'_p \rangle} \{ -\langle h_p, x' \rangle + \langle Th_p, x' \rangle \} \geqq 0 \tag{11}$$

for $p \geqq s$ *and every* $x \in K$, $x' \in K'$, *then process* (4) *converges*

$$w = \lim_{p \to \infty} w_p = x\,. \tag{12}$$

Proof is similar to that given by D. Marsal for the case of at most countable systems of linear equations in sequence spaces.

Note that sequences $\{h_p\}$, $\{z'_p\}$ required in (2) always exist, e.g. $h_p = 0$, $z'_p = 0$ for $p = 0, 1, \ldots$ From (2) it follows that

$$\langle f, z'_p \rangle + \langle h_p, z'_p \rangle \langle x, z'_p \rangle \leqq \langle x, z'_p \rangle\,,$$

consequently

$$\frac{\langle f, z'_p \rangle}{1 - \langle h_p, z'_p \rangle} - \langle x, z'_p \rangle \leqq 0$$

and thus

$$f + \frac{\langle f, z'_p \rangle}{1 - \langle h_p, z'_p \rangle} h_p \leqq f + \langle x, z'_p \rangle h_p\,.$$

According to (2)

$$f + \langle x, z'_p \rangle h_p \leqq f + Tx = x$$

and this implies in particular that

$$g_0 = f + \frac{\langle f, z'_0 \rangle}{1 - \langle h_0, z'_0 \rangle} h_0 \leqq x\,.$$

Let us consider the following sequence

$$f_p = x_p - Tx_p\,, \tag{13}$$

where

$$x_p = x_{p-1} - u_{p-1}\,, \quad p = 1, 2, \ldots, x_0 = x\,. \tag{14}$$

It is easy to see that

$$x_p = x - \sum_{j=0}^{p-1} u_j, \quad p = 1, 2, \ldots, \tag{15}$$

and that

$$w_p = \sum_{j=0}^{p} u_j = x - x_p \leqq x, \tag{16}$$

i.e. (7).

Since $u_j \in K$, $j = 0, 1, \ldots$, $w_{p+1} \geqq w_p$ for $p \geqq 0$ and we conclude that the first relation in (6) holds.

From this it follows that $\lim_{p \to \infty} u_p = 0$ and, according to (16) we obtain $t \in K$ as the limit

$$t = \lim_{p \to \infty} x_p .$$

It is readily seen that $f_p = v_p$, $p = 0, 1, \ldots$, and hence

$$v = \lim_{p \to \infty} v_p = \lim_{p \to \infty} f_p = t - Tt .$$

This proves the validity of the second relation in (6). We then have $v = 0$ if and only if $t = Tt$ and this is equivalent to $t = 0$. Thus, according to (15),

$$x = t + \lim_{p \to \infty} w_p = t + w$$

and (9) follows.

Let us suppose that there is an $x' \in K'$ for which $\langle v, x' \rangle > 0$. Then we would have

$$w + v^+ = w + (t - Tt)^+ \leqq w + t = x$$

a lower bound for x and so the process could be continued. Thus (8) is proved.

It remains to prove (12). Let us assume that (10) and (11) hold simultaneously. For some $s \geqq 0$ we have $v_s \in K$. Then also $g_s \in K$ and consequently, $u_s = g_s$. Finally

$$v_{s+1} = Tv_s + \frac{\langle v_s, z_s' \rangle}{1 - \langle h_s, z_s' \rangle} \{-h_s + Th_s\}$$

and, according to (11) $v_{s+1} \in K$. Comparing that with (8) we see that $v \in K$ and thus $v = 0$. By (9) we obtain (12) and this concludes the proof.

Remark. If T in (1) is unbounded (say a differential operator), but possesses a bounded inverse T^{-1} leaving invariant a cone K, a standard procedure ([2, sections 5 and 6]) allows us to apply Marsal's method directly to (1) without inverting T to obtain an equation with a bounded operator.

References

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Souhrn

O MARSALOVĚ METODĚ PRO ROVNICE S KLADNÝMI OPERÁTORY

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V článku se vyšetřuje rovnice (*) $x = Tx + f$ v komplexním Banachově prostoru X, v němž uspořádání je dáno normálním reprodukujícím kuželem K. Za předpokladu, že (*) má právě jedno řešení x^* v K je ukázáno, že jistá posloupnost $\{w_p\}$ (daná iteracemi – což je analogie Marsalovy metody) konverguje k řešení x^*. Práce je rozšířením Marsalových výsledků.

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