Aplikace matematiky

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Aplikace matematiky, Vol. 4 (1959), No. 2, 109–125

Persistent URL: http://dml.cz/dmlcz/102650

STATISTICAL QUALITY CONTROL OF OUT-OF-ROUNDNESS OF MACHINED PARTS

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(Received March 27th, 1958.) DT: 330.655

In the article a method is proposed for the Statistical quality control of deviations from roundness of machined parts. The method is based on the assumption of Weibull's distribution, further on the uniformly most powerful test for verifying the exceeding of the prescribed tolerance and on the test using the j : th value from the top in a random sample.

1. INTRODUCTION

In current literature, various questions relating to deviations from roundness of cylindrically shaped parts have been treated, since it is evident that such deviations play an important role in the attainment of high quality of precise products. The basic deviations from roundness are ovality and more generally out-of-roundness [5]. In practice the method of measuring deviations from roundness depends on the assumed geometrical shape of the cross-section of the machined part. When measuring on the basis of two-point contact (Fig. 1), the detail in question is rotated between two parallel planes and the difference between the maximum and minimum diameter of the detail is noted (ovality). When measuring on the basis of three-point contact (Fig. 2), the detail is rotated in a prismatic base and the difference between the maximum and minimum measured value is noted. In this case the third point of contact is that of the measuring indicator and lies in the plane of symmetry of the prismatic base (out-of-roundness).

On the basis of experimental material it can be assumed, that the deviation from roundness ξ has Weibull's distribution

$$f_W(x; \sigma, \beta) = \frac{\beta}{\sigma} x^{\beta-1} e^{-\frac{x^\beta}{\sigma}}, \quad x > 0\,, \quad \sigma > 0\,, \quad \beta > 0\,, \tag{1.1}$$

with parameters σ and β (e. g. KUTAJ [1] recommends this distribution for $\beta = 2$). Thus ξ^β has the exponential distribution

$$f_E(x; \sigma) = \frac{1}{\sigma} e^{-\frac{x}{\sigma}}, \quad x > 0, \quad \sigma > 0, \tag{1.2}$$

with mean value σ and variance σ^2.

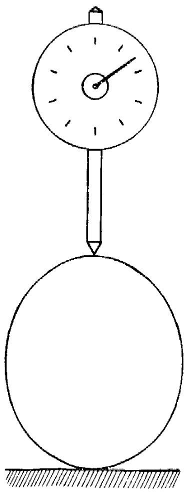

Fig. 1.

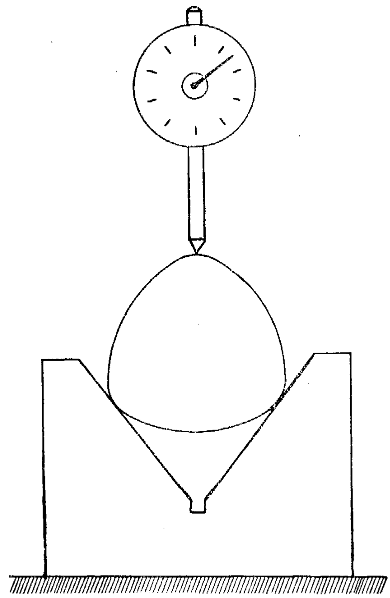

Fig. 2.

The problem is, assuming the parameter β to be known, to test the null hypothesis $H_0 : \sigma \leq \sigma_0$ with respect to the alternative hypothesis $H_1 : \sigma > \sigma_0$, which corresponds to the exceeding of the upper limit to the out-of-roundness.

2. THE UNIFORMLY MOST POWERFUL TEST

We begin with the simple hypotheses $H_0 : \sigma = \sigma_0$ and $H_1 : \sigma = \sigma_1$, $\sigma_1 > \sigma_0$. The most powerful test of level α must satisfy the Neyman-Pearson condition

$$\lambda = \frac{\prod_{k=1}^{n} f_W(x_k, \sigma_1)}{\prod_{k=1}^{n} f_W(x_k, \sigma_0)} \geqq c_\alpha, \tag{2.1}$$

where c_α depends on the level α. Inserting (1.1) in (2.1) we obtain

$$\lambda = \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{\frac{1}{n}\sum x_k^\beta \left(\frac{n}{\sigma_0} - \frac{n}{\sigma_1}\right)},$$

i. e. λ is an increasing function of $\frac{1}{n}\sum_{k=1}^{n} x_k^\beta$ for all $\sigma_1 > \sigma_0$, so that the region (2.1) is equivalent to the region

$$\frac{1}{n}\sum_{k=1}^{n} x_k^\beta > z_{\sigma_0, n, \alpha},$$

where $z_{\sigma_0, n, \alpha}$, in view of the fact that $\frac{2}{\sigma_0}\sum_{k=1}^{n} \xi_k^\beta$ has the chi-square distribution with $2n$ degrees of freedom, is equal to

$$z_{\sigma_0, n, \alpha} = \frac{\sigma_0}{2n}\chi^2_{2n,\alpha}, \tag{2.2}$$

where $\chi^2_{2n,\alpha}$ is the 100α percent value of the chi-square distribution ($\int_{\chi^2_{2n,\alpha}}^{\infty} \chi^2_{2n}(v)\,dv = \alpha$). Since the critical region is independent of σ_1, $\sigma_1 > \sigma_0$, the test based on the statistic

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k^\beta \tag{2.3}$$

is the uniformly most powerful test of H_0 with respect to the alternative hypothesis $H_1 : \sigma > \sigma_0$. Moreover, if $\mathbf{P}\left(\frac{1}{n}\sum_{k=1}^{n} \xi_k^\beta > z_{\sigma_0, n, \alpha} \mid \sigma = \sigma_0\right) = \alpha$, then for $\sigma \leqq \sigma_0$ the relation $\mathbf{P}\left(\frac{1}{n}\sum_{k=1}^{n} \xi_k^\beta > z_{\sigma_0, n, \alpha} \mid \sigma\right) \leqq \alpha$ holds, so that the test is uniformly most powerful also for the extended null hypothesis $H_0 : \sigma \leqq \sigma_0$.

