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ON GENERAL FEEDBACK SYSTEMS CONTAINING DELAYERS

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The paper deals with general nonlinear feedback systems which contain delaying elements. Theorems concerning the existence of the over-all transfer operator and the boundedness and stability of the response are given.

0. Consider a system $\mathfrak{A}$ which has two inputs u and Ψ, and two outputs y and Φ. We shall assume that to each pair of signals (u, Ψ) applied to the inputs of $\mathfrak{A}$ there corresponds a uniquely determined pair of responses (y, Φ) appearing at the outputs of $\mathfrak{A}$. The signals and responses are usually functions or vector-valued functions of time. Furthermore, let $\mathfrak{X}$ be a system having an input $\tilde{\Phi}$ and an output $\tilde{\Psi}$; we shall again assume that the response $\tilde{\Psi}$ appearing at the output of $\mathfrak{X}$ is uniquely determined by the signal $\tilde{\Phi}$ acting on the input. If we form an interconnection of $\mathfrak{A}$ and $\mathfrak{X}$ described by the block-diagram in Fig. 1, i.e. if we impose the constraints $\Phi = \tilde{\Phi}$ and $\Psi = \tilde{\Psi}$, we obtain a new system which is usually referred to as a feedback system. Its part consisting of the system $\mathfrak{X}$ is then called the feedback-loop. (Cf. [1], p. 614.) Observe that the new system has in fact a single input u and a single output y, because the introduced feedback-loop imposes a constraint on Φ and Ψ.

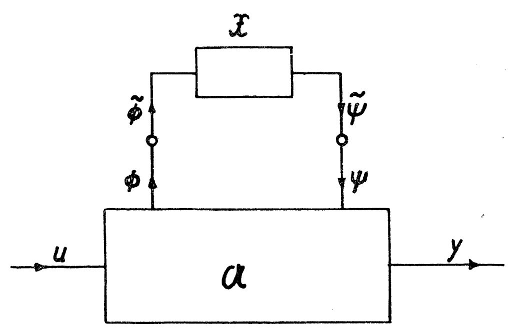

Fig. 1.

Having built up a feedback system, we face the following problems:

1. whether the new system is meaningful at all, i.e. if for every signal u there exist uniquely determined quantities y, Φ and Ψ which satisfy the equations governing the subsystems $\mathfrak{A}$ and $\mathfrak{X}$;

2. whether the response y is bounded in a certain sense for any bounded signal u, i.e. the system does not blow up for some signal;

3. whether the response y is insensitive to small changes of the corresponding signal u, i.e. a small change of u gives raise to a small change of y.

If the requirement 1 is satisfied we will say that the system has an over-all transfer operator. (This terminology is borrowed from the network-theory.) If 2 is met, we will say that we have an input-output boundedness. Finally, if requirement 3 is satisfied, the feedback system will be called input-output stable.

In the sequel we are going to discuss such feedback systems, where either $\mathfrak{X}$ or $\mathfrak{A}$ delays signals (in a certain generalized sense). Engineering applications of such linear systems may be found in [2].

1. Let us now turn to the exact treatment.

Let $\mathfrak{E}$ be a linear set and let F be a non-empty system of mappings from the interval $[0, \infty)$ into $\mathfrak{E}$. For the sake of simplicity, the elements of F will be called functions.

Observe that taking particularly $\mathfrak{E} = E$ (set of all real numbers), we obtain a set of ordinary functions defined on $[0, \infty)$ for F; similarly, putting $\mathfrak{E} = E^n$, we obtain a set of n-vector-valued functions defined on $[0, \infty)$.

Referring to the introduction, F will play the role of the set of all possible signals and responses.

Remark 1. In practice the signals and responses are usually vector-valued functions, but it may happen that u, y, Φ and Ψ have distinct dimensions. However, augmenting the considered feedback system by fictitious feedback-loops with zero transmission or fictitious inputs and outputs, we can always adopt the fact that all entities are from a single space F.

For the time being, no other assumptions on the system F but the following axiom AI* are needed.

AI*. *If v is a mapping from $[0, \infty)$ into $\mathfrak{E}$ such that, for every $\mu > 0$, there exists a function $v_\mu \in F$ with $v(t) = v_\mu(t)$ on $[0, \mu)$, then $v \in F$.*

Now, we are ready for stating the following proposition.

Theorem 1. *Let A and B be operators mapping $F \times F$ into F, and let X be an operator mapping F into itself. Furthermore, let the operators A and X satisfy the following conditions:*

1. *For any $u \in F$ and $v_1, v_2 \in F$ such that $v_1(t) = v_2(t)$ on $[0, \mu)$, (μ arbitrary), we have $\{A(u, v_1)\}(t) = \{A(u, v_2)\}(t)$ on $[0, \mu)$.*
2. *There exist a fixed number $T > 0$ and a fixed element $a \in F$ such that,*
   a) *if $x_1(t) = x_2(t)$ on $[0, \mu)$ for $x_1, x_2 \in F$, then $(Xx_1)(t) = (Xx_2)(t)$ on $[0, \mu + T)$,*
   b) *for any $x \in F$ we have $(Xx)(t) = a(t)$ on $[0, T)$.*

Then there exists a unique operator W mapping F into itself such that, for any $u \in F$, uniquely determined elements Φ, $\Psi \in F$ exist which satisfy the equations

$$\Phi = A(u, \Psi)\,, \tag{1}$$

$$\Psi = X\Phi\,, \tag{2}$$

$$y = B(u, \Psi) \tag{3}$$

with $y = Wu$.

Before turning to the proof, let us make a few comments on the above facts. Referring to the discussion carried out in the introduction we see that external behaviour of the system $\mathfrak{A}$ may be described by the operators A and B; obviously, A relates the response Φ to the pair of signals (u, Ψ), and B the response y to the same pair (u, Ψ). Analogously, the operator X relates the input and output quantities on the system $\mathfrak{X}$. Then, the equations (1), (2) and (3) govern the feedback interconnection plotted in Fig. 1. Thus, if the assumptions of Theorem 1 are met, we have exactly the situation described in problem 1; in particular, any signal u determines uniquely the response y of the entire feedback system and we have $y = Wu$. Consequently, W is called the over-all transfer operator.

Let us still explain the meaning of conditions 1 and 2 stated in Theorem 1. Condition 1 obviously states that the operator A is unanticipative in variable v; to be more specific, the values of the response Φ in an interval $[0, \mu)$ are determined only by u and values of Ψ in $[0, \mu)$, i.e. at any instant t_0 the response Φ is independent of the future $t > t_0$ of Ψ.

As for condition 2, it is a generalization of the delaying property of the system $\mathfrak{X}$; in other words, X is a generalized shifting operator. It can be easily verified that an example of such operator is furnished by the operator P_T defined by $(P_T x)(t) = x(t - T)$ for $t \geqq T$ and $(P_T x)(t) = \Theta$ (zero element of F) for $0 \leqq t < T$, $x \in F$.

Proof of Theorem 1. Choose a fixed element $u \in F$ and for any integer $n \geqq 1$, define the element Φ_n by

$$\Phi_n = A(u, X\Phi_{n-1})\,, \quad \Phi_0 = a\,. \tag{4}$$

The definition is clearly meaningful, since $\Phi_n \in F$ for any n. We are going to show that

$$\Phi_{n+1}(t) = \Phi_n(t) \quad \text{on} \quad [0, nT)\,. \tag{5}$$

Actually, (5) is evidently true for $n = 1$. We have $\Phi_1 = A(u, Xa)$ and $\Phi_2 = A(u, X\Phi_1)$; however, $(Xa)(t) = a(t) = (X\Phi_1)(t)$ on $[0\ T)$ by 2b), and consequently, by 1, $\Phi_1(t) = \Phi_2(t)$ on $[0, T)$. Next, suppose that (5) is true for some n. Then, by 2a), $(X\Phi_{n+1})(t) = (X\Phi_n)(t)$ on $[0, (n+1)T)$; consequently, by 1, $\{A(u, X\Phi_{n+1})\}(t) = \{A(u, X\Phi_n)\}(t)$ on $[0, (n+1)T)$, i.e. $\Phi_{n+2}(t) = \Phi_{n+1}(t)$ on the same interval and (5) is proved.

