Czechoslovak Mathematical Journal

Jaroslav Lukeš; Ivan Netuka; Jiří Veselý

Sixty years of Josef Král

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 4, 751–765

Persistent URL: http://dml.cz/dmlcz/102507

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

SIXTY YEARS OF JOSEF KRÁL

JAROSLAV LUKEŠ, IVAN NETUKA, JIŘÍ VESELÝ, Praha

Ten years ago, in December 1981, a "Harmonic Afternoon" was taking place at the Faculty of Mathematics and Physics of Charles University. The session included lectures from various domains of the potential theory. Among the participants of the seminar was also the fresh quinquagenarian Josef Král, who had devoted a considerable part of his research activities to the theory of harmonic functions, and

whose personal qualities are a paragon of harmony. Apparently everyone who has known him would confirm that no disharmony ever occurred at any meeting with him. The authors of this paper are convinced that it would be possible even to prove the following conjecture: *If a majority of people were like Josef Král, no controversies could occur and the world would be the most harmonic of all possible ones.*

Born on December 23, 1931 in a village Dolní Bučice near Čáslav, Josef Král graduated from the Faculty of Mathematics and Physics of Charles University in 1954 and became Assistant of its Department of Mathematics and soon also a research student (aspirant). In 1960 he defended his dissertation *On Lebesgue area of closed surfaces* and was granted the CSc. (Candidate of Science) degree. In 1965 he joined the Mathematical Institute of the Czechoslovak Academy of Sciences as a research worker in the Department of Partial Differential Equations, and in 1980 he was appointed head of the Department of Mathematical Physics. Meanwhile, in 1967, he had submitted the dissertation *Fredholm Method in Potential Theory* for the DrSc. (Doctor of Science) degree which he successfully defended. Approximately at the same time he also submitted his habilitation thesis *Heat Flows and the Fourier Problem*. Considering the extraordinarily high level of the thesis as well as the prominence both of his research work and his teaching activities at the Faculty, the Scientific Board of the Faculty proposed to appoint J. Král Professor in 1969. However, it took twenty years (sic!) before the changes in the country made it possible for J. Král to be actually appointed Professor for mathematical analysis in 1990.

Although J. Král has been affiliated to the Mathematical Institute for more than 25 years, he has never broken his contacts with the Faculty. His teaching activities have been of extraordinary extent. All the time he has lectured courses – both elementary and advanced – in the theory of integral and differential equations, measure theory, potential theory. He was supervisor of a number of diploma theses, member of committees for final, rigorous and doctoral examinations, author and co-author of a four-volume lecture notes on potential theory ([63], [71], [79], [81]). He has been frequently invited to give talks at conferences and universities abroad, and spent longer periods as visiting professor at Brown University in Providence, U.S.A. (1965–66), University Paris VI, France (1974), and University in Campinas, Brazil (1978).

In 1967 Josef Král founded a seminar in mathematical analysis, directed above all to potential theory. He educated a number of students and formed a group of collaborators round himself, which has been called "The Prague Harmonic Group" by friends and colleagues. The results of the group soon found international response and contacts were started with many world-famous specialists in the potential theory. Among those who came to Prague were prominent personalities as M. Brelot, H. Bauer, A. Cornea, G. Choquet or B. Fuglede, and still others came to Prague in 1987 when an international conference devoted specially to the potential theory was held here.

Let us now have a more detailed look at the research activities and scientific results of Josef Král. Its essential part belongs to mathematical analysis, in particular to the theory of measure and integral and to the potential theory. The early papers of Josef Král appear in the general scientific atmosphere of the late fifties, being strongly influenced by prominent mathematicians of the time, especially J. Mařík, V. Jarník and E. Čech. The papers concern primarily the geometric measure theory.

In papers [1], [2], [4], [5], [7], [57], [13], [78] Král studied curvilinear and surface integrals. As an illustration let us present a result following from [2], which was included in the lecture notes [63]: Let f: $[a, b] \to \boldsymbol{R}^2$ be a continuous closed parametric curve of finite length, $f([a, b]) = K$, and let $\text{ind}_f\, z$ denote the index of a point $z \in \boldsymbol{R}^2 \setminus K$ with respect to the curve f. For p integer set $G_p = \{z \in \boldsymbol{R}^2 \setminus K;\ \text{ind}_f\, z = p\}$, $G = \bigcup_{p \neq 0} G_p$. Let $\omega = G \to \bar{\boldsymbol{R}}$ be a locally integrable function and $v = (v_1 v_2)$: $K \cup G \to \boldsymbol{R}^2$ a continuous vector function. If

$$\int_{\partial R} (v_1\, dx + v_2\, dy) = \int_R \omega\, dx\, dy$$

for every closed square $R \subset G$ with positively oriented boundary ∂R, then for every $p \neq 0$ there exists an appropriately defined improper integral $\iint_{G_p} \omega\, dx\, dy$, and the series

$$\sum_{p=1}^{\infty} p\left(\iint_{G_p} \omega\, dx\, dy - \iint_{G_{-p}} \omega\, dx\, dy\right)$$

(which need not converge) is summable by Cesàro's method of arithmetic means to the sum $\int_f (v_1\, dx + v_2\, dy)$.

