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CONTINUOUS SELECTIONS OF FINITE-SET VALUED MAPPINGS

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1. INTRODUCTION

The question of existence of special (e.g. continuous or measurable) single-valued selections is one of the main questions in the theory of set-valued mappings (or multifunctions or correspondences). In the present paper we are interested in continuous selections only. Many results about continuous selections are known in the case of convex-valued mappings. Let us mention the most famous one which asserts that any lower semi-continuous mapping with nonempty closed convex values from a paracompact space into a Banach space admits a continuous selection (Michael [9], see also [10], [2], [8]).

The non-convex case is much more complicated. For example, there exists a continuous set-valued mapping F of the interval $[0, 1]$ into $\mathbb{R}^2$ such that its values are homeomorphic to $[0, 1]$, but F has no continuous selection ([7], see also [1], [2]).

The simplest class of set-valued mappings with non-convex values is probably the class of finite-valued mappings. As the main result of this paper we prove the existence of continuous selections (even in the strong sense, see below) of continuous finite-valued mappings from a locally connected treelike space X into an arbitrary Hausdorff topological space (Theorem 3.3). This result, together with a counter-example (Theorem 4.2), enables us to give a characterization of spaces X having the property that each continuous finite-valued $F: X \to 2^Y$ (with Y Hausdorff) admits a continuous selection, in a certain class of locally connected normal spaces (Theorem 4.4), which contains e.g. the family of compact locally connected spaces.

Let us start with some definitions and notation. By 2^A we denote the set of all subsets of a set A, the symbol $\# A$ means the number of elements of A ($\# A \in \in \{0, 1, 2, \ldots, \infty\}$). Let X, Y be topological spaces. We shall identify a set-valued mapping $F: X \to 2^Y$ with its graph as a subset of $X \times Y$, for instance $(x, y) \in F$ and $y \in F(x)$ will mean the same. For $F: X \to 2^Y$ (or equivalently: for $F \subset X \times Y$) we denote $D(F) = \{x \in X;\ F(x) \neq \emptyset\}$ and $F/M = F \cap (M \times Y)$ for $M \subset X$. (Clearly $D(F/M) = D(F) \cap M$.)

F is *upper semi-continuous*, shortly *u.s.c.* (*lower semi-continuous*, shortly *l.s.c.*) *at* $x_0 \in D(F)$ if for any open $V \subset Y$ with $F(x_0) \subset V$ ($F(x_0) \cap V \neq \emptyset$) there exists a neighborhood U of x_0 such that $F(x) \subset V$ ($F(x) \cap V \neq \emptyset$, respectively) for any $x \in U$. F is *continuous at* x_0 if it is both u.s.c. and l.s.c. at x_0. $F: X \to 2^Y$ is *continuous* if $D(F) = X$ and F is continuous at each point of X. F is *continuous on* $M \subset X$ if $M \subset D(F)$ and F/M is continuous as a mapping $M \to 2^Y$.

An $F: X \to 2^Y$ admits a selection if there exists a continuous (single-valued) mapping $f: X \to Y$ (*selection* of F) such that $f(x) \in F(x)$ for any $x \in X$. We shall say that F admits *selections in the strong sense* if for any $(x_0, y_0) \in F$ there exists a selection f of F with $f(x_0) = y_0$.

Let us briefly mention some known and elementary facts about selections of finite-valued mappings $F: X \to 2^Y$ with $D(F) = X$. Simple examples show that neither the upper semi-continuity nor the lower semi-continuity are sufficient for the existence of a selection. Even if we suppose a constant number of values, i.e. $\# F(x) = n$ for any $x \in X$, the selections need not exist if F is u.s.c.. For example, $X = [-1, 1]$, $Y = [-2, 2]$, $n = 2$, $F(x) = \{\operatorname{sgn}(x), x + \operatorname{sgn}(x)\}$ for $x \neq 0$, $F(0) = \{-1, 1\}$. With lower semi-continuity of F the situation is different, because together with a constant number of values it easily implies that F is continuous. The following theorem holds.