3. TEST BASED ON THE RANGE BETWEEN THE $n + 1 - j$: TH AND i : TH VALUE OF THE n ORDER STATISTICS

The test based on the statistic (2.3) is too complicated for practical work and therefore we propose a further statistic

$$\varrho_{(ij)} = \xi^\beta_{(n+1-j)} - \xi^\beta_{(i)}, \quad i < n + 1 - j, \tag{3.1}$$

where $\xi^\beta_{(k)} = V_k(\xi_1^\beta, \xi_2^\beta, \ldots, \xi_n^\beta)$ and the function $V_k(x_1, x_2, \ldots, x_n)$ denotes the kth of the values $x_1, x_2, \ldots, x_n$ arranged in ascending order of magnitude and $\xi^\beta_{(0)} = 0$. From among all the statistics(3.1) we choose that particular one which gives the largest power when testing the null hypothesis $H_0 : \sigma \leqq \sigma_0$

against the alternative hypothesis $H_1 : \sigma > \sigma_0$. Using the conception of the asymptotically largest power, it is shown in section 6 that this property is possessed by the statistic

$$\varrho_{(0j)} = \varrho_{(j)} = \xi^{\beta}_{(n+1-j)} , \quad j \approx 0.203n ,$$

the asymptotic relative efficiency of which is equal to $\frac{m}{n} \approx 0.65$ (m and n are the sample sizes which give the same power when using the test based on (2.3) and $\varrho_{(j)}$ respectively).

4. STATISTICAL QUALITY CONTROL

As the first suitable statistic for the statistical quality control of out-of-roundnesses we have indicated the random variable (2.3), which assumes the observed value

$$z = \frac{1}{n} \sum_{k=1}^{n} x^{\beta}_{k} , \tag{4.1}$$

where x_k, according to our application, are the observed values of the deviation from roundness ξ and n is the size of the random sample of machined parts. We take as the upper control limit the value $z_{\sigma_0, n, \alpha}$, for which the equation

$$\mathbf{P}\left(\frac{1}{n} \sum \xi^{\beta}_{k} > z_{\sigma_0, n, \alpha}\right) = \mathbf{P}\left(\frac{1}{n\sigma_0} \sum \xi^{\beta}_{k} > z_{1, n, \alpha}\right) = \alpha , \quad \left(z_{1, n, \alpha} = \frac{z_{\sigma_0, n, \alpha}}{\sigma_0}\right) , \tag{4.2}$$

holds, where we choose the values $\alpha = 0.01$ or 0.05. (The lower control limit equals zero.) By (4.2) we then have

$$z_{\sigma_0, n, \alpha} = \sigma_0 \, . \, z_{1, n, \alpha} \, . \tag{4.3}$$

The values $z_{1, n, \alpha}$ (Tab. 1) are obtained from (2.2) using the tables of the critical values of the chi-square distribution.

It remains to determine the value of the parameter σ_0 in (4.3). There are two ways. Either we estimate the parameter σ_0 by means of the unbiased estimate (4.1) when n is sufficiently large in which case

$$z_{\sigma_0, n, \alpha} = z \, . \, z_{1, n, \alpha} , \tag{4.4}$$

or we start out from the specified percent of defectives 100ε % and from the upper limit to the out-of-roundness T_ε. According to (1.1) and (1.2) we may write

$$\int_{T_\varepsilon}^{\infty} f_W(x; \sigma, \beta) \, \mathrm{d}x = \int_{T_\varepsilon^{\beta}}^{\infty} f_E(x; \sigma) \, \mathrm{d}x = \varepsilon \, .$$

It then follows that

$$T_\varepsilon^\beta = k_\varepsilon \cdot \sigma\,, \quad k_\varepsilon = -\lg \varepsilon\,,$$

(the coefficient k_ε is tabulated for certain values of ε in Table 2), so that

$$z_{\sigma_0, n, \alpha} = \frac{T_\varepsilon^\beta}{k_\varepsilon} \cdot z_{1, n, \alpha} = DT_\varepsilon^\beta\,, \tag{4.5}$$

where the coefficient $D = \frac{z_{1,n,\alpha}}{k_\varepsilon}$ is tabulated in Table 3).