Due to (5), define a mapping Φ from $[0, \infty)$ into $\mathfrak{E}$ by $\Phi(t) = \Phi_n(t)$, where $nT > t$. Since for any $\mu > 0$ the mapping Φ coincides with Φ_m with $m > \mu/T$ on $[0\ \mu)$, we have $\Phi \in F$ by the axiom AI*; hence the value $X\Phi$ is defined and belongs to F.

Let us now show that Φ satisfies the equation

$$\Phi = A(u, X\Phi) . \tag{6}$$

Actually choose a $t^* \geqq 0$. Then we can find an integer $n \geqq 1$ such that $t^* \in [0, nT)$, and $\Phi_{n+1}(t) = \Phi_n(t) = \Phi(t)$ on $[0, nT)$; hence, $\Phi(t^*) = \Phi_{n+1}(t^*)$. On the other hand, by (4) we have $\Phi_{n+1} = A(u, X\Phi_n)$; however, $(X\Phi_n)(t) = (X\Phi)(t)$ on $[0, (n+1)T)$ by 2a), so that, by 1, $\{A(u, X\Phi_n)\}(t) = \{A(u, X\Phi)\}(t)$ on $[0, (n+1)T)$. Consequently, $\Phi_{n+1}(t) = \{A(u, X\Phi)\}(t)$ on $[0, (n+1)T)$, and particularly, $\Phi_{n+1}(t^*) = \{A(u, X\Phi)\}(t^*)$; hence $\Phi(t^*) = \{A(u, X\Phi)\}(t^*)$, i.e. (6) holds.

As a next step show that the function Φ is the unique solution of (6). Thus, suppose that a $\tilde{\Phi} \in F$ exists such that

$$\tilde{\Phi} = A(u, X\tilde{\Phi}) . \tag{7}$$

Then, due to 2b), both $X\Phi$ and $X\tilde{\Phi}$ are equal to $a \in F$ on $[0, T)$, and consequently, by 1, $\Phi(t) = \tilde{\Phi}(t)$ on $[0, T)$. Assume that, for some $n \geqq 1$, $\Phi(t) = \tilde{\Phi}(t)$ on $[0, nT)$; then $(X\Phi)(t) = (X\tilde{\Phi})(t)$ on $[0, (n+1)T)$ by 2a), so that, by 1, $\Phi(t) = \tilde{\Phi}(t)$ on $[0, (n+1)T)$. Hence $\Phi(t) = \tilde{\Phi}(t)$ on $[0, kT)$ with any $k \geqq 1$ i.e. $\Phi = \tilde{\Phi}$.

Thus, equation (6) defines an operator Q mapping F into itself, i.e., for any $u \in F$ there exists a unique $\Phi \in F$ denoted by Qu which satisfies (6). However, since the equations (1) and (2) are equivalent to (6), it suffices to set $W = B(., XQ.)$, i.e., $Wu = B(u, XQu)$ for any $u \in F$ and the proof is completed.

Theorem 2. *Let the conditions of Theorem 1 be satisfied, and, in addition, let the operators A and B satisfy the assumptions*:

3. *For any $v \in F$ and any $u_1, u_2 \in F$ such that $u_1(t) = u_2(t)$ on $[0, \mu)$, we have $\{A(u_1, v)\}(t) = \{A(u_2, v)\}(t)$ on $[0, \mu)$.*
4. *For any $u_1, u_2, v_1, v_2 \in F$ such that $u_1(t) = u_2(t)$ and $v_1(t) = v_2(t)$ on $[0, \mu)$ we have $\{B(u_1, v_1)\}(t) = \{B(u_2, v_2)\}(t)$ on $[0, \mu)$.*

Then the operator W is unanticipative, i.e., $u_1, u_2 \in F$, $u_1(t) = u_2(t)$ on $[0, \mu)$ implies that $(Wu_1)(t) = (Wu_2)(t)$ on $[0, \mu)$.

The interpretation of Theorem 2 is straightforward; if the system $\mathfrak{A}$ is unanticipative, i.e. both A and B are unanticipative in both variables, then the over-all transfer operator of the feedback system is unanticipative, too.

Proof of Theorem 2. First observe that conditions 1 and 3 imply the validity of condition 4 for the operator A. Thus, let $u_1, u_2 \in F$ be such that $u_1(t) = u_2(t)$ on $[0, \mu)$; denote Φ_1 and Φ_2 the corresponding solutions of (6), i.e., we have $\Phi_1 = A(u_1, X\Phi_1)$ and $\Phi_2 = A(u_2, X\Phi_2)$. Let m be the largest integer such that $mT \leqq \mu$. Without loss of generality we may assume that $m > 0$.

Since $(X\Phi_1)(t) = (X\Phi_2)(t) = a(t)$ on $[0, T)$, it follows by the above observation that $\Phi_1(t) = \Phi_2(t)$ on $[0, T)$. Suppose that $\Phi_1(t) = \Phi_2(t)$ on $[0, kT)$ with $1 \leqq k < m$; then $(X\Phi_1)(t) = (X\Phi_2)(t)$ on $[0, (k+1)T)$ by 2a), and consequently, $\{A(u_1, X\Phi_1)\}(t) = \{A(u_2, X\Phi_2)\}(t)$ on $[0, (k+1)T)$, i.e., $\Phi_1(t) = \Phi_2(t)$ on the same interval. Hence, $\Phi_1(t) = \Phi_2(t)$ on $[0, mT)$. Then, $(X\Phi_1)(t) = (X\Phi_2)(t)$ on $[0, (m+1)T)$, i.e. also on $[0, \mu)$, so that, $\Phi_1(t) = \{A(u_1, X\Phi_1)\}(t) = \{A(u_2, X\Phi_2)\}(t) = \Phi_2(t)$ on $[0, \mu)$.

Thus, due to (2) and 2a), $\Psi_1(t) = \Psi_2(t)$ on $[0, \mu)$; finally, by (3) and condition 4, $y_1(t) = y_2(t)$ on $[0, \mu)$. Hence, the proof.

Using the pattern of proof of Theorem 1, we can easily prove the following proposition.

Theorem 3. *Let A and B be operators mapping $F \times F$ into F and let X be an operator mapping F into itself. Furthermore, let the following conditions be satisfied*:

1*. *There exists a fixed number $T > 0$ such that,*
  a) *for any $u, v_1, v_2 \in F$ we have $\{A(u, v_1)\}(t) = \{A(u, v_2)\}(t)$ on $[0, T)$,*
  b) *for any $u, v_1, v_2 \in F$ such that $v_1(t) = v_2(t)$ on $[0, \mu)$ we have $\{A(u, v_1)\}(t) = \{A(u, v_2)\}(t)$ on $[0, \mu + T)$.*

2*. *If $v_1, v_2 \in F$ and $v_1(t) = v_2(t)$ on $[0, \mu)$, then $(Xv_1)(t) = (Xv_2)(t)$ on $[0, \mu)$.*

Then a unique operator W mapping F into itself exists such that, for any $u \in F$, uniquely determined elements $\Phi, \Psi \in F$ exist which satisfy the equations (1), (2) *and* (3) *with $y = Wu$.*

Moreover, if in addition the operator A is unanticipative in variable u and B is unanticipative in both variables u and v (i.e. if conditions 3 *and* 4 *hold), then W is also unanticipative.*

(The proof is left to the reader.)