1.1. Theorem (S. Banach and S. Mazur [3]). *Let X, Y be metric spaces and let X be simply connected (i.e. any two paths in X with the same initial points and the same terminal points are homotopic in X with a homotopy which leaves the initial and terminal points constant) and locally pathwise connected. Then for any l.s.c. $F: X \to 2^Y$ with $\# F(x) = n < \infty$ for any $x \in X$, the graph of F is a disjoint union of n graphs of continuous single-valued mappings. Consequently, F admits selections in the strong sense.*

The assumption of simple connectedness of X cannot be omitted in the above theorem, as the following example illustrates. Let us remark that the same simple idea is used in Theorem 4.2.

1.2. Example. Let $S = \{z \in \mathbb{C}; |z| = 1\}$ and let $\varphi: S \to 2^S$ be "the complex square root", i.e. $\varphi(z) = \{s \in \mathbb{C}; s^2 = z\}$ for $z \in S$. Then φ is continuous and $\# \varphi(z) = 2$ for any $z \in S$. It is easy to see that φ has no selection.

If continuous finite-valued mappings (with non-constant number of values) are considered, the situation is more complicated. However, if the space of values is $\mathbb{R}$ (or homeomorphic to $\mathbb{R}$), a selection always exists.

1.3. Proposition. *Let X be any topological space and let $F: X \to 2^{\mathbb{R}}$ be continuous and finite-valued. Then the functions $f_1(x) = \min F(x)$, $f_2(x) = \max F(x)$ are continuous selections of F.*

Example 1.2 shows that the real line $\mathbb{R}$ cannot be replaced by much more general spaces. For some results in this direction see [4].

We omit an easy proof of Proposition 1.3. Let us note that it is a consequence of a more general well-known theorem: Let X, Y be topological spaces, let $F: X \to 2^Y$ be continuous and compact-valued, and let $\varphi: F \to \mathbb{R}$ be a continuous function (defined on the graph of F). Then the "marginal mapping" $F_1(x) = \{y \in F(x); \varphi(x, y) = \min \varphi(\{x\} \times F(x))\}$ is u.s.c. (cf. [2], [8]).

It is not known to the author for which spaces X in Proposition 1.3 there exist selections in the strong sense. Corollary 3.4 shows that any locally connected space in which the intersection of every pair of connected subsets is connected, is an example of such a space.

Let us recall the definition and main properties of treelike spaces. Our terminology is taken from L. E. Brouwer's book [5]. Some authors use the term "dendritic space" for treelike spaces.

1.4. Definition. A topological space X is *treelike* if it is connected and each two of its distinct points are separated by a third point, or equivalently ([5]): for any two distinct points a and b there is a point p such that a and b belong to different components of $X \setminus \{p\}$.

1.5. Theorem. ([5, Th. 4 in Ch. III, Th. 2 and Prop. 2 in Ch. II]).

For a connected locally connected space X the following assertions are equivalent.

(i) *X is treelike.*

(ii) *X is a T_1-space and the intersection of any family of its connected subsets is connected.*

(iii) *X is a Hausdorff space and the intersection of any two of its connected subsets is connected.*

(iv) *X is a Hausdorff space and $\#(\bar{A} \cap \bar{B}) \leqq 1$ whenever A and B are disjoint connected subsets of X.*

It is easy to see that any connected subspace of a treelike space is treelike [5]. Compact treelike spaces (called trees) are always locally connected [12].

Recall that an *arc* is a continuum with exactly two non-cut points (which are called endpoints of the arc), or equivalently: a non-degenerate orderable continuum. A space is *arcwise connected* if each two distinct points are the endpoints of some arc contained in the space.