Transformation of integrals was studied in [55], [3] and [61]. The last paper deals with the transformation of the integral with respect to the k-dimensional Hausdorff measure on a smooth k-dimensional surface in $\boldsymbol{R}^m$ to the Lebesgue integral in $\boldsymbol{R}^k$ (in particular, it implies the Substitution Theorem for Lebesgue integral in $\boldsymbol{R}^m$). Substitution Theorem for one-dimensional Lebesgue-Stieltjes integrals is proved in [3]. As a special case one then obtains a Banach-type theorem on the variation of a composed function which, as Solomon Marcus pointed out (Zentralblatt f. Math. *80* (1959), p. 271), implies the negative answer to one problem of H. Steinhaus from The New Scottish Book. To this category belongs also [6], where Král constructed an example of a mapping T: $D \to \boldsymbol{R}^2$ absolutely continuous in the Banach sense on a plane domain $D \subset \boldsymbol{R}^2$, for which the Banach indicatrix $N(\cdot, T)$ on $\boldsymbol{R}^2$ has an integral strictly greater than the integral over D of the absolute value of Schauder's generalized Jacobian $J_s(\cdot, T)$. In this way Král solved the problem posed by T. Radó in his monograph Length and Area (Amer. Math. Soc. 1948, (i) on p. 419). The papers [56], [9], [10], [11], [12], [15] deal with surface measures; [9] and [10] are in fact parts of the above mentioned CSc. dissertation, in which Král (independently of W. Fleming) solved the problem on the relation between the Lebesgue area and perimeter in the three-dimensional space, proposed by H. Federer in Proc. Amer. Math. Soc. *9* (1958), 447–451. In [11] a question of E. Čech from The New Scottish Book, concerning the area of a convex surface in the sense of A. D. Alexandrov, was answered.

Papers [14] and [43] are from the theory of integral. The former yields a certain generalization of the Fatou Lemma: *If $\{f_n\}$ is a sequence of integrable functions on a space X with a σ-finite measure μ such that for each measurable set $M \subset X$*

the sequence $\{\int_M f_n \, d\mu\}$ *is bounded from above, then the function* $\liminf f_n^+$ *is* μ*-integrable* (although the sequence $\{\int_X f_n^+ \, d\mu\}$ need not be bounded). In the latter paper Král proved a theorem on dominated convergence for nonabsolutely convergent GP-integral answering a question of J. Mawhin from Czechoslovak Math. J. *106* (1981), 614–632.

In [16] J. Král studied the relation between the length of a generally discontinuous mapping f: $[a, b] \to P$ with values in a metric space P and the integral of the Banach indicatrix with respect to the linear measure on $f([a, b])$. For continuous mappings f the result gives the affirmative answer to a question formulated by G. Nöbeling in 1949.

In [27] it is proved that functions satisfying the integral Lipschitz condition coincide with functions of bounded variation in the sense of Tonelli-Cesari, the paper [78] presents a counterexample to the converse of the Green Theorem. Finally, [52] provides an elementary characterization of harmonic functions in a circle representable by the Poisson integral of a Riemann-integrable function.

Still another paper from the measure and integration theory is [33], in which Král gives an interesting solution of the mathematical problem on hair (formulated by L. Zajíček): *For every open set* $G \subset \boldsymbol{R}^2$ *there is a set* $H \subset G$ *of full measure and a mapping assigning to each point* $x \in H$ *an* arc $A(x) \subset G$ *with the end point* x, *such that* $A(x) \cap A(y) = \emptyset$ *provided* $x \neq y$.

METHOD OF INTEGRAL EQUATIONS IN POTENTIAL THEORY

In [58] Král began to study the methods of integral equations and their application to the solution of the boundary-value problems of the potential theory. The roots of the method reach back into the last century and are connected, among others, with the names of C. Neumann, H. Poincaré, A. M. Ljapunov, I. Fredholm or J. Plemelj. The generally accepted opinion of the necessity of strongly restrictive assumptions on smoothness of the boundary for the essential properties of integral equations, expressed for example in the monographs of F. Riesz and B. Sz.-Nagy, R. Courant and D. Hilbert, or B. Epstein, culminated in the belief that for the planar case this method had reached the natural limits of its applicability in the results of J. Radon, being unsuitable for domains with nonsmooth boundaries. Let us note that, nonetheless, the method itself offers some advantages: when used, it beautifully exhibits the duality of the Dirichlet and the Neumann problem, provides an integral representation of the solution and – as was shown lately – is suitable also from the viewpoint of numerical calculations.

In order to describe Král's results it is suitable to define an extremely useful quantity introduced by himself, the so called cyclic variation. If $G \subset \boldsymbol{R}^m$ is an arbitrary open set with a compact boundary and $z \in \boldsymbol{R}^m$, let us denote by $p(z; \theta)$ the halfline with the initial point z and direction $\theta \in \Gamma := \{\theta \in \boldsymbol{R}^m; \ |\theta| = 1\}$. For every $p(z; \theta)$ we calculate the number of points that are essential hits of $p(z; \theta)$ at ∂G; they are the points from $p(z; \theta) \cap \partial G$ in each neighborhood U of which on this halfline there are sufficiently many (in the sense of one-dimensional (Hausdorff) measure H_1) points from both G and $\boldsymbol{R}^m \setminus G$, that is

$$H_1(U \cap p(z; \theta) \cap G) > 0\,, \quad H_1(U \cap p(z; \theta) \cap (\boldsymbol{R}^m \setminus G)) > 0\,.$$

Let us denote by $n_r(z, \theta)$ the number of the hits of $p(z; \theta)$ at ∂G whose distance from z is at most r, and define the value of $v_r^G(z)$ as the average number of hits $n_r(z, \theta)$ with respect to all possible halflines with the origin at z, i.e.

$$v_r^G(z) = \int_\Gamma n_r(z, \theta)\, \mathrm{d}\sigma(\theta)\,,$$

the integral being taken with respect to the (normalized) surface measure σ on Γ. For $r = \infty$ we write more briefly $v^G(z) := v_\infty^G(z)$. From the viewpoint of application of the method of integral equations it is appropriate to consider the following questions: (1) how general are the sets for which it is possible to introduce in a reasonable way the double-layer potential (the kernel is derived from the fundamental solution of the Laplace equation) or, as the case may be, the normal derivative of the single-layer potential defined by a charge placed on the boundary ∂G; (2) under what conditions is it possible to extend this potential (continuously) from the domain onto its boundary; (3) when is it possible to solve operator equations defined by this extension?