As the second suitable statistic for the statistical quality control of out-of-roundness we have indicated the random variable $\varrho_{(j)} = \xi^\beta_{(n+1-j)}$ for $j \approx 0{\cdot}203n$, the observed value of which is

$$r_{(j)} = V_{n+1-j}\,(x_1^\beta, x_2^\beta, \ldots, x_n^\beta)\,, \tag{4.6}$$

where x_k are the observed values of the deviation from roundness ξ. We start out again from the relation for the control limit

$$r_{(j), \sigma_0, n, \alpha} = \sigma_0 \cdot r_{(j), 1, n, \alpha} \tag{4.7}$$

the validity of which we can prove as in (4.2), where $r_{(j),1,n,\alpha} = \frac{r_{(j),\sigma_0,n,\alpha}}{\sigma_0}$. The values $r_{(j),1,n,\alpha}$ for $j = 1, 2$ are tabulated (Tab. 1) on the basis of the relation

$$\mathbf{P}\left(\frac{\varrho_{(j)}}{\sigma} < x\right) = I_{F_E(x,1)}\ (n - j + 1, j)\,, \tag{4.8}$$

where $I_x(a, b)$ is the Incomplete Beta-Function and $F_E(x, 1)$ is the distribution function of $\frac{\xi^\beta}{\sigma}$. The value of the parameter σ_0 may, for sufficiently large n, be estimated by the unbiased estimate

$$\frac{\varrho_{(j)}}{\mathbf{E}(\varrho_{(j)} \mid \sigma = 1)}\,,$$

using the relation (6.8). If we estimate σ_0 on the basis of the specified percentage defectives 100ε %, then we have

$$r_{(j),\sigma_0,n,\alpha} = \frac{T_\varepsilon^\beta}{k_\varepsilon}\, r_{(j),1,n,\alpha} = D_j^* \cdot T_\varepsilon^\beta\,, \tag{4.9}$$

where the coefficient $D_j^* = \frac{r_{(j),1,n,\alpha}}{k_\varepsilon}$ for $j = 1, 2$ is tabulated in Table 4. From the equation

$$\mathbf{P}(\varrho_{(j)} > r_{(j),\sigma_0,n,\alpha}) = \mathbf{P}\,(\sqrt[\beta]{\varrho_{(j)}} > \sqrt[\beta]{r_{(j),\sigma_0,n,\alpha}})$$

it follows, that for values of the statistic $\sqrt[\beta]{\varrho_{(j)}}$ (see (4.6))

$$\sqrt[\beta]{r_{(j)}} = [V_{n+1-j}(x_1^\beta, x_2^\beta, \ldots, x_n^\beta)]^{\frac{1}{\beta}} = V_{n+1-j}\,(x_1, x_2, \ldots, x_n)\,, \tag{4.10}$$

which are thus calculated directly from the observed values of the deviations from roundness ξ, we obtain the control limit

$$s_{(j),\sigma_0,n,\alpha} = \sqrt[\beta]{\sigma_0 \cdot r_{(j),1,n,\alpha}} \,.$$

Inserting (4.9) we obtain

$$s_{(j),\sigma_0,n,\alpha} = \sqrt[\beta]{D_j^*} \,.\, T_\varepsilon \,.$$

The advantage of this method consists in the fact that the deviations from roundness are plotted on the control chart direct in the technical units and that the calculation of values (4.10) is quite simple. On the other hand the change in the value of the parameter β evokes the change in the control limit.

According to the results in section 6, the well known statistics $\varrho_{(1)}$ or $\varrho_{(2)}$ ($\sqrt[\beta]{\varrho_{(1)}}$ or $\sqrt[\beta]{\varrho_{(2)}}$) for sample sizes $n = 5$ or 10 respectively are very convenient for practical purposes.

Table 1

n	$z_{1,n,\alpha}$		$r_{(1),1,n,\alpha}$		$r_{(2),1,n,\alpha}$	
	$\alpha = 0{\cdot}01$	$\alpha = 0{\cdot}05$	$\alpha = 0{\cdot}01$	$\alpha = 0{\cdot}05$	$\alpha = 0{\cdot}01$	$\alpha = 0{\cdot}05$
3	2·802	2·099	5·700	4·077	2·833	2·000
4	2·511	1·938	5·987	4·363	3·174	2·327
5	2·321	1·831	6·211	4·585	3·429	2·573
6	2·185	1·752	6·392	4·766	3·635	2·769
7	2·081	1·692	6·547	4·920	3·802	2·933
8	2·000	1·644	6·681	5·053	3·934	3·075
9	1·934	1·604	6·798	5·170	4·087	3·195
10	1·878	1·571	6·904	5·275	4·212	3·309

Table 2

ε	k_ε
0·005 00	5·298 32
0·010 00	4·605 17
0·020 00	3·912 02
0·049 79	3·000 00
0·050 00	2·995 73
0·135 34	2·000 00
0·367 88	1·000 00

Table 3

n	D			
	$\varepsilon = 0{\cdot}005$		$\varepsilon = 0{\cdot}01$	
	$\alpha = 0{\cdot}01$	$\alpha = 0{\cdot}05$	$\alpha = 0{\cdot}01$	$\alpha = 0{\cdot}05$
3	0·529	0·396	0·608	0·456
4	0·474	0·366	0·545	0·421
5	0·438	0·346	0·504	0·398
6	0·412	0·331	0·474	0·380
7	0·393	0·319	0·452	0·367
8	0·377	0·310	0·434	0·357
9	0·365	0·303	0·420	0·348
10	0·355	0·296	0·408	0·341