2. AUXILIARY PROPOSITIONS

In this section we state several lemmas. Main results of the present paper are contained in the next two sections. Throughout all this section, Y will denote any Hausdorff topological space.

2.1. Lemma. *Let X be connected and let $F: X \to 2^Y$ be continuous and compact-valued. If A, B are two disjoint open sets in $X \times Y$ such that*

$$F \cap A \neq \emptyset, \quad F \cap B \neq \emptyset \quad \text{and} \quad F \subset A \cup B,$$

then $D(F \cap A) = D(F \cap B) = X$.

Proof. The lower semi-continuity of F easily implies that $D(F \cap A)$, $D(F \cap B)$ are both open in X. If $D(F \cap A) \neq X$ then $D(F \cap A)$ cannot be closed since X is connected. Let x_0 be an arbitrary element of $\overline{D(F \cap A)} \setminus D(F \cap A)$. Since $\{x_0\} \times F(x_0) \subset B$, there exists a neighborhood U of x_0 such that $\{x\} \times F(x) \subset B$ for all $x \in U$, because $F(x_0)$ is compact and F is u.s.c. at x_0. But this is a contradiction with $x_0 \in \overline{D(F \cap A)}$. ∎

2.2. Lemma. *Let $F: X \to 2^Y$, $x_0 \in D(F) = X$, $\# F(x_0) = n$. Then the following are equivalent.*

(i) *F is continuous at x_0.*

(ii) *There exist pairwise disjoint open subsets $V_1, \ldots, V_n$ of Y and a neighborhood U of x_0 such that the sets $F_i = F \cap (U \times V_i)$ $(i = 1, \ldots, n)$ satisfy:*

$$F/U = \bigcup_i F_i\,, \quad D(F_i) = U \quad \text{and } F_i \text{ is continuous at } x_0\,.$$

Proof. Let $F(x_0) = \{y_1, \ldots, y_0\}$. If (i) holds, take arbitrary disjoint open sets $V_1, \ldots, V_n$ in such a way that $y_i \in V_i$ for $i = 1, \ldots, n$. By the continuity of F at x_0, there exists a neighborhood U of x_0 such that

$$F(x) \subset \bigcup_{i=1}^{n} V_i\,, \quad F(x) \cap V_i \neq \emptyset \quad \text{for} \quad i = 1, \ldots, n\,.$$

whenever $x \in U$. With these $V_1, \ldots, V_n$, U the condition (ii) is satisfied. If (ii) holds, observe that necessarily each V_i contains exactly one element of $F(x_0)$. Now (i) easily follows from the continuity of F_i's. ∎

2.3. Lemma (Intersection lemma). *Let X be locally connected and let $\mathscr{F}$ be a family of continuous finite-valued mappings from X into Y, linearly ordered by inclusion. Then $f = \bigcap \mathscr{F}$ is continuous.*

Proof. Observe that $D(f) = X$ and that for any $x_0 \in X$ there exists an $F \in \mathscr{F}$ such that $F(x_0) = f(x_0)$. Let $n = \# F(x_0)$. For this F find $V_1, \ldots, V_n$ and U as in Lemma 2.2. We can suppose U to be connected. Define $F_i = F \cap (U \times V_i)$, $f_i = f \cap (U \times V_i)$ for $i = 1, \ldots, n$. Now $D(f_i) = U$, $i = 1, \ldots, n$. In fact, for any $x \in U$ there exists $F_x \in \mathscr{F}$ with $F_x \subset F$ and $F_x(x) = f(x)$. By Lemma 2.1, $D(F_x \cap (U \times V_i)) = U$ for $i = 1, \ldots, n$, and hence $x \in D(f_i)$ for $i = 1, \ldots, n$.