The answer to the first two questions has the form of necessary and sufficient conditions formulated in terms of the function v^G. In [58] Král definitively solved problem (2) with the simultaneous use of the so called radial variation; the both quantities have their inspiration source in the Banach indicatrix. In this place let us notice that the Dirichlet problem is easily formulated even for domains with nonsmooth boundaries, while for the Neumann problem such sets show principal difficulties from the very beginning, regardless of the method used. Therefore it was necessary to pass in the formulation from the description in terms of a point function in the boundary condition to a description using the potential flow induced by the charge on the boundary.

By the method of integral equations, the Dirichlet and Neumann problems are solved undirectly: the solution is looked for in the form of a double-layer or single-layer potential. By the theorems on jumps these problems are reduced to the solution of the operator equations

$$\overline{W}^G f = g \quad \text{and} \quad \overline{N}\overline{U}\mu = \nu$$

where f, g are respectively the sought and the given function, μ and ν are respectively the sought and the given charge on the boundary ∂G. The potentials considered are harmonic functions on G, with the function g and the charge ν giving the boundary conditions. For simplification let us consider only the more transparent, even if mathematically less interesting, case of the Dirichlet problem, which corresponds to the former from the above equations. Let us consider three quantities of the same nature which are connected with the solvability of problems (1)–(3) and which are all derived from the cyclic variation introduced above:

(a) $v^G(x)$,

(b) $V^G := \sup\{v^G(\xi);\ \xi \in \partial G\}$,

(c) $v_0^G := \limsup_{r \to 0+} \{v_r^G(\xi);\ \xi \in \partial G\}$.

While in [58] the starting point is the set $G \subset \boldsymbol{R}^2$ bounded by a curve K of finite length, the subsequent papers [22], [64] already from the beginning consider an arbitrary open set G with compact boundary ∂G. In [58] Král solved problem (2), which opened the way to a generalization of Radon's results for curves with bounded rotation. The radial variation of a curve is also introduced here, and both variations are used in [60], [17] for studying angular limits of the double-layer potential. The results explicitly determine the value of the limit and give geometrically visualizable criteria which are necessary and sufficient conditions of existence of these limits. The mutual relation of the two quantities and their relation to the length and boundary rotation of curves is studied in [59], [17]. For the plane case the results are collected in a very comprehensive paper in two parts [20], [21], where the interrelations of the results are explained and conditions of solvability of the resulting operator equations are given.

Let us present these conditions explicitly for the dimension $m \geqq 3$. If $G \subset \boldsymbol{R}^m$ is a set with a smooth boundary ∂G, then the double-layer potential Wf with a continuous density f defined on ∂G is defined by the formula

$$Wf(x) := \int_{\partial G} f(y) \frac{(y - x)\, n(y)}{|x - y|^m} \, \mathrm{d}H_{m-1}(y), \quad x \in \boldsymbol{R}^m \setminus \partial G,$$

where $n(y)$ is the vector of the (outer) normal to G at the point $y \in \partial G$. For $x \notin \partial G$ the value $W\varphi(x)$ can be defined distributively for an arbitrary open G with compact boundary and for every smooth function φ; this value is the integral with respect to a certain measure (dependent on x) iff the quantity (a) is finite. Then $Wf(x)$ can naturally be defined for a sufficiently general f by the integral of f with respect to this measure.

Consequently, if we wish to define a generalized double-layer potential on G, the value of (a) must be finite for all $x \in G$. In fact, it suffices that $v^G(x)$ be finite on a finite set of points x from G which, however, must not lie in a single hyperplane; then the set G already has a finite perimeter. On its essential boundary (a certain essential part of boundary) the normal can be defined approximatively. This fact favourably projects in the situation: the formula for calculation of Wf remains valid if the classical normal occuring in it is replaced by the generalized normal in Federer's sense. If the quantity V^G from (b) is finite, then the same holds also for the function in (a) everywhere in G, and Wf can be continuously extended from G to $\bar{G}$ for every f continuous on ∂G. This condition is again a necessary and sufficient one; hence the solution of the Dirichlet problem can be obtained by solving the first of the above mentioned operator equations. Here the operator $\overline{W}^G f$ is defined by the limit values of $W^G f$ at the points of the boundary ∂G. A similar situation, which we will not describe in detail, occurs for the dual equation with the operator $\overline{N}\overline{U}^G$.

These results (generalizing the previous ones to the multidimensional case) can be found in [22], [64], where also the solvability of the equations in question is studied. Here Král deduced a sufficient condition of solvability depending on the magnitude of the quantity in (c), by means of which he explicitly expresses the so called Fredholm radius of certain operators related to those appearing in the equations considered. It is worth mentioning that the mere smoothness of the boundary does not guarantee the finiteness of the quantities in (b) or (a); cf. [23].

Considering the numerous similar properties of the Laplace equation and the heat equation it is natural to ask whether Král's approach (fulfilling the plan traced out by Plemelj) can be used also for the latter. Replacing in the definition of v^G the pencil of halflines filling the whole space $\boldsymbol{R}^m$ by a pencil of parabolic arcs filling the halfspace of $\boldsymbol{R}^{m+1}$ that is in time "under" the considered point (x, t) of the timespace, we can arrive at analogous results also for the heat equation. Only a deeper insight into the relation and distinction of the equations enables us to feel that the procedure had to be essentially modified in order to obtain comparable results; see [65], [24]. It is to be mentioned that the cyclic variation introduced by Král has proved to be a good tool for the study of further problems, for instance those connected with the Cauchy integral; see [25], [28].