Table 4

n	D_1^*				D_2^*			
	$\varepsilon = 0.005$		$\varepsilon = 0.01$		$\varepsilon = 0.005$		$\varepsilon = 0.01$	
	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.01$	$\alpha = 0.05$
3	1·076	0·770	1·238	0·885	0·535	0·377	0·615	0·434
4	1·130	0·823	1·300	0·947	0·599	0·439	0·689	0·505
5	1·172	0·865	1·349	0·996	0·647	0·486	0·745	0·559
6	1·207	0·900	1·388	1·035	0·686	0·523	0·789	0·601
7	1·236	0·929	1·422	1·068	0·718	0·554	0·826	0·637
8	1·261	0·954	1·451	1·097	0·742	0·580	0·854	0·668
9	1·283	0·976	1·476	1·123	0·771	0·603	0·887	0·694
10	1·303	0·996	1·499	1·146	0·795	0·625	0·915	0·719

Table 5

Values $h^2(p, q)$

q \ p	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8
0·1	0·543	0·494	0·442	0·385	0·324	0·256	0·181	0·096
0·2	0·582	0·512	0·439	0·362	0·280	0·192	0·097	
0·3	0·543	0·462	0·377	0·288	0·196	0·099		
0·4	0·473	0·384	0·292	0·197	0·100			
0·5	0·389	0·295	0·198	0·100				
0·6	0·296	0·199	0·100					
0·7	0·199	0·100						
0·8	0·100							

5. LIMITING DISTRIBUTION OF THE STATISTIC $\varrho(i_n, j_n)$

In the next section we shall use the limiting property of the statistic (3.1), which is stated in the following

Theorem 1: *Let i_n and j_n depend on n so, that*

$$\lim_{n \to \infty} j_n = \lim_{n \to \infty} (n + 1 - i_n - j_n) = \infty . \tag{5.1}$$

Then the statistic $\varrho(i_n j_n) = \xi^{\beta}_{(n+1-j_n)} - \xi^{\beta}_{(i_n)}$ *has the asymptotic normal distribution* $N\left(\sigma \sum_{s=j_n}^{n-i_n} \frac{1}{s}, \sigma^2 \sum_{s=j_n}^{n-i_n} \frac{1}{s^2}\right)$.

Lemma: *Let* $\eta_1, \eta_2, \ldots \eta_n, \ldots$ *be mutually independent random variables equally distributed with mean value* a, *finite variance* $b^2 > 0$ *and the distribution function* $F(x)$.

We introduce the relation

$$\eta_n^* = \sum_{s=1}^{n} c_{ns} \eta_s , \qquad c_{ns} > 0 .$$

If, for $n \to \infty$,

$$\frac{\max\limits_{1 \leq s \leq n} c_{ns}^2}{\sum\limits_{s=1}^{n} c_{ns}^2} \to 0 , \tag{5.2}$$

then the random variable η_n^* *has the asymptotic normal distribution* $N\,(a \sum_{s=1}^{n} c_{ns}, b^2 \sum_{s=1}^{n} c_{ns}^2)$.

Proof. The random variables

$$\eta'_{ns} = c_{ns} \eta_s , \qquad s = 1, 2, \ldots, n , \quad n = 1, 2, \ldots$$

have the mean values ac_{ns}, the variances $b^2 c_{ns}^2$ and the distribution functions $F\left(\frac{x}{c_{ns}}\right)$. The Lindeberg condition [8] has, therefore, for $\eta_n^* = \sum_{s=1}^{n} \eta'_{ns}$ the form $L_n \to 0$, where

$$L_n = \frac{1}{B_n^2} \sum_{s=1}^{n} \int\limits_{|x - ac_{ns}| > \tau B_n} (x - ac_{ns})^2 \, \mathrm{d}F\left(\frac{x}{c_{ns}}\right) = 0 , \quad \tau > 0 ,$$

and $B_n^2 = \mathbf{D}^2(\eta_n^*) = b^2 \sum_{s=1}^{n} c_{ns}^2$. Substituting for B_n we obtain by simple calculation the relation

$$L_n = \frac{1}{b^2 \sum\limits_{s=1}^{n} c_{ns}^2} \sum_{s=1}^{n} c_{ns}^2 \int\limits_{c_{ns}|x-a| > \tau b \sqrt{\Sigma c_{ns}^2}} (x - a)^2 \, \mathrm{d}F(x) \leq$$

$$\leq \frac{1}{b^2} \int\limits_{|x-a| > \tau b \frac{\sqrt{c^2_{ns}}}{\max\limits_{1 \leq s \leq n} c_{ns}}} (x - a)^2 \, \mathrm{d}F(x) .$$

Now, if (5.2) holds, then

$$\tau b \frac{\sqrt{\sum c_{ns}^2}}{\max\limits_{1 \leq s \leq n} c_{ns}} \to \infty ,$$

and as n tends to infinity (the variance b^2 is finite) we have

$$\int\limits_{|x-a| > \tau b \frac{\sqrt{\sum c^2_{ns}}}{\max c_{ns}}} (x - a)^2 \, \mathrm{d}F(x) \to 0 .$$

Thus the lemma is proved.