By Lemma 2.2, f is continuous at x_0, because the continuity of F_i at x_0 immediately implies the continuity of f_i at x_0. ∎

2.4. Lemma (Union lemma). *Let X be a locally connected treelike space and let $F: X \to 2^Y$ be continuous and finite-valued. Let $\mathscr{M}$ be a family of nonempty subsets of F, linearly ordered by inclusion, such that $D(M)$ is connected and M is continuous on $D(M)$ for any $M \in \mathscr{M}$. Then $N = \bigcup \mathscr{M}$ is continuous on the (connected) set $D(N)$.*

Proof. Clearly $D(N) = \bigcup\{D(M);\ M \in \mathscr{M}\}$ is connected and $N \subset F$. Since any connected subspace of a treelike space is treelike, we can suppose $D(N) = X$. Observe that for any $x_0 \in X$ there exists an $M \in \mathscr{M}$ such that $M(x_0) = N(x_0)$. The lower semi-continuity of M at x_0 immediately implies that N is l.s.c. at x_0. Let us prove that N is also u.s.c. at x_0. If $N(x_0) = F(x_0)$ then N is obviously u.s.c. at x_0 since F has the same property.

Let $N(x_0) = \{y_1, \ldots, y_k\}$, $F(x_0) = \{y_1, \ldots, y_n\}$, $1 \leqq k < n$. Let $W \subset Y$ be open and such that $N(x_0) \subset W$. Find $V_1, \ldots, V_n$ and U as in Lemma 2.2 for F. We can suppose that U is connected and that $V_i \subset W$ for $i = 1, \ldots, k$. Let $x \in U$ and $M_x \in \mathscr{M}$ be such that $M_x \supset M$ and $M_x(x) = N(x)$. By Theorem 1.5, the set $U \cap D(M_x)$ is connected. If $N(x) \cap V_j = M_x(x) \cap V_j \neq \emptyset$ for some $j > k$, then by Lemma 2.1 $D(M_x \cap (U \times V_j)) = D(M_x) \cap U$, and hence also $M(x_0) \cap V_j \neq \emptyset$, which contradicts the inclusion

$$M(x_0) \subset \bigcup_{i=1}^{k} V_i \,.$$

Therefore $N(x) \subset \bigcup_{i=1}^{k} V_i \subset W$ for any $x \in U$ and N is u.s.c. at x_0. ∎

2.5. Lemma (First extension lemma). *Let X be a locally connected treelike space and let $J \subset X$ be connected. Let $F: X \to 2^Y$ be continuous and compact-valued, and let $f \subset F$ be continuous on $D(f) = J$. If $x_0 \in \bar{J} \setminus J$ and $\# F(x_0) < \infty$, then there exists an $\bar{f}$ such that $f \subset \bar{f} \subset F$, $\bar{f}$ is continuous on $D(\bar{f}) = J \cup \{x_0\}$ and $\bar{f}/J = f$.*

Proof. Denote $F(x_0) = \{y_1, \ldots, y_n\}$ and find $V_1, \ldots, V_n$ and U as in Lemma 2.2. We can suppose U to be connected. Then also the set $U' = U \cap J$ is connected by Theorem 1.5. Put $\beta = \{i; 1 \leqq i \leqq n, f \cap (U' \times V_i) \neq \emptyset\}$ and define

$$\bar{f}(x) = f(x) \quad \text{for} \quad x \in J \,, \quad \bar{f}(x_0) = \{y_i; i \in \beta\} \,.$$

Let us denote $U_0 = (J \cup \{x_0\}) \cap U = U' \cup \{x_0\}$. Lemma 2.1 implies

$$D(f \cap (U' \times V_i)) = U' \quad \text{for} \quad i \in \beta \quad \text{and} \quad D(f \cap (U' \times V_i)) = \emptyset \quad \text{for} \quad i \notin \beta \,,$$

and hence $\bar{f}/U_0 = \bigcup_{i \in \beta} \bar{f}_i$ and $D(\bar{f}_i) = U_0$ for $i \in \beta$, where $\bar{f}_i = \bar{f} \cap (U_0 \times V_i)$. The continuity of $\bar{f}_i$ follows immediately from the fact that $\bar{f}_i \subset F_i := F \cap (U \times V_i)$ and $\bar{f}_i(x_0) = F_i(x_0) = \{y_i\}$, $i \in \beta$. Now, Lemma 2.2 ensures the continuity of $\bar{f}$ at x_0 (with respect to $J \cup \{x_0\}$). ∎

