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Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 3, 538–548

Persistent URL: http://dml.cz/dmlcz/102487

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

ON A PLANAR CONSTRUCTION OF QUASIGROUPS

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(Received October 2, 1990)

For a group G and a quasigroup $Q(\cdot)$ on the same underlying set define $\operatorname{dist}(G, Q) = \operatorname{card}\{(a, b) \in G^{(2)};\ a \,.\, b \neq ab\}$ and $\operatorname{gdist}(n) = \min \operatorname{dist}(G, Q)$, where G and Q are of order $n \geqq 2$, $G \neq Q(\cdot)$.

Some estimates of gdist (n) can be found in [1] and [2]. The upper bound of gdist (n) given in this paper seems to be the best one known up to now.

1. CONFIGURATIONS

Let $\mathbf{P}$ denote the set of all $(i, j, k) \in \mathbf{Z}^{(3)}$, $i + j + k = 0$, $\mathbf{Z}$ being the set of all integers. For any $i \in \mathbf{Z}$ let $b_i = \{(i, j, k) \in \mathbf{P};\ j, k \in \mathbf{Z}\}$, $c_i = \{(j, i, k) \in \mathbf{P};\ j, k \in \mathbf{Z}\}$ and $d_i = \{(j, k, i) \in \mathbf{P}; j, k \in \mathbf{Z}\}$. A finite subset s of b_i (or c_i or d_i) is called a *segment* iff it has at least two distinct elements and $z \in s$ holds for any $x, y, z \in \mathbf{P}$ such that $z = ax + (1 - a)\, y$, $0 < a < 1$, $x, y \in s$. Two segments s, t are called *parallel* if $s \subset b_i$, $t \subset b_j$ (or $s \subset c_i$, $t \subset c_j$ or $s \subset d_i$, $t \subset d_j$) for some $i, j \in \mathbf{Z}$.

By a *configuration* we mean any finite set S of segments where $s \cap t = \emptyset$ for any parallel $s, t \in S$, $s \neq t$. The *meet* $S \wedge T$ of configurations S, T is defined by $S \wedge T = \{s \cap t;\ s \in S,\ t \in T,\ \operatorname{card}(s \cap t) \geqq 2\}$. The *join* $S \vee T$ is formed by uniting the overlapping segments of $S \cup T$. (Formally, define $\sim$ as the transitive envelope of the relation r, where $(s, t) \in r$ iff $s, t \in S \cup T$, $s \cap t \neq \emptyset$ and s, t are parallel. Segments of $S \vee T$ are the unions of equivalence classes of $\sim$.) Further $S \leqq T$ iff for any $s \in S$ there is $t \in T$ with $s \subseteq t$.

Obviously, each segment has exactly two *extreme points*. If $x = (x_i)$, $y = (y_i)$, $1 \leqq i \leqq 3$ are the extreme points of a segment s and $s \subseteq b_j$ (or $s \subseteq c_j$ or $s \subseteq d_j$) for some $j \in \mathbf{Z}$, we define $\Delta(s) = |x_2 - y_2| = |x_3 - y_3|$ (or $\Delta(s) = |x_1 - y_1| = |x_3 - y_3|$ or $\Delta(s) = |x_1 - y_1| = |x_2 - y_2|$).

We call $x \in \mathbf{P}$ a *vertex* of the configuration S, if there is such $s \in S$ that x is an extreme point of s, or $x \in s \cap t$ for some $s, t \in S$, $s \neq t$. For a configuration S define $[S] = \{x \in \mathbf{P};\ x \in s \text{ for some } s \in S\}$ to be the set of its *points*.

For $(i, j, k) \in \mathbf{Z}^{(3)}$, $i + j + k \neq 0$ we define the *triangle* τ with coordinates (i, j, k) to be the configuration of those (three) segments which have their extreme points in $\{(i, j, -i - j), (i, -i - k, k), (-j - k, j, k)\}$. We put $\Delta(\tau) = |i + j + k|$

and call the segments of the triangle its *sides*. We have $\Delta(\tau) = \Delta(s)$ for any side s of the triangle τ. If $\Delta(\tau) = 1$, τ is called a *basic* triangle.