Proof of Theorem 1. The statistic (3.1) can be written [2]

$$\varrho_{(i_n j_n)} = \sum_{s=j_n}^{n-i_n} \frac{\xi_s^\beta}{s} , \tag{5.3}$$

where the ξ_s^β have the distribution (1.2). If we put $\eta_s = \xi_s^\beta$, $a = b = \sigma > 0$ and

$$c_{ns} = \frac{1}{s} \text{ for } j_n \leq s \leq n - i_n ,$$
$$= 0 \text{ otherwise} ,$$

then the condition (5.2) has the form

$$\frac{\frac{1}{j_n^2}}{\sum\limits_{s=j_n}^{n-i_n} \frac{1}{s^2}} \to 0 . \tag{5.4}$$

But

$$\frac{\frac{1}{j_n^2}}{\sum\limits_{s=j_n}^{n-i_n} \frac{1}{s^2}} \leq \frac{\frac{1}{j_n^2}}{\sum\limits_{s=j_n}^{n-i_n} \frac{1}{s(s+1)}} = \frac{\frac{1}{j_n^2}}{\frac{1}{j_n} - \frac{1}{n+1-i_n}} =$$
$$= \frac{n+1-i_n}{j_n(n+1-i_n-j_n)} = \frac{1}{j_n} + \frac{1}{n+1-i_n-j_n} , \tag{5.5}$$

so that (5.4) really follows when $n \to \infty$ from (5.5). From (5.3) we then have, that the mean value $\mathbf{E}(\varrho(i_n j_n)) = \sigma \sum\limits_{s=j_n}^{n-i_n} \frac{1}{s}$ and the variance $\mathbf{D}^2(\varrho(i_n j_n)) = \sigma^2 \sum\limits_{s=j_n}^{n-i_n} \frac{1}{s^2}$ thus proving Theorem 1.

6. ASYMPTOTIC RELATIVE EFFICIENCY OF THE STATISTIC $\varrho_{(i_n j_n)}$

From the point of view of the application (section 4) it is important to select i_n and j_n so, that we obtain a test with as large a power as possible. Let us find the asymptotic solution when $n \to \infty$, which is a guide to the solution for small n occurring in practice. We shall let the alternative hypothesis approach the null hypothesis as $n \to \infty$ in such a way, that the power converges to a number < 1, thus avoiding the convergence of the power to 1 as $n \to \infty$ and hence the vanishing of the difference between possible tests.

Let us assume, that for $n = 1, 2, \ldots$ we have the alternative hypothesis

$$\sigma_{1n} = \sigma_0 + \frac{d_1}{\sqrt{n}}, \quad d_1 > 0, \tag{6.1}$$

and that we wish to test it by the test (we denote it as Test 1) with the critical region

$$\varrho(i_n j_n) = \xi^{\beta}_{(n+1-j_n)} - \xi^{\beta}_{(i_n)} > c_1(n, i_n, j_n, \alpha), \tag{6.2}$$

where $c_1(n, i_n, j_n, \alpha)$ is selected so that the level is equal α.

Theorem 2. *If*

$$\lim_{n\to\infty} \frac{j_n}{n} = q, \quad \lim_{n\to\infty} \frac{i_n}{n} = p, \tag{6.3}$$

and if (5.1) *holds, then the power of Test* 1 *under the alternative hypotheses* (6.1) *converges to the number*

$$1 - \Phi\left(t_\alpha - \frac{d_1}{\sigma_0} \cdot \frac{\lg \frac{1-p}{q}}{\left(\frac{1-p-q}{q(1-p)}\right)^{\frac{1}{2}}}\right), \tag{6.4}$$

where Φ is the standardised normal distribution function and t_α the solution of the quation $1 - \Phi(t_\alpha) = \alpha$. The expression (6.4) *is maximal for $p = 0$, $q = 0.203\ldots$*

Proof. First of all the choice of σ does not effect the statistic $\dfrac{\varrho_{(i_n j_n)} - \mathbf{E}(\varrho_{(i_n j_n)})}{\mathbf{D}(\varrho_{(i_n j_n)})}$, so that according to Theorem 1 the statistic $\varrho_{(i_n j_n)}$ has under the hypothesis (6.1) the asymptotic normal distribution $N\left(\sigma_{1n} \sum_{s=j_n}^{n-i_n} \frac{1}{s}, \sigma_{1n}^2 \sum_{s=j_n}^{n-i_n} \frac{1}{s^2}\right)$ and as $n \to \infty$ we have

$$\frac{c_1(n, i_n, j_n, \alpha) - \mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma_0)}{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_0)} \to t_\alpha. \tag{6.5}$$

Now

$$\lim_{n\to\infty} \mathbf{P}\{\varrho_{(i_n j_n)} > c_1(n, i_n, j_n, \alpha \mid \sigma_{1n}\} =$$

$$= \lim_{n\to\infty} \mathbf{P}\left\{\frac{\varrho_{(i_n j_n)} - \mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma_{1n})}{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_{1n})} > \frac{c_1 - \mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma_{1n})}{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_{1n})} \mid \sigma_{1n}\right\} =$$

$$= \lim_{n\to\infty}\left[1 - \Phi\left(\frac{c_1 - \mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma_{1n})}{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_{1n})}\right)\right] = 1 - \Phi\left(\lim_{n\to\infty} \frac{c_1 - \mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma_{1n})}{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_{1n})}\right) =$$

$$= 1 - \Phi\left[\lim_{n\to\infty}\left(\frac{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_0)}{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_{1n})} \cdot \frac{c_1 - \mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma_0)}{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_0)} + \right.\right.$$

$$\left.\left. + \frac{\mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma_0) - \mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma_{1n})}{\mathbf{D}(\varrho_{(i_n j_n)} \mid \sigma_{1n})}\right)\right]. \tag{6.6}$$