2.6. Lemma (Second extension lemma). *Let X be a locally connected treelike space and let J be a closed connected proper subset of X. Let $F: X \to 2^Y$ be continuous and compact-valued, and let $f \subset F$ be continuous on $D(f) = J$. If $\# F(x) < \infty$ whenever x belongs to the boundary ∂J of J, then there exists a connected set $K \subset X$ containing J as a proper subset, and an $\bar{f}$ such that $f \subset \bar{f} \subset F$, $\bar{f}$ is continuous on $D(\bar{f}) = K$ and $\bar{f}/J = f$.*

Proof. Let W be a component of $X \setminus J$. The local connectedness of X implies that W is open in X ([6]). If $\bar{W} \subset X \setminus J$, then the set W is closed, because W is closed in $X \setminus J$. But this contradicts the connectedness of X. Therefore $\bar{W} \cap J \neq \emptyset$. By Theorem 1.5 $\bar{W} \cap J = \{x_0\}$ for some $x_0 \in X$. Clearly $x_0 \in \partial J$.

Denote $F(x_0) = \{y_1, \ldots, y_n\}$ and find $V_1, \ldots, V_n$ and U as in Lemma 2.2. We can suppose U to be connected. Denote $F_i = F \cap (U \times V_i)$ and define

$$\bar{f}(x) = f(x) \quad \text{for} \quad x \in J\,,$$

$$\bar{f}(x) = \bigcup \{F_i(x);\ F_i(x_0) \subset f(x_0)\} \quad \text{for} \quad x \in U \cap W\,.$$

By Lemma 2.2, $\bar{f}$ is continuous on $U \cap \overline{W} = (U \cap W) \cup \{x_0\}$, which (together with the continuity of $\bar{f}$ on J) implies that $\bar{f}$ is continuous on the connected set $K = J \cup (U \cap W) = D(\bar{f})$, because x_0 is the only common point of the sets J, $\overline{K \setminus J}$. ■

3. EXISTENCE OF SELECTIONS

3.1. Theorem. *Let X be locally connected and let Y be a Hausdorff space. Let $F: X \to 2^Y$ be continuous, finite-valued and such that the following property is satisfied*:

(P) *for any continuous (on X) $G \subset F$ and any $(x, y) \in G$, there exists a continuous $H \subset G$ (possibly set-valued) such that $H(x) = \{y\}$.*

Then F admits selections in the strong sense.

Proof. Let $(x_0, y_0) \in F$. Let $\mathscr{A}$ be the family of all continuous $G \subset F$ with $(x_0, y_0) \in G$. Then $\mathscr{A}$ is nonempty and partialy ordered by inclusion. By the Intersection lemma 2.3 and the Zorn-Kuratowski lemma ([6]), there exists a minimal element $f \in \mathscr{A}$. The property (P) implies that f is single-valued, and hence it is the selection with $f(x_0) = y_0$. ■

3.2. Theorem. *Let X be a locally connected treelike space, Y a Hausdorff space, and let $F: X \to 2^Y$ be continuous and finite-valued. Then for any $(x, y) \in F$ there exists a continuous $H \subset F$ with $H(x) = \{y\}$.*