Any finite non-empty set of basic triangles will be called a *region* and for any region L we define $\text{Compl}(L) = \bigvee \tau;\ \tau \in L$ to be its *complete configuration*. A configuration S is said to be *covered* by L iff $S \leqq \text{Compl}(L)$.

For any triangle τ denote by $\text{Reg}(\tau)$ the set of all basic triangles contained in τ (these are the basic triangles with vertices $\sum a_r v_r$, $0 \leqq a_r \leqq 1$, $\sum a_r = 1$, $1 \leqq r \leqq 3$, v_r being vertices of τ).

A region L is said to be *triangulated* by a configuration S iff there are triangles τ_i, $i \in I$ such that

(R1) $\text{Reg}(\tau_i) \cap \text{Reg}(\tau_j) = \emptyset$ for any $i, j \in I,\ i \neq j$,

(R2) $L = \bigcup \text{Reg}(\tau_i),\ i \in I$ and

(R3) $S \wedge \text{Compl}(L) = \bigvee \tau_i,\ i \in I$.

Loosely spoken, the triangles τ_i partition L, are contained in S and cover any segment of S restricted to L.

For any segment $s \subseteq b_r$, $r \in \mathbf{Z}$ define half-planes s^+, s^- by $s^+ = \{(i, j, k) \in \mathbf{P};\ i \geqq r\}$ and $s^- = \{(i, j, k) \in \mathbf{P};\ i \leqq r\}$. Similarly define s^+ and s^- for $s \subseteq c_r$ and $s \subseteq d_r$. The half-planes s^+ and s^- will be known as *half-planes determined by the segment* s.

A configuration S is said to be *binary at* x, iff there exists a half-plane U determined by $s \in S$ such that $x \in s$ and $[t] \subseteq U$ for all $t \in S$ with $x \in t$. A configuration S is said to be *binary*, if it is binary at all its vertices.

A configuration S is called a *t-configuration* iff it is binary and there is a triangle $\tau \leqq S$ such that $\text{Reg}(\tau)$ covers S and S triangulates τ.

Restating (R1 – 3) we get that S is a t-configuration provided it is binary and there are a triangle τ and triangles τ_i, $i \in I$ such that

(T1) $\text{Reg}(\tau_i) \cap \text{Reg}(\tau_j) = \emptyset$ for any $i, j \in I,\ i \neq j$,

(T2) $\text{Reg}(\tau) = \bigcup \text{Reg}(\tau_i),\ i \in I$ and

(T3) $S = \bigvee \tau_i,\ i \in I$.

The concept of the t-configuration is crucial for this paper and its study will be resumed in the subsequent sections. In the rest of this section some easy observations and auxiliary results will be formulated.

First, note that the definitions of a segment and a configuration might be extended to the infinite case. Define an i-segment s to be a subset of b_i (or c_i or d_i) such that it has at least two distinct elements and $z \in s$ holds for any $x, y, z \in \mathbf{P}$ with $z = ax + (1 - a)\,y$, $0 < a < 1$, $x, y \in s$. Define an i-configuration to be any set S of i-segments, where $s \cap t = \emptyset$ for any parallel and distinct $s, t \in S$. The definitions of $\wedge$ and $\vee$ might be directly extended to include the meet and join of i-configurations. Now, it is easy to check that the i-configurations form a distributive lattice with the minimum element $\emptyset$ and the maximum element $\mathbf{I} = \{b_i, c_i, d_i;\ i \in \mathbf{Z}\}$. The sublattice of all configurations is generated by the atoms of this lattice. For any i-configuration S

there exists (a unique) i-configuration S^c such that $S \wedge S^c = \emptyset$ and $S \vee S^c = \mathbf{I}$. The set of i-configurations is therefore a Boolean algebra and we shall henceforward use this fact. For configurations S and T we shall write $S \setminus T$ instead of $S \wedge T^c$.