The theorem just stated clearly concerns such feedback systems whose part $\mathfrak{A}$ delays signals Ψ, and part $\mathfrak{X}$ is unanticipative.

Let us now present two actual feedback systems as examples.

Example 1. Consider the classical feedback configuration plotted in Fig. 2. Assume that F, in addition to axiom AI*, is a linear set, that the operator N mapping F into itself is unanticipative and X satisfies the condition 2. From Fig. 2 it is apparent that we have here

$$\Phi = N(u + \Psi)\,; \quad y = \Phi = N(u + \Psi)\,; \quad \Psi = X\Phi\,.$$

Hence, in the language of Theorem 1,

$$A(u, v) = B(u, v) = N(u + v) \quad \text{for any} \quad (u, v) \in F \times F\,.$$

Due to the unanticipativity of N, the operator A satisfies the conditions 1 and 3, and B the condition 4. Consequently, the considered system has an unanticipative over-all transfer operator.

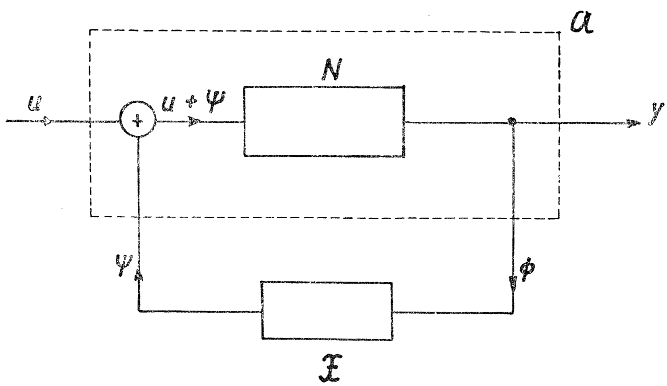

Fig. 2.

Example 2. Let F be the set of all n-vector-valued functions defined on $[0, \infty)$ which are locally integrable. It is obvious that F satisfies the axiom AI*. Let $P(\xi_1, \xi_2, \ldots, \xi_k)$, $R(\eta_1, \eta_2, \ldots, \eta_l)$ and $T(\zeta_1, \zeta_2, \ldots, \zeta_m)$ be n-vector-valued functions of n-vector arguments ξ_i, η_j and ζ_p such that, for any choice of $\xi_i(t)$, $\eta_j(t)$, $\zeta_p(t) \in F$, $i = 1, 2, \ldots, k$, $j = 1, 2, \ldots, l$, $p = 1, 2, \ldots, m$, we have

$$P(\xi_1(t), \ldots, \xi_k(t)), \quad R(\eta_1(t), \ldots, \eta_l(t)), \quad T(\zeta_1(t), \ldots, \zeta_m(t)) \in F .$$

Furthermore, let $H_i(t, u, v)$, $i = 1, 2, \ldots, k$, $M_j(t, u)$, $j = 1, 2, \ldots, l$ and $S_p(t, u, v)$, $p = 1, 2, \ldots, m$ be defined for $t \geqq 0$ and $u, v \in E^n$ and be such that

$$H_i(t, u(t), v(t)), \quad M_j(t, u(t)), \quad S_p(t, u(t), v(t)) \in F$$

for any $u(t), v(t) \in F$. Finally, let $K_i(t, \tau, u, v)$, $i = 1, 2, \ldots, k$, $N_j(t, \tau, u)$, $j = 1, 2, \ldots$ $\ldots, l$ and $Q_p(t, \tau, u, v)$, $p = 1, 2, \ldots, m$ be defined for $0 \leqq \tau \leqq t < \infty$ and $u, v \in E^n$ and be such that

$$\int_0^t K_i(t, \tau, u(\tau), v(\tau))\, d\tau, \quad \int_0^t N_j(t, \tau, u(\tau))\, d\tau, \quad \int_0^t Q_p(t, \tau, u(\tau), v(\tau))\, d\tau \in F$$

for any choice of $u(t), v(t) \in F$.

Let the feedback system be described by operators A, B and X, which are defined by

$$\{A(u, v)\}(t) = P(\xi_1(t), \xi_2(t), \ldots, \xi_k(t)),$$

where

$$\xi_1(t) = H_i(t, u(t), v(t)) + \int_0^t K_i(t, \tau, u(\tau), v(\tau))\, d\tau\,, \quad t \geqq 0\,, \quad u, v \in F\,,$$

$$\{B(u, v)\}\,(t) = T(\zeta_1(t), \zeta_2(t), \ldots, \zeta_m(t))\,,$$

where

$$\zeta_p(t) = S_p(t, u(t), v(t)) + \int_0^t Q_p(t, \tau, u(\tau), v(\tau))\, d\tau\,, \quad t \geqq 0\,, \quad u, v \in F\,,$$

and

$$\{Xv\}\,(t) = R(\eta_1(t), \eta_2(t), \ldots, \eta_l(t))\,,$$

where

$$\eta_j(t) = M_j(t, (P_{\tilde{T}_j}v)\,(t)) + \int_0^t N_j(t, \tau, (P_{\tilde{T}_j}v)\,(\tau))\, d\tau\,,$$

$t \geqq 0$, $v \in F$ and $\tilde{T}_j$, $j = 1, 2, \ldots, l$ are fixed positive numbers. (As above, $P_{\tilde{T}}$ denotes the shifting operator.)

It is obvious that, due to assumptions made above, the operators A and B actually map $F \times F$ into F and X maps F into itself. Moreover, it can be easily verified that operators A, B and X satisfy the conditions 1 through 4 given in Theorems 1 and 2. (Note that 2 is satisfied with $T = \min_{1,\ldots,l} \tilde{T}_j$ and $a = R(\tilde{\eta}_1(t), \tilde{\eta}_2(t), \ldots, \tilde{\eta}_l(t))$, where $\tilde{\eta}_j(t) = M_j(t, 0) + \int_0^t N_j(t, \tau, 0)\, d\tau$.) Hence, the considered system has an unanticipative over-all transfer operator.

Let us now turn to the discussion of the input-output boundedness and stability. Since here it is necessary to measure somehow the size of a signal or response, we have to introduce a metric or a norm into a subset of F. Thus, let us state the following requirement.

A2*. *Let F satisfy the axiom* AI* *and, in addition to it, let F contain a subset F^* which is a linear normed space possessing the property*:

If, for a function $v \in F$, a number $\Lambda > 0$ exists such that for any $\mu > 0$ there exists a $v_\mu \in F^*$ with $v_\mu(t) = v(t)$ on $[0, \mu)$ and $\|v_\mu\| \leqq \Lambda$, then $v \in F^*$ and $\|v\| \leqq \Lambda$.