Proof. Let $F(x) = \{y_1, \ldots, y_n\}$ with $y_1 = y$. Find $V_1, \ldots, V_n$ and U from Lemma 2.2. We can suppose that U is connected and $y = y_1 \in V_1$. Denote $F_1 = F \cap$ $\cap (U \times V_1)$. Then the family

$$\mathscr{M} = \{M;\ F_1 \subset M \subset F,\ D(M) \text{ is connected},$$
$$M \text{ is continuous on } D(M),\ M/U = F_1\}$$

is partialy ordered by inclusion and nonempty, since $F_1 \in \mathscr{M}$. The Union lemma 2.4 and the Zorn-Kuratowski lemma ([6]) imply the existence of a maximal element $H \in \mathscr{M}$. By the First extension lemma 2.5, $D(H)$ is closed. By the Second extension lemma 2.6, $D(H) = X$. ■

3.3. Theorem (Main result). *Let X be a locally connected treelike space and Y any Hausdorff topological space. Then each continuous finite-valued mapping $F: X \to 2^Y$ admits selections in the strong sense.*

Proof. By Theorem 3.2, each continuous finite-valued F satisfies the property (P) from Theorem 3.1. ■

3.4. Corollary. *Let X be a locally connected Hausdorff space in which the intersection of any two connected sets is connected. Then any continuous finite-valued mapping from X into a Hausdorff space Y admits selections in the strong sense.*

Proof. Each component of X is open, locally connected and treelike (Theorem 1.5). ■

4. A COUNTEREXAMPLE AND CHARACTERIZATIONS

Instead of the property (iii) of Theorem 1.5 we can consider a weaker property dealing only with closed subsets of the space X. Note that we do not require that X be connected or compact in the following definition.

4.1. Definition. A topological space X is called *hereditarily unicoherent* if the intersection of any pair of its closed connected subsets is connected.

The following theorem shows that for the existence of selections on a normal space X, it is necessary that X be hereditarily unicoherent. The idea of proof is based on Example 1.2.

4.2. Theorem (Counterexample). *Let X be a normal space in which there exist closed connected subsets C_1 and C_2 such that $C_1 \cap C_2$ is not connected (i.e. X is not hereditarily unicoherent). Then there exists a continuous finite-valued $F: X \to 2^B$ into the two-dimensional ball B such that $1 \leqq \# F(x) \leqq 2$ for any $x \in X$, and F has no continuous selection.*

Proof. Since $C_1 \cap C_2$ is closed and not connected, there exist nonempty closed sets $K, L \subset X$ such that $K \cap L = \emptyset$ and $K \cup L = C_1 \cap C_2$. Put $f_0(x) = 1$ for $x \in K$ and $f_0(x) = -1$ for $x \in L$. The mapping $f_0: C_1 \cap C_2 = K \cup L \to \mathbb{C}$ is continuous. Denote $S = \{z \in \mathbb{C};\ |z| = 1\}$, $S^+ = \{z \in S;\ \operatorname{Im}(z) \geqq 0\}$, $S^- = \{z \in S;\ \operatorname{Im}(z) \leqq 0\}$. S^+ and S^- are both homeomorphic to a compact interval and $f_0(K) \subset S^+, f_0(L) \subset \subset S^-$. Hence by the Tietze-Urysohn theorem ([6]), there exist $f_1: C_1 \to S^+$, $f_2: C_2 \to S^-$, continuous extensions of f_0. The mapping $f: C_1 \cup C_2 \to S$ defined by $f/C_i = f_i$ for $i = 1, 2$ is then well-defined and continuous. The connectedness of C_i clearly implies

$$(1) \qquad f_1(C_1) = S^+ \,, \quad f_2(C_2) = S^- \,,$$
$$f(C_1 \cup C_2) = S \subset B = \{z \in \mathbb{C};\ |z| \leqq 1\} \,.$$

There exists a continuous extension $\bar{f}: X \to B$ of f (note that B is homeomorphic to a closed square in $\mathbb{R}^2$ and consider Tietze's extension of components of f). Let $\varphi: B \to 2^B$ be the "comples square root", i.e. $\varphi(z) = \{s \in \mathbb{C};\ s^2 = z\}$. Observe that φ is a continuous finite-valued mapping with $\# \varphi(z) = 2$ for $z \neq 0$ and $\varphi(0) = \{0\}$. Finally, let us define $F: X \to 2^B$ by $F = \varphi \circ \bar{f}$.