The sums in the expressions for the moments

$$\mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma) = \sigma \sum_{s=j_n}^{n-i_n} \frac{1}{s}, \quad \mathbf{D}^2(\varrho_{(i_n j_n)} \mid \sigma) = \sigma^2 \sum_{s=j_n}^{n-i_n} \frac{1}{s^2}, \tag{6.7}$$

can be suitable modified. Using the Euler-MacLaurin sum formula for the function $f(x) = x^{-k}$ we have the relation

$$\sum_{s=n_1}^{n_2} \frac{1}{s^k} = \int_{n_1}^{n_2} x^{-k}\, \mathrm{d}x + \frac{1}{2n_1^k} + \frac{1}{2n_2^k} + k \int_{n_1}^{n_2} P_1(x)\, x^{-(k+1)}\, \mathrm{d}x,$$

where we shall assume, that the integers $k \geqq 1$ and $0 < n_1 < n_2$. The function $P_1(x)$ is determined as follows (m is an integer):

$$P_1(x) = m - x + \frac{1}{2}, \quad m < x < m + 1,$$
$$= 0, \qquad x = m,$$

so that $|P_1(x)| < \frac{1}{2}$ for all x. Therefore

$$\left|\int_{n_1}^{n_2} P_1(x)\, x^{-(k+1)}\, \mathrm{d}x\right| < \int_{n_1}^{n_2} \frac{1}{2x^{k+1}}\, \mathrm{d}x < \frac{1}{2kn_1^k},$$

thus

$$\sum_{s=n_1}^{n_2} \frac{1}{s_k} = \int_{n_1}^{n} \frac{1}{x^k}\, \mathrm{d}x + \varepsilon_n^{(k)},$$

where

$$|\varepsilon_n^{(k)}| < \frac{3}{2} \cdot \frac{1}{n_1^k}.$$

Inserting this result in (6.7) we have

$$\mathbf{E}(\varrho_{(i_n j_n)} \mid \sigma) = \sigma \left(\lg \frac{n - i_n}{j_n} + \varepsilon_n^{(1)} \right) ,$$
$$\mathbf{D}^2(\varrho_{(i_n j_n)} \mid \sigma) = \sigma \left(\frac{n - i_n - j_n}{j_n (n - i_n)} + \varepsilon_n^{(2)} \right) , \tag{6.8}$$

where $|\varepsilon_n^{(k)}| < \frac{3}{2} \cdot \frac{1}{j_n^k}$. Using this relation in (6.6), where we insert also the alternative hypothesis (6.1) and make use of the limits given in (6.3) and (6.5), we obtain (6.4). The expression (6.4) becomes a maximum, when the function

$$h(p, q) = \left(\frac{q(1 - p)}{1 - p - q} \right)^{\frac{1}{2}} \lg \frac{1 - p}{q} \tag{6.9}$$

is maximal. First of all the function $h(p, q)$ has no extreme values for $0 \leqq \leqq p < 1 - q < 1$. Now let $p = p_0$; then the equation $\frac{\mathrm{d}}{\mathrm{d}q} h(p_0, q) = 0$ may be written in the form

$$v_q = e^{2\left(1 - \frac{1}{v_q}\right)} .$$

We then find

$$v_q = \frac{1 - p_0}{q} = 4.920 \ldots .$$

The function $h(p_0, q)$ has thus a maximum $\hat{q}_{p_0} = \frac{1 - p_0}{v_q}$, which increases with decreasing p_0, so that it is largest for $p_0 = 0$, i. e.

$$\hat{q}_0 = \frac{1}{v_q} = 0.203 \ldots$$

Thus Theorem 2 is proved.

We now compare Test 1 with the uniformly most powerful test. Therefore we use the alternative hypothesis

$$\sigma_{1m} = \sigma_0 + \frac{d_2}{\sqrt{m}} , \quad d_2 > 0 , \quad m = 1, 2, \ldots , \tag{6.10}$$

and we shall test it by the uniformly most powerful test (we denote it as Test 2) with the critical region

$$\frac{1}{m} \sum_{k=1}^{m} \xi_k^\beta > c_2(m, \alpha) .$$

The asymptotic power of Test 2 is obtained in a way similar to that used in Theorem 2 in the form

$$1 - \Phi \left(t_\alpha - \frac{d_2}{\sigma_0} \right) . \tag{6.11}$$

Comparing (6.4) and (6.11) we see, that Test 1 has the same asymptotic power with respect to the alternative hypothesis (6.1) as Test 2 with respect to the alternative hypothesis (6.10), when (see (6.9))

$$\left(\frac{d_2}{d_1}\right)^2 = h^2(p, q) .$$

The expression $h^2(p, q)$ is the asymptotic relative efficiency (some values are given in Table 5), because if

$$\frac{m}{n} = \left(\frac{d_2}{d_1}\right)^2 ,$$

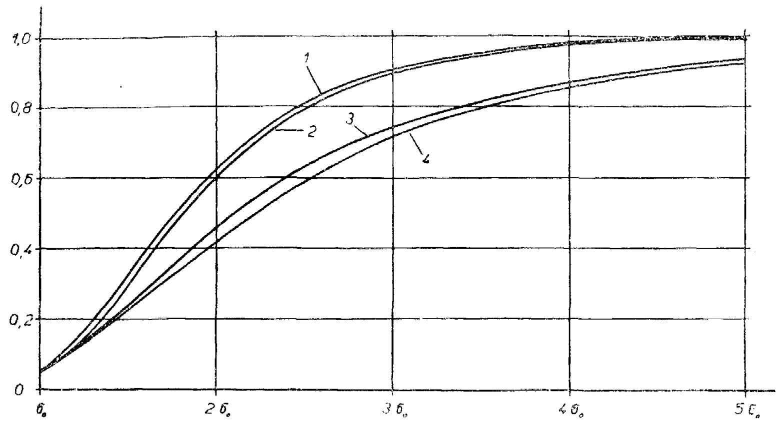

Fig. 3. The power curves of the Test 1 and 2.