Observe that the commonly used linear normed spaces satisfy the axiom A2*. Actually, let F be the set of all measurable n-vector functions defined on $[0, \infty)$.

a) Let $\mathscr{B} \subset F$ consist of all n-vector functions x such that the norm $|x(t)|$ is bounded in $[0, \infty)$, and let $\|x\| = \sup_{t \in [0,\infty)} |x(t)|$; (here, $|c|$ signifies a norm of a constant vector c); then, clearly, $\mathscr{B}$ is a linear normed space. Next, let $v \in F$ and let a number $\Lambda > 0$ exist such that, for any $\mu > 0$, we can find a $v_\mu \in \mathscr{B}$ with $v_\mu(t) = v(t)$ on $[0, \mu)$ and $\|v_\mu\| \leqq \Lambda$. Then

$$\sup_{[0,\mu)} |v(t)| = \sup_{[0,\mu)} |v_\mu(t)| \leqq \sup_{[0,\infty)} |v_\mu(t)| = \|v_\mu\| \leqq \Lambda\,;$$

thus, $\sup_{[0,\infty)} |v(t)| \leqq \Lambda$, i.e. $v \in \mathscr{B}$ and $\|v\| \leqq \Lambda$. Hence, $\mathscr{B}$ satisfies the axiom A2*.

The situation is the same in the case of the space $C \subset \mathscr{B}$ consisting of all continuous vector functions with the above norm.

b) Let $p \geqq 1$ and let $L_p \subset F$ consist of all vector functions x such that $\int_0^\infty |x(t)|^p \, dt < < \infty$; putting $\|x\| = (\int_0^\infty |x(t)|^p \, dt)^{1/p}$ for $x \in L_p$, then, as known, L_p is a linear normed space with the usual operations of addition and multiplication by a constant. Thus, let again $v \in F$ be such that a fixed $\Lambda > 0$ exists and for any $\mu > 0$ we can find a $v_\mu \in L_p$ with $v_\mu(t) = v(t)$ on $[0, \mu)$ and $\|v_\mu\| \leqq \Lambda$. Then we have

$$\int_0^\mu |v(t)|^p \, dt = \int_0^\mu |v_\mu(t)|^p \, dt \leqq \int_0^\infty |v_\mu(t)|^p \, dt = \|v_\mu\|^p \leqq \Lambda^p ;$$

consequently, $\int_0^\infty |v(t)|^p \, dt \leqq \Lambda^p$, i.e. $v \in L_p$ and $\|v\| \leqq \Lambda$. Hence, the axiom A2* is satisfied.

A similar reasoning will persuade us that the space L_∞, consisting of all essentially bounded vector functions x with the norm $\|x\| = \operatorname{ess\,sup}_{[0,\infty)} |x(t)|$, also satisfies A2*.

Now, we can state a proposition on boundedness.

Theorem 4. *Let the operators A, B and X satisfy the conditions of Theorem 1 and be such that A and B map $F^* \times F^*$ into F^* and X maps F^* into itself. Furthermore, let constants C_1, C_2 exist such that*

$$\|A(u, v_1) - A(u, v_2)\| \leqq C_1 \|v_1 - v_2\| \tag{8}$$

for any u, v_1, $v_2 \in F^*$, and

$$\|Xv_1 - Xv_2\| \leqq C_2 \|v_1 - v_2\| \tag{9}$$

for any v_1, $v_2 \in F^*$.

For every integer $n \geqq 1$ denote

$$\lambda_n = \sup_{(u, v_1, v_2) \in S_n} \frac{\|A(u, v_1) - A(u, v_2)\|}{\|v_1 - v_2\|}, \tag{10}$$

where

$$S_n = \{(u, v_1, v_2) : u, v_1, v_2 \in F^*, \ v_1 \neq v_2, \ v_1(t) = v_2(t) \ \text{on} \ [0, nT)\}, \tag{11}$$

and

$$\mu_n = \sup_{(v_1, v_2) \in \tilde{S}_n} \frac{\|Xv_1 - Xv_2\|}{\|v_1 - v_2\|}, \tag{12}$$

where

$$\tilde{S}_n = \{(v_1, v_2) : v_1, v_2 \in F^*, \ v_1 \neq v_2, \ v_1(t) = v_2(t) \ \text{on} \ [0, nT)\}. \tag{13}$$

Then, if $\lim_{n\to\infty} \lambda_n \,.\, \lim_{n\to\infty} \mu_n < 1$, the operator W maps F^* into itself.

Moreover, if a nonnegative function $\beta(\xi, \eta)$ nondecreasing in η exists such that

$$\|\mathscr{B}(u, v)\| \leqq (\|u\|, \|v\|) \tag{14}$$

for all $u, v \in F^*$, then, for any $u \in F^*$,

$$\|Wu\| \leqq \beta(\|u\|, M\|A(u, X\Theta)\| + \|X\Theta\|), \tag{15}$$

where $M > 0$ depends only on A and X, and Θ is the zero element of F^*.

Proof. First, from (11) and (13) it follows that $S_n \supset S_{n+1}$ and $\tilde{S}_n \supset \tilde{S}_{n+1}$ for any integer $n \geqq 1$; consequently, $\lambda_n \geqq \lambda_{n+1}$ and $\mu_n \geqq \mu_{n+1}$. Moreover, due to (8) and (9) we have $C_1 \geqq \lambda_1$ and $C_2 \geqq \mu_1$ so that the proper limits $\lim_{n\to\infty} \lambda_n$ and $\lim_{n\to\infty} \mu_n$ actually exist.

Next, choose a $\mu \in F^*$; referring to the proof of Theorem 1, let us construct the functions Φ_n by $\Phi_n = A(u, X\Phi_{n-1})$, $n = 1, 2, 3, \ldots$, $\Phi_0 = \Theta$ (zero element of F^*). Observe that here we take advantageously Θ for Φ_0; it is clear that all properties of functions Φ_n discussed in the proof of Theorem 1 are preserved and, in particular, the sequence Φ_n defines the solution Φ of the equation $\Phi = A(u, X\Phi)$ in the same way as before.

In view of the assumptions made above we have $\Phi_n \in F^*$ for every $n \geqq 0$. Next, put

$$v_n = \Phi_n - \Phi_{n-1}, \quad n = 1, 2, 3, \ldots \tag{16}$$

By (5) we have $v_n(t) = \Theta$ on $[0, (n-1)T)$; moreover, $v_n \in F^*$ and

$$\Phi_n = \sum_{i=1}^{n} v_i, \quad n = 1, 2, 3, \ldots \tag{17}$$

On the other hand, we can easily verify that

$$\|v_n\| \leqq C_2 \lambda_1 \lambda_2 \ldots \lambda_{n-1} \mu_1 \mu_2 \ldots \mu_{n-2} \|\Phi_1\|, \quad n = 2, 3, \ldots \tag{18}$$

with $\Phi_1 = A(u, X\Theta)$ and $\mu_0 = 1$. Actually, (18) is true for $n = 2$ because $\|v_2\| = \|\Phi_2 - \Phi_1\| = \|A(u, X\Phi_1) - A(u, X\Theta)\|$; however, since $X\Phi_1 = X\Theta = a$ on $[0, T)$ by 2b), we have by (10),

$$\|v_2\| \leqq \lambda_1 \|X\Phi_1 - X\Theta\| \leqq \lambda_1 C_2 \|\Phi_1 - \Theta\| = \lambda_1 C_2 \|\Phi_1\|.$$

Next, assume that (18) is true for some $n \geqq 2$; then we have

$$\|v_{n+1}\| = \|\Phi_{n+1} - \Phi_n\| = \|A(u, X\Phi_n) - A(u, X\Phi_{n-1})\|.$$

Since $\Phi_n(t) = \Phi_{n-1}(t)$ on $[0, (n-1)T)$, it follows by 2a) that $(X\Phi_n)(t) = (X\Phi_{n-1})(t)$ on $[0, nT)$, and consequently, by (10) and (12),

$$\|v_{n+1}\| \leqq \lambda_n \|X\Phi_n - X\Phi_{n-1}\| \leqq \lambda_n \mu_{n-1} \|\Phi_n - \Phi_{n-1}\| \leqq$$
$$\leqq C_2 \lambda_1 \lambda_2 \ldots \lambda_n \mu_1 \mu_2 \ldots \mu_{n-1} \|\Phi_1\|.$$

Hence, the estimate (18) holds for any $n \geqq 2$.