It should be also noticed that, apart from the lecture notes mentioned above, Král later in the monograph [38] presented a selfcontained survey of the above described results — this book provides the most comfortable way for a reader to get acquainted with the results for the Laplace equation. This monograph includes also some new results: for example, if the quantity in (c) is sufficiently small, then G has only a finite number of components — this is one of the consequences of the Fredholm method, cf. [73], [38]. Part of the publication is devoted to results of [35] concerning the contractivity of the Neumann operator, which is connected with the numerical solution of boundary value problems, a subject more than 100 years old. The solution is again a definitive one.

The subject of the papers [76], [80], [86], [47] belongs to the field of application of the method of integral equations; they originated in connection with some invitations to deliver lectures at conferences and symposia. Král further developed the above described methods and, for instance, in [86] indicated the applicability of the methods also to the "infinite-dimensional" Laplace equation.

The last period is characterized by Král's return to the original problems from a rather different viewpoint. The quantity in (c) may be relatively small for really complicated sets G, but can be unpleasantly large for some even very simple sets arising for example in $\boldsymbol{R}^m$ as finite unions of parallelepipeds. Even for this particular case the solution is already known. It turned out that an appropriate re-norming leads to a desirable reduction of the Fredholm radius (the tool used here is a "weighted" cyclic variation); see [44], [48].

A characteristic feature of Král's results concerning the boundary value problems is that the analytical properties of the operators considered are expressed in visualizable geometrical terms. For the planar case see, in particular, [46].

REMOVABLE SINGULARITIES

Let us now pass to Král's contribution to the study of removable singularities of solutions of partial differential equations.

Let $P(D)$ be a partial differential operator with smooth coefficients defined in an open set $U \subset \boldsymbol{R}^m$ and let $L(U)$ be a set of locally integrable functions on U. A relatively closed set $F \subset U$ is said to be removable for $L(U)$ with respect to $P(D)$ if the following implication holds: if $h \in L(U)$ is such that $P(D)\, h = 0$ on $U \setminus F$ (in the sense of distributions), then $P(D)\, h = 0$ on the whole set U.

As an example let us consider the case when $P(D)$ is the Laplace operator in $\boldsymbol{R}^m$, $m > 2$, and L is one of the following two sets of functions: (1) continuous functions on U; (2) functions satisfying the Hölder condition with an exponent $\gamma \in (0, 1)$. It is known from classical potential theory that in the case (1) a set is removable for $L(U)$ iff it has zero Newton's capacity. For the case (2) L. Carleson (1963) proved that a set is removable for $L(U)$ iff its Hausdorff measure of dimension $\gamma + m - 2$ is zero.

In [67] Král obtained a result of Carleson's type for the solution of the heat equation. Unlike the Laplace operator, the heat operator fails to be isotropic. Anisotropy enters Král's result in two ways: first, Hölder condition is considered with the exponents γ and $\frac{1}{2}\gamma$ with respect to the spatial and the time variables, respectively, and second, anisotropic Hausdorff measure is used. Roughly speaking, the intervals used for covering have a length of edge s in the direction of the space coordinates, and s^2 along the time axis. The paper was the start of an extensive project the aim of which is to master removable singularities for more general differential operators and wider scales of function spaces.

Let M be a finite set of multiindices and let the operator

$$P(D) = \sum_{\alpha \in M} a_\alpha D^\alpha$$

have infinitely differentiable complex-valued coefficients on an open set $U \subset \boldsymbol{R}^m$. Let us choose a fixed m-tuple $n = (n_1, n_2, \ldots, n_m)$ of positive integers such that

$$|\alpha : n| = \sum_{k=1}^{m} \alpha_k / n_k \leqq 1$$

for every multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in M$.

Let us recall that an operator $P(D)$ is called semielliptic if $\xi = (\xi_1, \xi_2, \ldots, \xi_m) = 0$ is the only real-valued solution of the equation

$$\sum_{|\alpha : n| = 1} a_\alpha \xi^\alpha = 0\,.$$

(Of course, we set $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2} \dots \xi_m^{\alpha_m}$ for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$. The class of semielliptic operators includes, among others, the elliptic operators, the parabolic operators in the sense of Petrovskij (in particular, the heat operator), as well as the Cauchy-Riemann operator.

For n fixed and $\bar{n} = \max\{n_k;\ 1 \leqq k \leqq m\}$ the operator $P(D)$ is assigned the metric

$$\varrho(x, y) = \max\{|x_k - y_k|^{n_k/\bar{n}};\ 1 \leqq k \leqq m\}, \quad x, y \in \boldsymbol{R}^m.$$

To each measure function f, a Hausdorff measure on the metric space $(\boldsymbol{R}^m, \varrho)$ is associated in the usual way. Roughly speaking, this measure reflects the possibly different behaviour of $P(D)$ with respect to the individual coordinates, and it was by the measures of this type that J. Král succeeded in characterizing the removable singularities for a number of important and very general situations.

Removable singularities are studied in [30] (see also [72)] for anisotropic Hölder classes, in [75] for classes with a certain anisotropic modulus of continuity; in the latter case the measure function for the corresponding Hausdorff measure is derived from the modulus of continuity. In [77] Hölder conditions of integral type (covering Morrey's and Campanato's spaces as well as the BMO) are studied.