$1 - \mathbf{M}\left(\frac{1}{7}\sum_{k=1}^{7} \xi_k^\beta;\ 0.05\right)$; $2 - \mathbf{M}(\varrho_{(2)};\ 0.05;\ 10)$; $3 - \mathbf{M}\left(\frac{1}{4}\sum_{k=1}^{4} \xi_k^\beta;\ 0.05\right)$; $4 - \mathbf{M}(\varrho_{(1)};\ 0.05;\ 5)$.

then the alternative hypotheses (6.1) and (6.10) are equivalent and Test 1 for the sample size n has the same power as Test 2 for the sample size m, i. e. the fraction $\frac{m}{n}$ states to what number we could reduce the number of observations if we used the Test 2 in place of the Test 1.

From Theorem 2 and (6.9) it follows, that the largest asymptotic relative efficiency equals $h^2(0;\ 0.203\ \ldots) = 0.647\ldots$, i. e. when Test 1 has the critical region (6.2) in the form

$$\varrho_{(j)} = \xi^\beta_{(n+1-j_n)} > c_1(n, j_n, \alpha) , \quad j_n \approx 0.203n .$$

We use this result as an approximation for small m and n. Since m, n and j are integers, we shall proceed as follows: We start with the sample size n and put

$$j = [0.203n] ,$$

where $[x]$ is the largest integer $\leqq x$, and

$$m = [0.647n]^* ,$$

where $[x]^*$ is the smallest integer $\geqq x$. As we have experimentally ascertained, we obtain by this procedure the best agreement between the power curves of the two tests for small sample sizes. The power of Test 1 may be written (see sections 3 and 4)

$$\mathbf{M}(\varrho_{(j)}; \alpha, n) = \mathbf{P}(\varrho_{(j)} > r_{(j), \sigma_0, n, \alpha} \mid \sigma = k\sigma_0) , \quad (k > 1)$$

and the power of Test 2 equals (see sections 2 and 4)

$$\mathbf{M}\left(\frac{1}{m} \sum_{k=1}^{m} \xi_k^\beta; \alpha\right) = \mathbf{P}\left(\frac{1}{m} \sum_{k=1}^{m} \xi_k^\beta > z_{\sigma_0, n, \alpha} \mid \sigma = k\sigma_0\right) \quad (k > 1) .$$

In this way the power curves of both tests were obtained as shown in Fig. 3 for the pairs $\left[\mathbf{M}(\varrho_{(1)};\ 0.05;\ 5),\ \mathbf{M}\left(\frac{1}{4} \sum_{1}^{4} \xi_k^\beta;\ 0.05\right)\right]$ and $\left[\mathbf{M}(\varrho_{(2)};\ 0.05;\ 10),\ \mathbf{M}\left(\frac{1}{7} \sum_{1}^{7} \xi_k^\beta;\ 0.05\right)\right]$. From the course of these curves good agreement is evident thus indicating the relatively high efficiency of Test 1 (using the order statistic $\xi_{(n-j+1)}^\beta$) even when using small samples.

7. CONCLUSION

On the basis of tests performed on antifriction bearings it has been verified, that the distribution (1.1) is a suitable model for various deviations from geometrical shape. For these cases the coefficients for determining control limits for the statistical quality control of out-of-roundness are given. The detailed quality control procedure is omitted, since it is analogous to known methods of control by variables. Application of the method presented here together with that proposed by B. Pardubský [6] makes possible statistical quality control in the wide field, where deviations from geometrical shape of precise machined parts have to be restricted to a minimum.

Literature

[1] *Кутай:* Изучение точности производства с применением законов распределения существенно положительных величин, Вестник ленинградского университета, 1956.

[2] *Rényi:* On the Theory of Order Statistics, Acta Mathematica Ac. Sc. Hungaricae, 1953.

[3] *Noether:* A Theorem of Pitman, Annals of Math. Stat., 1955.

[4] *Stuart:* The Measurement of Estimation and Test Efficiency, ISI, 1957, No 62.
[5] *Pardubský:* Úchylky kruhovitosti a jejich měření pomocí prismatické podložky, Sborník konference o geometrických úchylkách tvaru a ploch, 1958.
[6] *Pardubský:* Úchylky geometrického tvaru valivých ložisek, Strojírenství, 1957.
[7] *Cramér:* Mathematical Methods of Statistics, 1946.
[8] *Гнеденко:* Курс теории вероятностей, 1950.
[9] *Janko:* Statistické tabulky, 1958.
[10] *Pearson:* Tables of the Incomplete Beta-Function.
[11] Математические таблицы, ИАН СССР.

Souhrn

STATISTICKÁ REGULACE ODCHYLEK NEOKROUHLOSTI TVARU SOUČÁSTÍ

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(Došlo dne 27. března 1958.)