Now, by (17), we have for any $n \geqq 1$,

$$\|\Phi_n\| \leqq \sum_{i=1}^{n} \|v_i\| \leqq (1 + C_2 \sum_{k=2}^{n-1} \prod_{j=1}^{k-2} \lambda_j \prod_{p=1}^{k-1} \mu_p) \|\Phi_1\| . \tag{19}$$

However, due to the assumption $\lim_{n\to\infty} \lambda_n \,.\, \lim_{n\to\infty} \mu_n < 1$,

$$M' = 1 + C_2 \sum_{k=2}^{\infty} \prod_{j=1}^{k-1} \lambda_j \prod_{p=1}^{k-2} \mu_p < \infty . \tag{20}$$

Consequently, for any $n \geqq 1$,

$$\|\Phi_n\| \leqq M' \|\Phi_1\| . \tag{21}$$

On the other hand, for any $\varkappa > 0$ we have $\Phi(t) = \Phi_n(t)$ on $[0, \varkappa)$ whenever $nT > \varkappa$; hence, by the axiom A2*, $\Phi \in F^*$ and $\|\Phi\| \leqq M' \|\Phi_1\|$. Thus, summarizing our results, for any $u \in F^*$ we have $\Phi \in F^*$, and consequently, $Wu = B(u, X\Phi) \in F^*$ by the above assumptions. Hence, the first assertion of Theorem 4 is proved.

As for the second assertion, it suffices to realize that $\|X\Phi\| \leqq \|X\Phi - X\Theta\| + \|X\Theta\| \leqq C_2\|\Phi\| + \|X\Theta\|$ and introduce this inequality into (14).

If in Theorem 4 the assumption "*A*, *B* and *X* satisfy the conditions of Theorem 1" is replaced by "*A*, *B* and *X* satisfy the conditions of Theorem 3", the assertion remains true without any change; the proof is the same except for some slight modifications and is left to the reader.

Remark 2. The proof just carried out suggests that the axiom A2* may be replaced by the following requirement:

A3*. *The set F contains a complete linear normed space F^*.*

Actually, define the elements Φ_n as before and v_n by (16); then in view of (19) and (20), the series $\sum_{i=1}^{\infty} v_i$ converges in the space F^*, because $\sum_{i=1}^{\infty} \|v_i\| < \infty$. (Cf. [3].) Hence, there exists an element $\Phi \in F^*$ such that $\|\Phi - \Phi_n\| \to 0$ as $n \to \infty$. However, we can easily show that Φ is a solution of (6); indeed, by (8) and (9) we have $\|A(u, X\Phi) - A(u, X\Phi_n)\| \leqq C_1 C_2 \|\Phi - \Phi_n\| \to 0$ as $n \to \infty$. Thus, $A(u, X\Phi_n) \to A(u, X\Phi)$ in F^*; the relation (4) concludes then the proof.

Note also that the particular spaces $\mathscr{B}$, L_p and L_∞ mentioned above are complete.

Let us now state a proposition on the input-output stability; to this purpose, we will require F to satisfy the axiom A3*.

Theorem 5. *Let the conditions of Theorem 4 be satisfied with the exception of* (8), *which is replaced by: Constants C, C_1 exist such that*

$$\|A(u_1, v_1) - A(u_2, v_2)\| \leqq C\|u_1 - u_2\| + C_1\|v_1 - v_2\| \tag{22}$$

for any $u_1, u_2, v_1, v_2 \in F^*$. Furthermore, let the operator B be continuous at every point $(u, v) \in F^* \times F^*$; then the operator W is continuous at every point $u \in F^*$.

If, in addition, there exist an $H > 0$ and $b \in F^*$ such that

$$\|A(u, b)\| \leqq H \tag{23}$$

for every $u \in F^*$ and B is uniformly continuous, then W is uniformly continuous, too.

Proof. We will use the notation and results of the preceding proofs. Let $u \in F^*$ and let Φ be the corresponding solution of the equation $\Phi = A(u, X\Phi)$. Then, due to Remark 2, $\|\Phi - \Phi_n\| \to 0$ as $n \to \infty$, i.e. $\Phi_n \to \Phi$ in F^*. Thus, by (17), $\Phi = \sum_{i=1}^{\infty} v_i$ and

$$\Phi - \Phi_n = \sum_{i=n+1}^{\infty} v_i ; \tag{24}$$

consequently, by (18)

$$\|\Phi - \Phi_n\| \leqq \left(\sum_{i=n+1}^{\infty} q_i\right) \|\Phi_1\| , \tag{25}$$

where $q_i = C_2 \prod_{j=1}^{i-1} \lambda_j \prod_{k=1}^{i-2} \mu_k$. Observe that the q_i's are independent of u.

Analogously, if $\tilde{u} \in F^*$ and $\tilde{\Phi}$ is defined by $\tilde{\Phi} = A(\tilde{u}, X\tilde{\Phi})$, we have

$$\|\tilde{\Phi} - \tilde{\Phi}_n\| \leqq \left(\sum_{i=n+1}^{\infty} q_i\right) \|\tilde{\Phi}_1\| . \tag{26}$$

However, $\Phi_1 = A(u, X\Theta)$ and $\tilde{\Phi}_1 = A(\tilde{u}, X\Theta)$ so that, by (22),

$$\|\tilde{\Phi}_1\| \leqq \|A(\tilde{u}, X\Theta) - A(u, X\Theta)\| + \|A(u, X\Theta)\| \leqq C\|\tilde{u} - u\| + \|A(u, X\Theta)\| .$$

Hence, by (25) and (26),

$$\|\Phi - \Phi_n\| + \|\tilde{\Phi} - \tilde{\Phi}_n\| \leqq \left(\sum_{i=n+1}^{\infty} q_i\right) \{2\|A(u, X\Theta)\| + C\|u - \tilde{u}\|\} . \tag{27}$$

Observe that if (23) is satisfied, then

$$\|A(u, X\Theta)\| \leqq \|A(u, X\Theta) - A(u, b)\| + \|A(u, b)\| \leqq C_1\|X\Theta - b\| + H ,$$

and consequently,

$$\|\Phi - \Phi_n\| + \|\tilde{\Phi} - \tilde{\Phi}_n\| \leqq \left(\sum_{i=n+1}^{\infty} q_i\right) \{2C_1\|X\Theta - b\| + 2H + C\|u - \tilde{u}\|\} . \tag{28}$$

On the other hand, we can write

$$\|\Phi - \tilde{\Phi}\| \leqq \|\Phi_n - \tilde{\Phi}_n\| + \|\Phi - \Phi_n\| + \|\tilde{\Phi} - \tilde{\Phi}_n\| ; \tag{29}$$

however, using repeatedly (22), (9) and the equations $\Phi_k = A(u, X\Phi_{k-1})$ and $\tilde{\Phi}_k = A(\tilde{u}, X\tilde{\Phi}_{k-1})$, we obtain

$$(30) \qquad \|\Phi_n - \tilde{\Phi}_n\| = \|A(u, X\Phi_{n-1}) - A(\tilde{u}, X\tilde{\Phi}_{n-1})\| \leqq$$
$$\leqq C\|u - \tilde{u}\| + C_1\|X\Phi_{n-1} - X\tilde{\Phi}_{n-1}\| \leqq C\|u - \tilde{u}\| + C_1C_2\|\Phi_{n-1} - \tilde{\Phi}_{n-1}\| \leqq$$
$$\leqq \ldots \leqq C\|u - \tilde{u}\| \{1 + C_1C_2 + C_1^2C_2^2 + \ldots + C_1^{n-2}C_2^{n-2}\} + C_1^{n-1}C_2^{n-1}\|\Phi_1 - \tilde{\Phi}_1\|.$$