The papers [39] and [42] go still farther: spaces of functions are investigated whose prescribed derivatives satisfy conditions of the above mentioned types.

For general operators Král proved that the vanishing (or, as the case may be, the σ-finiteness) of an appropriate Hausdorff measure is a sufficient condition of removability for the given set of functions. (Let us point out that, when constructing the appropriate Hausdorff measure, the metric ϱ reflects the properties of the operator $P(D)$, while the measure function reflects the properties of the class of the functions considered.)

It is remarkable that for semielliptic operators with constant coefficients Král proved that the above sufficient conditions are also necessary. An additional restriction for the operators is used to determine precise growth conditions for the fundamental solution and its derivatives. The method of potential theory (combined with a Frostman-type assertion on the distribution of measure), which is applied in the proof of necessary conditions, is very well explained in [99]. In the same work also the results on removable singularities for the wave operator are presented; see [53] and [50] dealing with related topics.

In the conclusion of this section let us demonstrate the completeness of Král's research by the following result for elliptic operators with constant coefficients, which is a consequence of the assertions proved in [42]: the removable singularities for functions that togehter with their certain derivatives belong to a suitable Campanato space, are characterized by the vanishing of the classical Hausdorff measures, whose dimension (in dependence on the function space) fills in the whole interval between 0 and m.

The theory of harmonic spaces started to develop in the sixties. Its aim was to build up an abstract potential theory that would include not only the classical potential theory but would also make it possible to study wide classes of partial differential equations of elliptic and parabolic types. Further development showed that the theory of harmonic spaces represents an appropriate link between the partial differential equations and the stochastic processes.

In the abstract theory the role of the Euclidean space is played by a locally compact topological space (this makes it possible to cover manifolds and Riemann surfaces and simultaneously to exploit the theory of Radon measures), while the solution of a differential equation is replaced by a sheaf of vector spaces of continuous functions satisfying certain natural axioms. One of them, for example, is the axiom of basis, which guarantees the existence of a basis of the topology consisting of sets regular for the Dirichlet problem, or the convergence axiom, which is a suitable analogue of the classical Harnack's theorem.

Král's intention probably was not to systematically work in the theory of harmonic spaces. However, he realized that this modern and developing branch of the potential theory must not be neglected. In his seminar he gave a thorough report on Bauer's monograph *Harmonische Räume und ihre Potentialtheorie*, and later on the monograph of C. Constantinescu and A. Cornea *Potential Theory on Harmonic Spaces*.

In Král's list of publications there are four papers dealing with harmonic spaces. In [32] an affirmative answer is given to the problem of J. Lukeš concerning the existence of a nondegenerate harmonic sheaf with Brelot's convergence property on a connected space which is not locally connected. The paper [26] provides a complete characterization of sets of ellipticity and absorbing sets on one-dimensional harmonic spaces. All noncompact connected one-dimensional Brelot harmonic spaces are described in [31]. In [29], harmonic spaces with the following continuation property are investigated: Each point is contained in a domain D such that every harmonic function defined on an arbitrary subdomain of D can be harmonically continued onto the whole D. It is shown that a Brelot space X has this property iff it has the following simple topological structure: for every $x \in X$ there exist arcs $C_1, C_2, \ldots, C_n$ such that $\bigcup \{C_j;\ 1 \leqq j \leqq n\}$ is a neighborhood of x and $C_j \cap C_k = \{x\}$ for $1 \leqq j < k \leqq n$.

The papers [41] and [37] are devoted to potentials of measures. In [41] it is shown that for the kernels K satisfying the domination principle, the following continuity principle is valid: If ν is a charge whose potential $K\nu$ is finite, and if the restriction of $K\nu$ to the support of the charge ν is continuous, then the potential $K\nu$ is necessarily continuous on the whole space. In the case of a measure (i.e. non-negative charge) this is the classical Evans-Vasilesco theorem. However, this theorem does not yield (by passing to the positive and negative parts) the above assertion, since "cancellation of discontinuities" may occur.

In [37] a proof is given of a necessary and sufficient condition for measures ν on $\boldsymbol{R}^m$ to have the property that there exists a nontrivial measure ϱ on $\boldsymbol{R}$ such that the heat potential of the measure $\nu \otimes \varrho$ locally satisfies the anisotropic Hölder condition.

In [45] the size of the set of fine strict maxima of functions defined on $\boldsymbol{R}^m$ is studied. Let us recall that the fine topology in the space $\boldsymbol{R}^m$, $m > 2$, is defined as the coarsest one among all topologies for which all potentials are continuous. For $f: \boldsymbol{R}^n \to \boldsymbol{R}$ let us denote by $M(f)$ the set of all points $x \in \boldsymbol{R}^m$ which have a fine neighborhood V such that $f < f(x)$ on $V \setminus \{x\}$. It is shown in [45] that the set $M(f)$ has zero Newton capacity provided f is a Borel function.

In [40] Král proved the following theorem of Radó's type for harmonic functions (and in this way verified Greenfield's conjecture): If h is a continuously differentiable function on an open set $G \subset \boldsymbol{R}^m$ and h is harmonic on the set $G_h = \{x \in G;\, h(x) \neq 0\}$, then h is harmonic on the whole set G. In this case the set G_h on which h is harmonic, satisfies $h(G \setminus G_h) \subset \{0\}$. For various function spaces, Král [40] characterized, in terms of suitable Hausdorff measures, the sets $E \subset \boldsymbol{R}$ for which the condition $h(G \setminus G_h) \subset E$ guarantees that h is harmonic on the whole set G.

An analogue of Radó's theorem for differential forms and for the solutions of elliptic differential equations is proved in [51].