For a region L define its *boundary* by $\text{Bound}(L) = \bigvee(\sigma \wedge \tau)$, where $\sigma \in L$ and $\tau \notin L$ are basic triangles. Clearly, $\text{Bound}(\text{Reg}(\tau)) = \tau$ for any triangle τ.

1.1. Lemma. *Let L be a region, let $s \in \text{Bound}(L)$ and let $x \in s$ be an extreme point of s. Then there exists $t \in \text{Bound}(L)$ such that $s \neq t$ and $x \in t$.*

Proof. Consider all six basic triangles with the vertex x.

1.2. Lemma. *For any regions L, M we have* $\text{Bound}(L \cap M) \leqq \text{Bound}(L) \vee \vee \text{Bound}(M)$.

Proof. If σ and τ are basic triangles with $\sigma \in L \cap M$ and $\tau \notin L \cap M$, then $\sigma \in L$ and $\tau \notin M$.

1.3. Lemma. *For any two distinct triangles σ, τ with* $\text{Reg}(\sigma) \cap \text{Reg}(\tau) \neq \emptyset$ *there exists a segment s such that either $\{s\} \leqq \sigma$, $\{s\} \wedge \tau = \emptyset$ and $\{s\} \leqq \leqq \text{Compl}(\text{Reg}(\tau))$, or $\{s\} \leqq \tau$, $\{s\} \wedge \sigma = \emptyset$ and $\{s\} \leqq \text{Compl}(\text{Reg}(\sigma))$.*

Proof. Consider $\varrho = \text{Bound}(\text{Reg}(\sigma) \cap \text{Reg}(\tau))$ and denote $\alpha = \varrho \setminus \tau$. By 1.2 $\varrho \leqq \text{Bound}(\text{Reg}(\sigma)) \vee \text{Bound}(\text{Reg}(\tau)) \leqq \sigma \vee \tau$. Hence $\alpha \leqq \sigma$, $\alpha \wedge \tau = \emptyset$ and $\alpha \leqq \varrho \leqq \text{Compl}(\text{Reg}(\sigma) \cap \text{Reg}(\tau)) \leqq \text{Compl}(\text{Reg}(\tau))$. If $\alpha \neq \emptyset$, choose any segment $s \in \alpha$. Let now $\varrho \setminus \tau = \emptyset = \varrho \setminus \sigma$ and let x, y be the extreme points of a segment $t \in \varrho$. Then $\varrho \leqq \sigma \wedge \tau$ and hence by 1.1 x and y are vertices of both σ and τ. This implies $\Delta(\sigma) = \Delta(\tau)$ and $\sigma = \tau$, a contradiction.

1.4. Lemma. *Let τ be a triangle and for $x \in \mathbf{P}$ let* $\text{r}(x) = \text{card}\{\sigma \in \text{Reg}(\tau);$ *x is a vertex of $\sigma\} > 0$. Then $\text{r}(x) = 1$ if x is a vertex of τ, $\text{r}(x) = 3$ if $x \in [\tau]$ is not a vertex of τ and $\text{r}(x) = 6$ if $x \notin [\tau]$.*

Proof. This is obvious.

1.5. Lemma. *Let K, L be such regions and S, T such configurations that*

(i) *S triangulates K and T triangulates L,*

(ii) *$K \cap L = \emptyset$,*

(iii) *$S \setminus T$ is covered by K and $T \setminus S$ is covered by L.*

Then $L \cup K$ is triangulated by $S \vee T$ and $(S \setminus T) \vee (T \setminus S)$ is covered by $L \cup K$.