Since $\|\Phi_1 - \tilde{\Phi}_1\| = \|A(u, X\Theta) - A(\tilde{u}, X\Theta)\| \leqq C\|u - \tilde{u}\|$, we have finally,

$$(31) \qquad \|\Phi_n - \tilde{\Phi}_n\| \leqq C\{1 + \sum_{i=1}^{n-1} C_1^i C_2^i\} \|u - \tilde{u}\|.$$

Now, let $u \in F^*$ be a fixed element, and let $\varepsilon > 0$. Since the series $\sum_{i=1}^{\infty} q_i$ converges, then, by (27), we can find n so large that, for any $\tilde{u}$ with $\|u - \tilde{u}\| < 1$, we will have $\|\Phi - \Phi_n\| + \|\tilde{\Phi} - \tilde{\Phi}_n\| < \frac{1}{2}\varepsilon$. If, for this n, we let $\|u - \tilde{u}\| < \frac{1}{2}\varepsilon \cdot C^{-1}\{1 + \sum_{i=1}^{n-1} C_1^i C_2^i\}^{-1}$, then we will have $\|\Phi - \tilde{\Phi}\| < \varepsilon$ by (29). Hence, the operator Q assigning to each $u \in F^*$ the solution $\Phi \in F^*$ of $\Phi = A(u, X\Phi)$ is continuous at u. Since X is continuous by (9), then the operator W defined by $Wu = B(u, XQu)$ is continuous, too. Hence, the proof.

Finally, if (23) holds and B is uniformly continuous, then, according to (28), for an $\varepsilon > 0$ we can find $n > 0$ such that $\|\Phi - \Phi_n\| + \|\tilde{\Phi} - \tilde{\Phi}_n\| < \frac{1}{2}\varepsilon$ for any pair, $u, \tilde{u} \in F^*$ with $\|u - \tilde{u}\| < 1$. Then $\|\Phi_n - \tilde{\Phi}_n\|$ can again be made less than $\frac{1}{2}\varepsilon$ for any $u, \tilde{u} \in F^*$ with $\|u - \tilde{u}\| < \delta$, where δ is sufficiently small. Hence, $\|\Phi - \tilde{\Phi}\| < \varepsilon$ whenever $\|u - \tilde{u}\| < \min[1, \delta]$; the rest of the proof is obvious.

Remark 3. Theorem 5 obviously remains true if the operators A, B and X satisfy the assumptions of Theorem 3 instead of those of Theorem 1.

Let us now discuss the interpretation of the above results in the language of feedback systems. The meaning of Theorem 4 is certainly straightforward; if the parts $\mathfrak{A}$ and $\mathfrak{X}$ of a feedback system satisfy the conditions of Theorem 4 and if, for example, $F^* = \mathscr{B}$, then every bounded signal u produces a bounded response y. If even condition (14) is satisfied, a boundary for $\|y\|$ is available. As for the conditions imposed on $\lim_{n\to\infty} \lambda_n$ and $\lim_{n\to\infty} \mu_n$, they express the fact that, crudely speaking, both systems $\mathfrak{A}$ and $\mathfrak{X}$ attenuate the signals sufficiently as $t \to \infty$. (To be more accurate, $\mathfrak{A}$ attenuates the signals in the path input Ψ – output Φ.)

Next, it is obvious that the concept of continuity of the operator W grasps the idea about the input – output stability; actually, if $u \in F^*$ is a signal acting on a feedback system, which satisfies the conditions of Theorem 5, and $y = Wu$ is the corresponding response, then for every $\varepsilon > 0$ a $\delta > 0$ exists such that $\|y - \tilde{y}\| = \|Wu - W\tilde{u}\| < \varepsilon$ whenever $\|u - \tilde{u}\| < \delta$, i.e. a small change of u affects y only slightly. If, in particular, $F^* = \mathscr{B}$, then we have the input – output stability of Liapunov's type.

Observe also that if condition (23) is satisfied, we have a uniform stability, i.e. the responses y are uniformly insensitive to small changes of the corresponding input signals.

Let us make the following observation. Referring to the classical feedback configuration discussed in Example 1, we see that if the operator N describing the system $\mathfrak{A}$ satisfies the condition $\|Nx_1 - Nx_2\| \leqq \alpha\|x_1 - x_2\|$ for any $x_1, x_2 \in F^*$ with a fixed $\alpha > 0$, then condition (22) (and consequently, (8) too), is satisfied. The number λ_n in (10) may then be defined by

$$\lambda_n = \sup_{(v_1, v_2) \in \tilde{S}_n} \frac{\|Nv_1 - Nv_2\|}{\|v_1 - v_2\|}$$

with $\tilde{S}_n$ given by (13). Hence, under the above condition on N we have the input-output boundedness and stability provided the assumptions on X are met.

Due to its major importance in practice, let us also briefly discuss the case of a linear feedback system, i.e. if the behaviour of $\mathfrak{A}$ and $\mathfrak{X}$ is governed by linear operators defined on F.

Thus, for the present purposes, assume that F is a linear set and that, for any u, $v \in F$,

$$A(u, v) = A_1 u + A_2 v\,, \quad B(u, v) = B_1 u + B_2 v\,, \tag{32}$$

where A_1, A_2, B_1, B_2 and X are linear operators on F. If $(A_1 u)(t) = \Theta$ on $[0, \mu)$ whenever $u(t) = \Theta$ on $[0, \mu)$, and if the same is true for A_2, B_1, B_2, then clearly conditions 1, 3 and 4 in Theorems 1 and 2 are satisfied. As for condition 2, it is equivalent to the requirement: if $v(t) = \Theta$ on $[0, \mu)$, then $(Xv)(t) = \Theta$ on $[0, \mu + T)$. (Observe that here we necessarily have $a = \Theta$.)

Thus, under the mentioned requirements, the feedback system has an unanticipative over-all transfer operator W. A little thought will persuade us that W is linear, too. Moreover, we can give an explicit formula for W. Actually, by (32) and (1), (2), (3) we have

$$\Phi = A_1 u + A_2 X \Phi\,, \quad y = Wu = B_1 u + B_2 X \Phi\,. \tag{33}$$

Using a similar method of proof as in proving Theorem 1, we can easily verify that the first equation (33) yields

$$\Phi = \Big(I + \sum_{i=1}^{\infty} (A_2 X)^i\Big) A_1 u\,, \tag{34}$$

where I is the identity operator on F. Note that the infinite sum in (34) is to be understood in the following sense (no convergence concept is defined in F!): for any v and $t \geqq 0$,

$$\Big\{\Big(\sum_{i=1}^{\infty} (A_2 X)^i\Big) v\Big\}(t) = \Big\{\Big(\sum_{i=1}^{m} (A_2 X)^i v\Big)\Big\}(t)\,,$$

where $mT > t$. Indeed, it is clear that, due to the conditions imposed on X and A_2, $\{(A_2X)^k v\}(t) = \Theta$ on $[0, kT)$ for any $v \in F$, and consequantly, the infinite sum reduces in fact to a finite sum for any finite $t \geqq 0$.