The papers [82], [36] do not directly belong to the potential theory, being only loosely connected with it. They are devoted to the estimation of the analytic capacity by means of the linear measure. For a compact set $Q \subset \boldsymbol{C}$ and for $z \in \boldsymbol{C}$ let us denote by $v^Q(z)$ the average number of points of intersection of the halflines originating at z with Q and set $V(Q) = \sup \{v^Q(z);\, z \in Q\}$. Let us present the main result of [36]: If $Q \subset \boldsymbol{C}$ is a continuum and $K \subset Q$ is compact, then the following inequality holds for the analytic capacity $\gamma(K)$ and the linear measure $m(K)$:

$$\gamma(K) \geqq \frac{1}{2\pi} \frac{1}{2\,V(Q) + 1}\, m(K)\,.$$

We do hope that we have succeeded in at least indicating the depth and elegance of Král's mathematical results. Many of them are of definitive character and provide final solution of important problems. The way in which Král presents his results shows his conception of mathematical exactness, perfection and beauty.

His results and their international response, together with his extraordinarily successful activities in mathematical education, have placed Josef Král among the most prominent Czechoslovak mathematicians of the post-war period. His modesty, devotion and humble respect in front of the immensity of Mathematics have made him an exceptional person.

His personal qualities combined with his talent, but above all with extreme industriousness, persistence and dedication to Mathematics, form the background of Král's successful career. Since even for Josef Král (whose name means "King" in Czech), there was no royal way to Mathematics.

PUBLICATIONS CONTAINING NEW RESULTS WITH COMPLETE PROOFS

[1] Der Greensche Satz, Czechoslovak Math. J. *7* (1957), 235—247, with *J. Mařík.*
[2] On curvilinear integrals in the plane (Russian), Czechoslovak Math. J. *7* (1957), 584—598.
[3] Transformation of the Stieltjes-Lebesgue integral (Russian), Czechoslovak Math. J. *8* (1958), 83—93, with *J. Marek.*
[4] On Lipschitzian mappings from the plane into Euclidean spaces, Czechoslovak Math. J. *8* (1958), 257—266.
[5] Note to the Gauss-Ostrogradskij formula (Czech), Časopis Pěst. Mat. *84* (1959), 283—292.
[6] Note on strong generalized jacobians, Czechoslovak Math. J. *9* (1959), 429—439.
[7] Closed systems of mappings and the surface integral (Russian), Czechoslovak Math. J. *10* (1960), 27—67.
[8] To the problem No. 2 proposed by J. Mařík (Czech), Časopis Pěst. Mat. *85* (1960), 205.
[9] Note on the sets whose characteristic function has a generalized measure for its partial derivative (Czech), Časopis Pěst. Mat. *86* (1961), 178—194.
[10] Note on perimeter of the Cartesian product of two sets (Czech), Časopis Pěst. Mat. *86* (1961), 261—268.
[11] On a problem concerning area of a convex surface (Czech), Časopis Pěst. Mat. *86* (1961), 277—287.
[12] On Lebesgue area of simple closed surfaces (Russian), Czechoslovak Math. J. *12* (1962), 44—68.
[13] Note on the Stokes formula for 2-dimensional integrals in *n*-space, Mat.-fyz. časopis Slov. Akad. Vied. *12* (1962), 280—292, with *B. Riečan.*
[14] Note on sequences of integrable functions, Czechoslovak Math. J. *13* (1963), 114—126, with *J. Jelínek.*
[15] A note on perimeter and measure, Czechoslovak Math. J. *13* (1963), 139—147.
[16] Note on linear measure and the length of path in a metric space (Czech), Acta Univ. Carolin.-Math. Phys. *1* (1963), 1—10.
[17] Some inequalities concerning the cyclic and radial variations of a plane path-curve, Czechoslovak Math. J. *14* (1964), 271—280.
[18] On the logarithmic potential of the double distribution, Czechoslovak Math. J. *14* (1964), 306—321.
[19] Non-tangential limits of the logarithmic potential, Czechoslovak Math. J. *14* (1964), 455—482.
[20] The Fredholm radius of an operator in potential theory I, Czechoslovak Math. J. *15* (1965), 454—473.
[21] The Fredholm radius of an operator in potential theory II, Czechoslovak Math. J. *15* (1965), 565—588.
[22] The Fredholm method in potential theory, Trans. Amer. Math. Soc. *125* (1966), 511—547.
[23] Smooth functions with infinite cyclic variation (Czech), Časopis Pěst. Mat. *93* (1968), 178—185.
[24] Flows of heat and the Fourier problem, Czechoslovak Math. J. *20* (1970), 556—598.
[25] On the modified logarithmic potential, Czechoslovak Math. J. *21* (1971), 76—98, with *J. Lukeš.*
[26] Elliptic points in one-dimensional harmonic spaces, Comment. Math. Univ. Carolin. *12* (1971), 453—483, with *J. Lukeš* and *I. Netuka.*
[27] Functions satisfying an integral Lipschitz conditions, Acta Univ. Carolin.-Math. Phys. *12* (1971), 51—54.
[28] Integrals of the Cauchy type, Czechoslovak Math. J. *22* (1972), 663—682, with *J. Lukeš.*
[29] Indefinite harmonic continuation, Časopis Pěst. Mat. *98* (1973), 87—94, with *J. Lukeš.*