Let us denote by Q_i the i-th closed quadrant in the complex plane $\mathbb{C}$ ($i = 1, 2, 3, 4$), i.e. $Q_i = \{z \in \mathbb{C} \setminus \{0\};\ \operatorname{Arg}(z) \cap [(i-1)\pi/2,\ i\pi/2] \neq \emptyset\} \cup \{0\}$.

Suppose that F has a selection. Then also $F/C_1 \cup C_2 = \varphi \circ f$ must have a selection, let us denote it by g. Now, by (1)

(2) $$(\varphi \circ f)(C_1) = \varphi(S^+) = S \cap (Q_1 \cup Q_3) \quad \text{and}$$
$$(\varphi \circ f)(C_2) = \varphi(S^-) = S \cap (Q_2 \cup Q_4)\,.$$

The connectedness of C_i implies that $g(C_i)$ is a connected subset of $(\varphi \circ f)(C_i)$ and hence

(3) $$g(C_1) \subset S \cap Q_1 \quad \text{or} \quad g(C_1) \subset S \cap Q_3\,, \quad \text{and also}$$
$$g(C_2) \subset S \cap Q_2 \quad \text{or} \quad g(C_2) \subset S \cap Q_4\,.$$

Since $g(K) \subset (\varphi \circ f)(K) = \{1, -1\}$ and $g(K) \subset g(C_1)$, we have by (3) that g is constant on K. Similarly, g is constant on L. Let $g(K) = \{1\}$. Then by (3) we have $g(C_1) \subset S \cap Q_1$ and $g(C_2) \subset S \cap Q_4$. Conseqeently, $g(L) \subset g(C_1) \cap g(C_2) \subset S \cap \cap Q_1 \cap Q_4 = \{1\}$, which is a contradiction with $g(L) \subset (\varphi \circ f)(L) = \{\mathrm{i}, -\mathrm{i}\}$. The assumption $g(K) = \{-1\}$ leads to a contradiction in an analogous way. ∎

It is not known to the author whether any normal connected and locally connected space which is hereditarily unicoherent, must be treelike. However, some results in this direction are known.

4.3. Theorem. *Let X be a connected locally connected space satisfying at least one of the following conditions.*

(I) *X is compact.*

(II) *X is locally compact.*

(III) *X is locally arcwise connected and separable.*

(IV) *Each connected subspace of X is arcwise connected.*

Then X is treelike if and only if X is hereditarily unicoherent and Hausdorff.

Proof. In all the above cases, the necessity is an immediate consequence of the implication (i) ⇒ (iii) in Theorem 1.5. Sufficiency:

(I) See [13, Theorem 9].

(II) By (I), any subcontinuum of X is treelike. Apply [14, Corollary to Theorem 3].

(III) A hereditarily unicoherent space does not contain any simple closed curve (i.e. the union of two arcs having just only their endpoints in common). Apply 14, Theorem 2].

(IV) In a hereditarily unicoherent space each two points can be connected by at most one arc. Apply [11, Theorem 3]. ∎

These results allow us to characterize some situations in which there exist selections of continuous finite-valued mappings.

4.4. Theorem. *Let X be a locally connected normal space satisfying at least one of the conditions* (I), (II), (III), (IV) *of Theorem* 4.3. *Then the following assertions are equivalent.*

(i) *X is hereditarily unicoherent.*

(ii) *Each component of X is treelike.*

(iii) *Any continuous finite-valued* $F: X \to 2^Y$, *where* Y *is a Hausdorff space, admits selections in the strong sense.*

(iv) *Any continuous finite-valued* $F: X \to 2^Y$, *where* Y *is a Hausdorff space, admits a selection.*

(v) *Any continuous finite-valued* $F: X \to 2^B$ *with* $\# F(x) \leqq 2$ *for any* $x \in X$ *has a selection* (B *denotes the two-dimensional ball*).