Substituting from (34) into the second equation (33), we obtain the sought formula for W, i.e.,

$$Wu = \{B_1 + B_2X(I + \sum_{i=1}^{\infty} (A_2X)^i) A_1\} u . \tag{35}$$

Next, let us discuss the input-output boundedness and stability. Assume that the operators A_1, A_2, B_1, B_2 and X are continuous on F^*, and consequently, bounded. Then the conditions (8), (9) and (22) are obviously satisfied; moreover, the numbers λ_n and μ_n given by (10) and (12), may be defined by

$$\lambda_n = \sup_{x \in R_n} \frac{\|A_2 x\|}{\|x\|} , \quad \mu_n = \sup_{x \in R_n} \frac{\|Xx\|}{\|x\|} , \tag{36}$$

where $R_n := \{x : x \in F^*, x \neq \Theta, x(t) = \Theta \text{ on } [0, nT)\}$. (The proof is obvious.)

Thus, let $\lambda_n \to \lambda$, $\mu_n \to \mu$ as $n \to \infty$ and let $\lambda\mu < 1$; then, as shown in the proof of Theorem 4, the solution Φ of the equation $\Phi = A(u, X\Phi)$ satisfies the condition $\|\Phi\| \leqq M'\|\Phi_1\|$, where $M' > 0$ is a constant independent of u. However, by (32), $\|\Phi_1\| = \|A(u, X\Theta)\| = \|A_1u + A_2X\Theta\| = \|A_1u\| \leqq \|A_1\| \|u\|$; hence, $\|\Phi\| \leqq$ $\leqq M' \|A_1\| \|u\|$. On the other hand, the second equation in (32) yields

$$\begin{aligned} \|Wu\| = \|y\| = \|B_1u + B_2X\Phi\| \leqq \|B_1\| \cdot \|u\| + \|B_2\| \cdot \|X\| \cdot \|\Phi\| \leqq \\ \leqq (\|B_1\| + M'\|A_1\| \cdot \|B_2\| \cdot \|X\|) \|u\| . \end{aligned} \tag{37}$$

Hence, W is a bounded operator on F^*, and consequently, is uniformly continuous. This means that we have input-output boundedness and uniform stability.

Finally, let us present two simple examples clarifying the application of Theorems 4 and 5.

Example 3. Consider a simplified version of Example 2, i.e. let $P = \xi_1$, $R = \eta_1$, $T = \zeta_1$, and let the functions H, M, S and K, N, Q (we drop the index for obvious reason) have properties stated there. Thus, the operators A, B and X are defined by

$$\begin{aligned} \{A(u, v)\}(t) &= H(t, u(t), v(t)) + \int_0^t K(t, \tau, u(\tau), v(\tau))\, d\tau , \\ \{B(u, v)\}(t) &= S(t, u(t), v(t)) + \int_0^t Q(t, \tau, u(\tau), v(\tau))\, d\tau , \\ \{Xv\}(t) &= M(t, (P_T v)(t)) + \int_0^t N(t, \tau, (P_T v)(\tau))\, d\tau , \\ & u, v \in F , \quad t \geqq 0 , \quad T > 0 . \end{aligned} \tag{38}$$

Let $F^* = \mathscr{B}$ and assume that the following conditions are fulfilled:
C_1: Nonnegative functions $h_2(t)$, $m(t)$ and constants h_1, s_1, s_2 exist such that

$$\begin{aligned}(39)\qquad &|H(t, \xi_1, \eta_1) - H(t, \xi_2, \eta_2)| \leqq h_1|\xi_1 - \xi_2| + h_2(t)\,|\eta_1 - \eta_2|\,,\\ &|S(t, \xi_1, \eta_1) - S(t, \xi_2, \eta_2)| \leqq s_1|\xi_1 - \xi_2| + s_2|\eta_1 - \eta_2|\,,\\ &|M(t, \xi_1) - M(t, \xi_2)| \leqq m(t)\,|\xi_1 - \xi_2|\end{aligned}$$

for every $\xi_1, \xi_2, \eta_1, \eta_2 \in E^n$, $t \in [0, \infty)$ and

$$h_2^0 = \sup_{[0,\infty)} h_2(t) < \infty\,, \quad m^0 = \sup_{[0,\infty)} m(t) < \infty\,,$$

$$\sup_{[0,\infty)} |H(t, 0, 0)| < \infty\,, \quad \sup_{[0,\infty)} |S(t, 0, 0)| < \infty\,, \quad \sup_{[0,\infty)} |M(t, 0)| < \infty\,.$$

C_2: Nonnegative functions k_1, k_2, q_1, q_2 and n_1 of arguments t, τ exist such that

$$\begin{aligned}(40)\qquad &|K(t, \tau, \xi_1, \eta_1) - K(t, \tau, \xi_2, \eta_2)| \leqq k_1(t, \tau)\,|\xi_1 - \xi_2| + k_2(t, \tau)\,|\eta_1 - \eta_2|\,,\\ &|Q(t, \tau, \xi_1, \eta_1) - Q(t, \tau, \xi_2, \eta_2)| \leqq q_1(t, \tau)\,|\xi_1 - \xi_2| + q_2(t, \tau)\,|\eta_1 - \eta_2|\,,\\ &|N(t, \tau, \xi_1) - N(t, \tau, \xi_2)| \leqq n_1(t, \tau)\,|\xi_1 - \xi_2|\end{aligned}$$

for every $\xi_1, \xi_2, \eta_1, \eta_2 \in E^n$, $0 \leqq \tau \leqq t < \infty$, and

$$\tilde{k}_i = \sup_{[0,\infty)} \int_0^t k_i(t, \tau)\,d\tau < \infty\,, \quad \tilde{q}_i = \sup_{[0,\infty)} \int_0^t q_i(t, \tau)\,d\tau < \infty\,, \quad i = 1, 2\,,$$

$$\tilde{n} = \sup_{[0,\infty)} \int_0^t n_1(t, \tau)\,d\tau < \infty\,, \quad \sup_{[0,\infty)} \int_0^t |K(t, \tau, 0, 0)|\,d\tau < \infty\,,$$

$$\sup_{[0,\infty)} \int_0^t |Q(t, \tau, 0, 0|\,d\tau < \infty\,, \quad \sup_{[0,\infty)} \int_0^t |N(t, \tau, 0)|\,d\tau < \infty\,.$$

First, we can easily verify that the operators A and B map $\mathscr{B} \times \mathscr{B}$ into $\mathscr{B}$ and X maps $\mathscr{B}$ into itself. Actually, by (39) we have

$$(41)\qquad |H(t, \xi, \eta)| \leqq h_1|\xi| + h_2^0|\eta| + |H(t, 0, 0)|\,,$$

and by (40)

$$(42)\qquad |K(t, \tau, \xi, \eta)| \leqq k_1(t, \tau)\,|\xi| + k_2(t, \tau)\,|\eta| + |K(t, \tau, 0, 0)|\,.$$

Hence, for any $u, v \in \mathscr{B}$ and $t \geqq 0$, (38) yields

$$
\begin{aligned}
(43) \qquad & \left|\{A(u, v)\}(t)\right| \leqq h_1|u(t)| + h_2^0|v(t)| + |H(t, 0, 0)| + \\
& + \int_0^t k_1(t, \tau)|u(\tau)| \, d\tau + \int_0^t k_2(t, \tau)|v(\tau)| \, d\tau + \int_0^t |K(t, \tau, 0, 0)| \, d\tau \leqq \\
& \leqq h_1\|u\| + h_2^0\|v\| + \sup_{[0,\infty)} |H(t, 0, 0)| + \tilde{k}_1\|u\| + \tilde{k}_2\|v\| + \\
& + \sup_{[0,\infty)} \int_0^t |K(t, \tau, 0, 0)| \, d\tau ,
\end{aligned}
$$

i.e. $|\{A(u, v)\}(t)|$ is bounded on $[0, \infty)$. In the same manner the boundedness of $|\{B(u, v)\}(t)|$ and $|\{Xv\}(t)|$ can be verified.