[30] Removable singularities of solutions of semielliptic equations, Rend. Mat. (6) *6* (1973), 763—783.
[31] Brelot spaces on one-dimensional manifolds (Czech), Časopis Pěst. Mat. *99* (1974), 179—185, with *J. Lukeš*.
[32] An example of a harmonic sheaf (Czech), Časopis Pěst. Mat. *101* (1976), 98—99.
[33] To a mathematical problem on hair (Czech), Časopis Pěst. Mat. *101* (1976), 305—307.
[34] On boundary behavior of double-layer potentials (Russian), Trudy Seminara S. L. Soboleva, vol. *2*, Akad. Nauk Novosibirsk, Novosobirsk, 1976, pp. 19—34.
[35] Contractivity of C. Neumann's operator in potential theory, J. Math. Anal. Appl. *61* (1977), 607—619, with *I. Netuka*.
[36] Analytic capacity and linear measure, Czechoslovak Math. J. *28* (1978), 445—461, with *J. Fuka*.
[37] Heat sources and heat potentials, Časopis Pěst. Mat. *105* (1980), 184—191, with *S. Mrzena*.
[38] Integral Operators in Potential Theory, Lecture Notes in Mathematics, vol. 823, Springer-Verlag, 1980, 170 pp.
[39] On singularities of solutions of partial differential equations (Russian), Trudy Seminara S. L. Soboleva, vol. 1, Akad. Nauk Novosibirsk, Novosibirsk, 1983, pp. 78—89.
[40] Some extension results concerning harmonic functions, J. London Math. Soc. (2)*28* (1983), 62—70.
[41] A note on continuity principle in potential theory, Comment. Math. Univ. Carolin. *25* (1984), 149—157.
[42] Semielliptic singularities, Časopis Pěst. Mat. *109* (1984), 304—332.
[43] Note on generalized multiple Perron integral, Časopis Pěst. Mat. *110* (1985), 371—374.
[44] Some examples concerning applicability of the Fredholm-Radon method in potential theory, Aplikace matematiky *31* (1986), 293—308, with *W. Wendland*.
[45] Fine topology in potential theory and strict maxima of functions, Exposition. Math. *5* (1987), 185—191, with *I. Netuka*.
[46] Boundary regularity and normal derivatives of logarithmic potentials, Proc. Roy. Soc. Edinburgh Sect. A *106* (1987), 241—258.
[47] On the applicability of the Fredholm-Radon method in potential theory and the panel method, Proceedings of the Third GAMM-Seminar Kiel 1987, Friedr. Vieweg und Sohn, Braunschweig, 1987, pp. 120—136, with *W. Wendland*.
[48] Layer potentials on boundaries with corners and edges, Časopis Pěst. Mat. *13* (1988), 169—178, with *T. Angell* and *R. E. Kleinman*.
[49] Problème de Neumann faible avec condition frontière dans L^1, Séminaire de Théorie du Potential, Paris No 9; Lecture Notes in Mathematics, vol. 1393, Springer-Verlag, 1988, pp. 145—160.
[50] Removable singularities of the functional wave equation, Z. Anal. A. *8* (6) (1989), 495—500, with *M. Chlebik*.
[51] Extension results of the Radó type, Proceedings of the Internat. Colloquium in Complex Analysis, Bucharest 1989, in print.
[52] Poisson integrals of Riemann integrable functions, with *W. F. Pfeffer*; in print.
[53] Removability of the wave singularities in the plane, with *M. Chlebik*; in print.

OTHER PUBLICATIONS

[54] Green's theorem (Czech), Časopis Pěst. Mat. *81* (1956), 476—479, with *M. Neubauer*.
[55] Transformations of multidimensional integrals (Czech), Časopis Pěst. Mat. *83* (1958), 365—368.
[56] On Lebesgue area of closed surfaces (Russian), Czechoslovak Math. J. *9* (1959), 470—471.

[57] On r-dimensional integrals in $(r + 1)$-space, Czechoslovak Math. J. *11* (1961), 626—628.
[58] On the logarithmic potential, Comment. Math. Univ. Carolin. *3* (1962), 3—10.
[59] On cyclic and radial variations of a plane path, Comment. Math. Univ. Carolin. *4* (1963), 3—9.
[60] On angular limit values of Cauchy-type integrals (Russian), Dokl. Akad. Nauk SSSR *155* (1964), 32—34.
[61] Integration with respect to Hausdorff measure on a smooth surface (Czech), Časopis Pěst. Mat. *89* (1964), 433—448, with *J. Mařík.*
[62] On double-layer potential in multidimensional space (Russian), Dokl. Akad. Nauk SSSR *159* (1964), 1218—1220.
[63] Potential Theory I (Lecture Notes) (Czech), Státní pedagogické nakladatelství, Praha, 1965, 159 pp.
[64] On the Neumann problem in potential theory, Comment. Math. Univ. Carolin. *7* (1966), 485—493.
[65] Flows of heat, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) *46* (1969), 61—63.
[66] Limits of double layer potentials, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) *48* (1970), 39—42.
[67] Hölder-continuous heat potentials, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) *51* (1971), 17—19.
[68] To Problem No. 4 proposed by I. Babuška (Czech), Časopis Pěst. Mat. *97* (1972), 207—208.
[69] Problem No. 1 (on semiregular sets) (Czech), Časopis Pěst. Mat. *97* (1972), 334.
[70] On removable singularities of solutions of the heat equation (Russian), Primeněnije funkcionalnych metodov k krajevym zadačam matematičeskoj fiziki, Akad. Nauk Novosibirsk, Novosibirsk, 1972, pp. 103—106.
[71] Potential Theory II (Lecture Notes) (Czech), Státní pedagogické nakladatelství, Praha, 1972, with *I. Netuka* and *J. Veselý*, 348 pp.
[72] Regularity of potentials and removability of singularities of partial differential equations, Proceedings of the Conference „Equadiff 3", J. E. Purkyně University, Brno, 1972, 1972, pp. 179—185.
[73] A note on the Robin problem in potential theory, Comment. Math. Univ. Carolin. *14* (1973), 767—771.
[74] Nonlinear evolution equations and potential theory, Proceedings of a Summer School (editor), Academia, Praha, 1975.
[75] Potentials and removability of singularities, Nonlinear evolution equations and potential theory, Academia, Praha, 1975, pp. 95—106.
[76] Potentials and boundary value problems, Proceedings of the Conference "5. Tagung über Probleme der Mathematischen Physik", Karl-Marx-Stadt, 1975, pp. 484—500, cf. Correction of misprints, Comment. Math. Univ. Carolin. 17 (1976), 205—206.
[77] Singularités non essentielles des solutions des équations aux dérivées partielles, Séminare de Théorie du Potentiel Paris (1972—74); Lecture Notes in Mathematics, vol. 518, Springer-Verlag, 1974, pp. 95—106.
[78] An example to a converse of Green's theorem (Czech), Časopis Pěst. Mat. *101* (1976), 205—206.
[79] Potential Theory III (Lecture Notes) (Czech), Státní pedagogické nakladatelství, Praha, 1976, with *I. Netuka* and *J. Veselý*, 216 pp.
[80] Potential theory and Neumann's method, Mitteilungen der Mathematischen Gesellschaft der DDR *3—4* (1976), 71—79.
[81] Potential Theory IV (Lecture Notes) (Czech), Státní pedagogické nakladatelství, Praha 1977, with *I. Netuka* and *J. Veselý*, 304 pp.