Proof. Let (σ_i), $i \in I$ and (τ_j), $j \in J$ be the hypothetized triangulating sets, $\bigcup \text{Reg}(\sigma_i) = K$ and $\bigcup \text{Reg}(\tau_j) = L$. The union of these sets obviously satisfies both (R1) and (R2). By (iii) $T \setminus S \leqq T \wedge \text{Compl}(L)$ and hence $(S \vee T) \wedge \wedge \text{Compl}(K) = (S \vee (T \setminus S)) \wedge \text{Compl}(K) \leqq (S \wedge \text{Compl}(K)) \vee (T \setminus S) \leqq \leqq (S \wedge \text{Compl}(K)) \vee (T \wedge \text{Compl}(L)) = M$. Similarly, $(S \vee T) \wedge \text{Compl}(L) \leqq \leqq M$, so that (R3) is satisfied as well. The rest is clear.

1.6. Lemma. *Let S and T be two binary configurations and let W be the set of vertices of $S \vee T$. Further, let $A = W \cap [S \setminus T]$ and $B = W \cap [T \setminus S]$. $S \vee T$ is binary iff it is binary at each point of $A \cap B$.*

Proof. Choose $x \in W$ and put $L = \{s; x \in s \text{ and } \Delta(s) = 1\}$. Obviously, $S \vee T$ is binary at x iff $(S \vee T) \wedge L$ is binary (at x). If $x \notin A$, then $(S \vee T) \wedge L = T \wedge L$ and if $x \notin B$, then $(S \vee T) \wedge L = S \wedge L$. $T \wedge L$ and $S \wedge L$ are binary by the hypothesis.

2. QUASIGROUPS

2.1. Lemma. *Suppose that a configuration S triangulates the region L. The triangulating set (τ_i), $i \in I$ is then determined uniquely.*

Proof. Let (τ_k), $k \in K$ be another triangulation and let $k \in K$, $i \in I$ be such that $\tau_i \neq \sigma_k$ and $\text{Reg}(\tau_i) \cap \text{Reg}(\sigma_k) \neq \emptyset$. We may assume by 1.3 that there is a segment s with $\{s\} \leqq \sigma_k \leqq S \wedge \text{Compl}(L)$, $\{s\} \wedge \tau_i = \emptyset$ and $\{s\} \leqq \text{Compl}(\text{Reg}(\tau_i))$. By (R1) $\text{Compl}(\text{Reg}(\tau_i)) \wedge \tau_j \leqq \tau_i$ for any $j \in I$, and hence $\{s\} \leqq \tau_i$ – which is a contradiction.

For the rest of this section let S be a t-configuration triangulated by (τ_i), $i \in I$. The triangle τ with $\text{Reg}(\tau) = \bigcup \text{Reg}(\tau_i)$, $i \in I$ is also determined uniquely and we denote it by $\tau(S)$. Further, we put $\deg(S) = \text{card}(I)$ and $\Delta(S) = \Delta(\tau)$. The number $\deg(S)$ will be known as the *degree* of the t-configuration S.

2.2. Lemma. *Let x be a vertex of S and let* $\mathrm{d}(x) = \text{card}\{s \in S; x \in s\}$. *Then* $\mathrm{d}(x) = 3$ *if x is no vertex of $\tau(S)$ and* $\mathrm{d}(x) = 2$ *if x is a vertex of $\tau(S)$.*

Proof. Suppose that x is no vertex of $\tau(S)$ and choose $x \in S$ and a half-plane U determined by s so that $x \in s$ and $[t] \subseteq U$ whenever $x \in t \in S$. Let σ_i, $1 \leqq i \leqq 3$ be all basic triangles with $x \in [\sigma_i] \subseteq U$. Because x is not a vertex of $\tau(S)$, we have $\sigma_i \in \text{Reg}(\tau(S))$ and hence $\sigma_i \in \text{Reg}(\tau_{j_i})$ for some $j_i \in I$, $1 \leqq i \leqq 3$. We put $\varrho_i = \tau_{j_i}$ and it follows from the definition of U that $[\varrho_i] \subseteq U$. For any $1 \leqq i \leqq 3$ consider now the number $r_i = \text{card}\{\eta \in \text{Reg}(\varrho_i); x \in [\eta]\}$. By $[\eta] \subseteq U$ we have $r_i \leqq 3$ and $r_i = 3$ implies $\sigma_k \in \text{Reg}(\varrho_i)$ for $1 \leqq k \leqq 3$. In such a case x would be no vertex of S and so we have $r_i = 1$ by 1.4. Hence x is a vertex of ϱ_i and ϱ_i, $1 \leqq i \leqq 3$ are pair-wise distinct triangles. The rest is clear.