Furthermore, if $u_1, u_2, v_1, v_2 \in \mathscr{B}$ and $t \geqq 0$, we obtain by (38), (39) and (40),

$$
\begin{aligned}
& |\{A(u_1, v_1)\}(t) - \{A(u_2, v_2)\}(t)| \leqq h_1\|u_1 - u_2\| + h_2^0\|v_1 - v_2\| + \\
& + \tilde{k}_1\|u_1 - u_2\| + \tilde{k}_2\|v_1 - v_2\| ,
\end{aligned}
$$

and consequently,

$$
(44) \qquad \|A(u_1, v_1) - A(u_2, v_2)\| \leqq (h_1 + \tilde{k}_1)\|u_1 - u_2\| + (h_2^0 + \tilde{k}_2)\|v_1 - v_2\| .
$$

Analogously we get

$$
(45) \qquad \|B(u_1, v_1) - B(u_2, v_2)\| \leqq (s_1 + \tilde{q}_1)\|u_1 - u_2\| + (s_2 + \tilde{q}_2)\|v_1 - v_2\| ,
$$

and

$$
\|Xv_1 - Xv_2\| \leqq (m^0 + \tilde{n})\|v_1 - v_2\| .
$$

Hence, the conditions (22), (8) and (9) in Theorems 5 and 4 are satisfied, and B is a continuous operator.

On the other hand, if $n \geqq 1$ is an integer and $u, v_1, v_2 \in \mathscr{B}$ are such that $v_1(t) = v_2(t)$ on $[0, nT)$, we obtain for any $t \geqq 0$,

$$
\begin{aligned}
& |\{A(u, v_1)\}(t) - \{A(u, v_2)\}(t)| \leqq h_2(t)|v_1(t) - v_2(t)| + \\
& + \int_{nT}^t k_2(t, \tau)|v_1(\tau) - v_2(\tau)| \, d\tau \leqq \\
& \leqq \|v_1 - v_2\| \sup_{t \in [nT, \infty)} \left\{h_2(t) + \int_{nT}^t k_2(t, \tau) \, d\tau\right\} .
\end{aligned}
$$

Hence, we have for the number λ_n in (10),

$$
(46) \qquad \lambda_n \leqq \tilde{\lambda}_n = \sup_{t \in [nT, \infty)} \left\{h_2(t) + \int_{nT}^t k_2(t, \tau) \, d\tau\right\} .
$$

Similarly we obtain for μ_n in (12),

$$\mu_n \leqq \tilde{\mu}_n = \sup_{t \in [nT, \infty)} \left\{ m(t) + \int_{nT}^{t} n_1(t, \tau) \, d\tau \right\} . \tag{47}$$

Consequently, if $\lim_{n \to \infty} \tilde{\lambda}_n \, . \, \lim_{n \to \infty} \tilde{\mu}_n < 1$, then the over-all transfer operator W maps $\mathscr{B}$ into itself and is continuous, i.e. the considered feedback system is input–output bounded and stable.

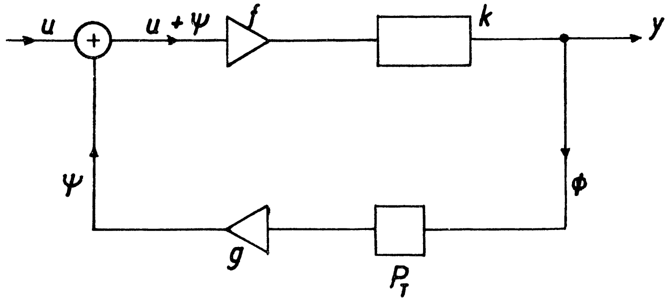

Fig. 3.

Example 4. Consider the classical feedback configuration whose block–diagram is plotted in Fig. 3. Let the forward-path be formed by a time-invariant memoryless gain f in tandem with a time-varying linear system k, and the feedback loop by a delayor P_T in tandem with a gain g. More specifically, the gain f is governed by an equation $x_2 = f(x_1)$, (x_1, x_2 is the input and output quantity, respectively), where f satisfies the Lipschitz condition $|f(\xi_1) - f(\xi_2)| \leqq p|\xi_1 - \xi_2|$, the linear system k by $x_2(t) = \int_0^t k(t, \tau) \, x_1(\tau) \, d\tau$, where $\sup_{[0,\infty)} \int_0^t |k(t, \tau)| \, d\tau < \infty$, and the gain g by $x_2 = g(x_1)$, where $|g(\xi_1) - g(\xi_2)| \leqq q|\xi_1 - \xi_2|$. Our task is to find a bound for p and q guaranteeing the input–output boundedness and stability of the system provided our attention is focussed on the behavior in the space $\mathscr{B}$.

Using our notation, we have

$$\{A(u, v)\} (t) = \{B(u, v)\} (t) = \int_0^t k(t, \tau) f(u(\tau) + v(\tau)) \, d\tau , \tag{48}$$
$$\{Xv\} (t) = g(\{P_T v\} (t))$$

for any $u, v \in \mathscr{B}$ and $t \geqq 0$.

Recalling the results of Example 3 we see that the considered system satisfies the conditions (22), (8) and (9). By (46) we obtain

$$\lambda_n \leqq \sup_{t \in [nT, \infty)} p \int_{nT}^{t} |k(t, \tau)| \, d\tau ,$$

and similarly, $\mu_n \leqq q$ for every n.

Hence,

$$pq < \left\{ \lim_{a \to \infty} \sup_{t \in [a, \infty)} \int_a^t |k(t, \tau)| \, d\tau \right\}^{-1}$$

is the sought condition for p and q ensuring the boundedness and stability.

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Souhrn

O OBECNÝCH ZPĚTNOVAZEBNÍCH SYSTÉMECH OBSAHUJÍCÍCH ZPOŽĎOVACÍ PRVKY

VÁCLAV DOLEŽAL

V článku jsou vyšetřovány obecné nelineární zpětnovazební soustavy, které obsahují zpožďovací prvky. Předně je ukázáno, že operátor přenosu takové soustavy existuje za poměrně slabých předpokladů. Dále je dokázána věta, že když operátory popisující soustavu jsou v jistém smyslu kausální, že pak operátor přenosu je rovněž kausální. Konečně jsou vysloveny věty o stabilitě typu vstup–výstup a ohraničenosti typu vstup—výstup. Použití vyložené teorie je ilustrováno na několika konkrétních případech zpětnovazebních soustav.

Резюме

ОБ ОБЩИХ СИСТЕМАХ С ОБРАТНОЙ СВЯЗЬЮ СОДЕРЖАЩИХ ЗАПАЗДЫВАЮЩИЕ ЭЛЕМЕНТЫ

ВАЦЛАВ ДОЛЕЖАЛ (VÁCLAV DOLEŽAL)

В статье рассматриваются общие нелинейные системы с обратной связью, которые содержат запаздывающие элементы. Прежде всего показано, что передаточный оператор такой системы существует при довольно слабых предположениях. Далее доказана теорема о том, что если операторы описывающие систему каузальны в определенном смысле, то и передаточный оператор каузален. Наконец даются теоремы об устойчивости типа вход—выход и ограниченности типа вхоф—выход. Приложения изложенной теории иллюстрируются на нескольких конкретных примерах систем с обратной связью.

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