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is a consequence of Theorem 4.3. Observe that each component of X is open in X. Consequently, the implication (ii) $\Rightarrow$ (iii) follows from Theorem 3.3. The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious. Finally, the implication (v) $\Rightarrow$ (i) is a consequence of Theorem 4.2. ∎

Let us explicitly mention another consequence of our results, giving the characterization of selectionability of finite-valued continuous mappings between convex sets in topological linear spaces. Let us recall that the dimension of a convex set is the dimension of its affine hull.

4.5. Proposition. *Let* K *be a convex subset of a Hausdorff locally convex topological linear space* X *and let* C *be a convex subset of a Hausdorff topological linear space* Y. *Then the following assertions are equivalent.*

(i) *Any continuous finite-valued* $F: K \to 2^C$ *has a selection.*

(ii) $\dim K \leqq 1$ *or* $\dim C \leqq 1$.

Proof. If $\dim C \leqq 1$ ($\dim K \leqq 1$) then F has a selection by Proposition 1.3 (by Theorem 3.3, respectively). If the dimensions of K and C are both greater than 1, each of them contains a triangle which is homeomorphic to the two-dimensional ball B. Let us denote these triangles by T_K and T_C. By Theorem 4.2, there exists a continuous finite-valued mapping G of T_K into T_C without selections. Since any plane in X is a retract of X and a triangle in a plane is a retract of the plane, there exists a retraction $r: K \to T_K$. Let i: $T_C \to C$ be the identity embedding. Then the mapping $F = i \circ G \circ r: K \to 2^C$ is finite-valued and continuous, but admits no selection. ∎

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References

[1] *G. Anichini, G. Conti* and *P. Zecca:* Approximation and selection for nonconvex multifunctions in infinite dimensional spaces, Bolletino U.M.I., to appear.

[2] *J.-P. Aubin* and *A. Cellina:* Differential Inclusions, Springer-Verlag, 1984.

[3] *S. Banach* and *S. Mazur:* Über mehrdeutige stetige Abbildungen, Studia Math. *5* (1934), 174—178.

[4] *C. R. Borges:* Continuous selections for one-to-finite continuous multi-functions, Questions Answers Gen. Topology *3* (1985/86), 103—109.

[5] *A. E. Brouwer:* Treelike Spaces and Related Connected Topological Spaces, Math. Centre Tracts 75, Amsterdam 1977.
[6] *R. Engelking:* General Topology, Monografie Matematyczne 60, Warszawa 1977.
[7] *A. F. Filippov:* Conditions for the existence of solutions of many valued differential equations, Differential Equations *13* (1977), 740—746.
[8] *E. Klein* and *A. C. Thompson:* Theory of Correspondences, A Wiley — Interscience publication, 1984.
[9] *E. Michael:* Continuous selections I, Ann. Math. *63* (1956), 361—382.
[10] *E. Michael:* Selected selection theorems, Amer. Math. Monthly *63* (1956), 233—238.
[11] *T. B. Muenzenberger, R. E. Smithson* and *L. E. Ward:* Characterizations of arboroids and dendritic spaces, Pacific J. Math. *102* (1982), 107—121.
[12] *L. E. Ward:* A note on dendrites and trees, Proc. Amer. Math. Soc. *5* (1954), 992—994.
[13] *L. E. Ward:* Mobs, trees, and fixed points, Proc. Amer. Math. Soc. *8* (1957), 798—804.
[14] *L. E. Ward:* On dendritic sets, Duke Math. J. *25* (1958), 505—513.
[15] *G. T. Whyburn:* Analytic Topology, Amer. Math. Soc. Colloq. Publ. *28*, New York 1942.

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