2.3. Lemma. *Let x be a vertex of S and let $x \in t_1$ and $x \in t_2$ for segments $t_1, t_2 \in S$, $t_1 \neq t_2$. Then there is exactly one segment $t_3 \in S$ such that there are segments $s_i \subseteq t_i$, $1 \leqq i \leqq 3$ with $\{s_1, s_2, s_3\} = \tau_j$ for some $j \in I$.*

Proof. If x is a vertex of $\tau(S)$, consider the only τ_j, $j \in I$ which has x as its vertex Suppose now that x is not a vertex of $\tau(S)$. To prove the existence, choose a segment s, a half-plane U and triangles σ_i, ϱ_i, $1 \leqq i \leqq 3$ in the same way as in the proof of the preceding lemma. For any $k \in \{1, 2\}$ there are two distinct $j \in \{1, 2, 3\}$ and a segment $s_{k,j}$ such that $s_{k,j} \subseteq t_k$ and $s_{k,j}$ is a side of ϱ_j. Hence there exists $j \in \{1, 2, 3\}$ such that $s_{1,j} \subseteq t_1$ and $s_{2,j} \subseteq t_2$. Then ϱ_j is the sought triangle. If $\sigma = \{s_1, s_2, s_3\}$ is another such triangle, then x is a vertex of σ, $[\sigma] \subseteq U$ and $\sigma_{j'} \in \text{Reg}(\sigma)$ for some $1 \leqq j' \leqq 3$. Therefore $\sigma = \varrho_{j'}$ and it is easy to see that $j' = j$.

For a positive $n \in \mathbf{Z}$ denote $\mathbf{Z}_n = \mathbf{Z}/n\mathbf{Z}$ and let $\phi = \phi_n$ be the natural projection $\phi: \mathbf{Z} \to \mathbf{Z}_n$. Further, assume that $n = \Delta(S)$. We easily obtain

2.4. Lemma. *If s, t are two parallel segments of S, $s \subseteq b_i$, $t \subseteq b_j$ (or $s \subseteq c_i$, $t \subseteq c_j$, or $s \subseteq d_i$, $t \subseteq d_j$), then $\phi(i) = \phi(j)$ implies $i = j$.*

We shall define now a new operation on $\mathbf{Z}_n$. We define $\phi(i) * \phi(j) = \phi(k)$ if there is a triangle τ_r, $r \in I$ with the coordinates $(i, j, -k)$, and $\phi(i) * \phi(j) = \phi(i) + \phi(j)$ if there is no such triangle.

Let τ_r, $r \in I$ have coordinates $(i, j, -k)$ and $\tau_{r'}$, $r' \in I$ coordinates $(i', j', -k')$. If $\phi(i') = \phi(i)$ (or $\phi(j') = \phi(j)$ or $\phi(k') = \phi(k)$), we get $i' = i$ (or $j' = j$ or $k' = k$) by 2.4. If $i' = i$ and $j' = j$, we have $k' = k$ by 2.3. The operation $*$ is therefore correctly defined.