Na základě experimentálního materiálu lze předpokládat, že odchylka neokrouhlosti ξ se řídí Weibullovým rozdělením (1.1) s parametry σ a β, z nichž β se během výroby nemění. Překročení horní mezní odchylky neokrouhlosti tvaru T_ε odpovídá alternativa $H_1 : \sigma > \sigma_0$ proti nulové hypotéze $H_0 : \sigma \leqq \sigma_0$. K testování nulové hypotézy jsou navrženy dva testy: stejnoměrně nejmohutnější test (1) pomocí statistiky $\frac{1}{n} \sum_{k=1}^{n} \xi_k^\beta$ a test (2) pomocí rozpětí $\varrho_{(ij)} = \xi^\beta_{(n+1-j)} - \xi^\beta_{(i)}$ mezi j-tou největší a i-tou nejmenší hodnotou v uspořádaném výběru $\xi_1^\beta, \ldots, \xi_n^\beta$.

Pro aplikaci testu (1) je vhodné užití kritického oboru

$$\frac{1}{n} \sum_{k=1}^{n} \xi_k^\beta > DT_\varepsilon^\beta$$

(koeficient D je tabelován v tab. 3) a pro aplikaci testu (2) užití velmi jednoduchého oboru

$$\sqrt[\beta]{\varrho_{(oj)}} = \xi_{(n+1-j)} > \sqrt[\beta]{D_j^*} \cdot T_\varepsilon \,, \quad j \approx 0{,}203n$$

(koeficient D_j^* je tabelován v tab. 4), kde ε je povolené procento zmetků. V souvislosti s volbou kriteria (2) jsou dokázány dvě věty:

Věta 1. *Nechť i_n a j_n závisí na n tak, že $\lim_{n\to\infty} j_n = \lim_{n\to\infty} (n + 1 - i_n - j_n) = \infty$. Potom má statistika $\varrho_{(i_n j_n)}$ asymptoticky normální rozdělení*

$$N\left(\sigma \sum_{s=j_n}^{n-i_n} \frac{1}{s} \,,\ \sigma^2 \sum_{s=j_n}^{n-i_n} \frac{1}{s^2}\right).$$

Věta 2. *Je-li*

$$\lim_{n\to\infty} \frac{j_n}{n} = q\,, \quad \lim_{n\to\infty} \frac{i_n}{n} = p$$

a je-li splněno (5.1), *pak mohutnost testu pomocí* $\varrho_{(i_n j_n)}$ *při alternativách* (6.1) *konverguje k číslu* (6.4), *kde* Φ *je normovaná normální distribuční funkce a* t_α *je řešením rovnice* $1 - \Phi(t_\alpha) = \alpha$. *Výraz* (6.4) *dosahuje maxima při* $p = 0$, $q = 0{,}203\ldots$

Резюме

СТАТИСТИЧЕСКИЙ КОНТРОЛЬ ОТКЛОНЕНИЙ ФОРМЫ ДЕТАЛЕЙ ОТ ОКРУЖНОСТИ

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(Поступило в редакцию 27/III 1958 г.)

На основании опытов можно предполагать, что отклонение формы ξ распределено по закону (1.1) (WEIBULL) с параметрами σ и β, из которых β не изменяется в течении производства. Превышению верхнего предельного отклонения формы соответствует альтернативная гипотеза $H_1 : \sigma > \sigma_0$ против нулевой гипотезы $H_0 : \sigma \leqq \sigma_0$. К проверке нулевой гипотезы предложены два критерия: равномерно наиболее мощный критерий (1), пользующийся статистикой $\frac{1}{n}\sum_{k=1}^{n} \xi_k^\beta$, и критерий (2), пользующийся размахом $\varrho_{(ij)} = \xi_{(n+1-j)}^\beta - \xi_{(i)}^\beta$ между $(n + 1 - j)$-ым и i-ым членами вариационного ряда $\xi_1^\beta, \xi_2^\beta, \ldots, \xi_n^\beta$.

К применению критерия (1) удобна критическая область

$$\frac{1}{n}\sum_{k=1}^{n} \xi_k^\beta > DT_\varepsilon^\beta$$

(коэффициент D табелирован в таб. 3), а к применению критерия (2) удобна очень простая область

$$\sqrt[\beta]{\varrho_{(oj)}} = \xi_{(n+1-j)} > \sqrt[\beta]{D_j^*}\,.\,T_\varepsilon\,, \quad j \approx 0{,}203n$$

(коэффициент D_j^* табелирован в таб. 4), где ε — дозволенный процент брака.

В связи с выбором критерия (2) доказаны следующие теоремы:

Теорема 1. *Пусть* i_n *и* j_n *зависит от* n *так, что* $\lim\limits_{n\to\infty} j_n = \lim\limits_{n\to\infty} (n + 1 - i_n - j_n) = \infty$. *Тогда статистика* $\varrho_{(i_n j_n)}$ *распределена по асимптотически нормальному закону*

$$N\left(\sigma \sum_{s=j_n}^{n-j_n} \frac{1}{s}, \quad \sigma^2 \sum_{s=j_n}^{n-i_n} \frac{1}{s^2}\right).$$

Теорема 2. *Если*

$$\lim_{n\to\infty} \frac{j_n}{n} = q, \quad \lim_{n\to\infty} \frac{i_n}{n} = p$$

и если выполнено (5.1), *то мощность критерия* $\varrho_{(i_n, j_n)}$ *для альтернативной гипотезы* (6.1) *сходится к числу* (6.4), *где* Φ — *нормированная нормальная функция распределения и* t_α *выполняет равенство* $1 - \Phi(t_\alpha) = \alpha$. *Выражение* (6.4) *максимально, когда* $p = 0$, $q = 0{,}203 \ldots$