[82] Analytic capacity, Proceedings of the Conference "Elliptische Differentialgleichungen, Rostock 1977", pp. 133—142.
[83] On removability of singularities in solving differential equations and on Neumann's principle (Russian), Matěrialy soveščanija po primeněniju metodov těorii funkcij i funkcionalnogo analiza k zadačam matěmatičeskoj fiziki (Alma-Ata 1976), Akad. Nauk Novosibirsk, Novísibirsk, 1978, pp. 132—140.
[84] Estimates of analytic capacity, Zapiski naučnych seminarov LOMI *81* (1978), 212—217.
[85] Boundary behavior of potentials, Proceedings of the Conference „Equadiff 4"; Lecture Notes in Mathematics, vol. 703, Springer-Verlag, 1979, pp. 205—212.
[86] Potential flows, Proceedings of the Conference „Equadiff 5"; Teubner-Texte zur Mathematik, vol. 47, Teubner-Ver., 1981, pp. 198—204.
[87] Negligible values of harmonic functions (Russian), Proceedings „Primeněnije metodov těorii funkcij i funkcional'nogo analiza k zadačam matěmatičeskoj fiziki", Jerevan, 1982, pp. 168—173.
[88] Some extension problems, Linear and Complex Analysis Problem Book; Lecture Notes in Mathematics, vol. 1043, Springer-Verlag, 1984, pp. 639—640.
[89] A conjecture concerning subharmonic functions (Czech), Časopis Pěst. Mat. *110* (1985), 415.
[90] Double layer potentials on boundaries with corners and edges, Comment. Math. Univ. Carolin. *27* (1986), 419, with *T. Angell* and *R. E. Kleinman*.
[91] In memory of Professor Marcel Brelot (Czech), Pokroky Mat. Fyz. Astronom. *33* (1988), 170—173, with *J. Lukeš, I. Netuka* and *J. Veselý*.
[92] On weak normal derivative of the double-layer potential with integrable density (Russian), Proceedings „Funkcional'nyje i čislennyje metody matematičeskoj fiziki", Kijev, 1988, pp. 107—110.
[93] Potential theory — Surveys and Problems, Lecture Notes in Mathematics, vol. 1344, Springer-Verlag, 1988, co-editor with *J. Lukeš, I. Netuka* and *J. Veselý*.
[94] Proceedings of the Conference on Potential Theory, Plenum Publishing Company, New York, 1988, co-editor with *J. Lukeš, I. Netuka* and *J. Veselý*.
[95] Contractivity of the operator of the arithmetical mean, Proceedings „Potential Theory — Surveys and Problems"; Lecture Notes in Mathematics, vol. 1344, Springer-Verlag, 1988, pp. 223—225, with *D. Medková*.
[96] Boundary regularity and potential-theoretic operators, Proceedings „Potential Theory — Surveys and Problems"; Lecture Notes in Mathematics, vol. 1344, Springer-Verlag, 1988, pp. 220—222.
[97] Fine maxima, Proceedings „Potential Theory — Surveys and Problems"; Lecture Notes in Mathematics, vol. 1344, Springer-Verlag, 1988, pp. 226—228, with *I. Netuka*.
[98] International Conference on Potential Theory (Czech), Pokroky Mat. Fyz. Astronom. *33* (1988), 108—110, with *J. Lukeš, I. Netuka* and *J. Veselý*.
[99] Hausdorff measures and removable singularities of solutions of partial differential equations (Czech), Pokroky Mat. Fyz. Astronom. *35* (1990), 319—330.
[100] Removable singularities and Hausdorff measures (Russian), Proceedings „Primeněnije metodov teorii funkcij i funkcional'nogo analiza k zadačam matematičeskoj fiziki", in print.
[101] To sixtieth anniversary of birthday of RNDr. Jaroslava Fuka, CSc. (Czech), Časopis Pěst. Mat. *114* (1989), 315—317.
[102] To seventieth anniversary of birthday of Professor Jan Mařík (Czech), Časopis Pěst. Mat. *115* (1990), 433—437, with *K. Karták* and *J. Matyska*.