If $u, v, w \in \mathbf{Z}_n$ are such that $u * v = u * w$ and $v \neq w$, then either $u * v \neq u + v$ or $u * w \neq u + w$. Assume the former case occurs, and let the triangle τ_r, $r \in I$ have the coordinates $(i, j, -k)$, where $\phi(i) = u$ and $\phi(j) = v$. If $u * w \neq u + w$, then there is a triangle $\tau_{r'}$, $r' \in I$ with coordinates $(i', j', -k')$ and $\phi(i') = \phi(i)$, $\phi(k') = \phi(k)$. As above, we get $i' = i$, $k' = k$ by 2.4 and $j' = j$ by 2.3. Then $v = \phi(j) = \phi(j') = w$ and we may assume $u * w = u + w$. Consider now the point $x = (i, k - i, -k) \in \mathbf{P}$, which is a vertex of τ_r. If x is not a vertex of τ, then we have $x \in s \subseteq c_{k-i}$ for some $s \in S$ by 2.2. Moreover, by 2.3 then there exist $r' \in I$, $k' \in \mathbf{Z}$ such that $\tau_{r'}$ has coordinates $(i, k - i, -k')$. From $u * v = u + w$ we obtain $w = (u * v) - u = \phi(k - i)$, and hence $\phi(k) = u * v = u * w = \phi(k')$. By 2.4 we get $k = k'$, which is a contradiction, as $i + (k - i) - k = 0 \neq i + (k - i) - k'$. If x is a vertex of τ, we use another vertex of τ to get a similar contradiction. Therefore $u * v = u * w$ implies $v = w$ and we see that $\mathbf{Z}_n(*)$ is cancellative. As it is finite, it is a quasigroup.

Let $(i, j, -k)$ be the coordinates of τ_r, $r \in I$. Then $-k \neq i + j$ and $\phi(i) + \phi(j) = \phi(k)$ iff $|i + j - k| = n$. This is the case iff $\operatorname{card}(I) = \deg(S) = 1$. (In such a case $\mathbf{Z}_n(*) = \mathbf{Z}_n(+)$.)

We denote $\mathbf{Z}_n(*)$ by $\mathbf{Z}(S)$ and conclude

2.5. Theorem. *Let S be a t-configuration with* $\deg(S) > 1$ *and* $\Delta(S) = n$. *Then $\mathbf{Z}(S)$ is a quasigroup on $\mathbf{Z}_n$ and* $\operatorname{dist}(\mathbf{Z}(S), \mathbf{Z}_n(+)) = \deg(S)$.

3. TRAPEZOIDS

A configuration ϑ will be called a *trapezoid* if there exist two distinct triangles σ and τ, which have a common vertex, $\operatorname{Reg}(\sigma) \subseteq \operatorname{Reg}(\tau)$ and $\vartheta = (\tau \setminus \sigma) \vee (\sigma \setminus \tau)$. Note that $\sigma \setminus \tau$ contains only one segment, we shall denote it by $\mathrm{s}(\vartheta)$. The extreme points x, y of $\mathrm{s}(\vartheta)$ are the *obtuse* vertices of ϑ and we denote by $\mathrm{m}(\vartheta) = (x + y)/2$ the middle point of $\mathrm{s}(\vartheta)$. The other vertices of ϑ are called *acute* and we put $\mathrm{p}(\vartheta) = \Delta(\sigma) = \Delta(\mathrm{s}(\vartheta))$ and $\mathrm{q}(\vartheta) = \Delta(\tau) - \Delta(\sigma)$. (The triangles τ and σ are clearly determined uniquely for any trapezoid ϑ.) The segments of ϑ are called its *sides* and we define $\operatorname{Reg}(\vartheta) = \operatorname{Reg}(\tau) \setminus \operatorname{Reg}(\sigma)$.

We will extend any trapezoid ϑ, $p(\vartheta) \neq q(\vartheta)$ to a configuration $Z(\vartheta)$. The configuration $Z(\vartheta)$ contains a trapezoid $z(\vartheta)$ distinct from ϑ and this makes a repeated use of the extension (it will be called the *Z-extension*) possible.

Let $s = s(\vartheta)$, t, u, v be the segments of ϑ and suppose that t is parallel to s. (We have $p(\vartheta) = \Delta(u) = \Delta(v)$.) To define $Z(\vartheta)$, three cases are distinguished:

A. Let $q(\vartheta) < p(\vartheta)$. Define a segment w to be parallel to v and with extreme points in $s \cap u$ and $w \cap t$. Denote by ϱ_1 the triangle with sides u and w. Define a segment y to be parallel to u and with extreme points in $w \cap t$ and $y \cap s$. Denote by ϱ_2 the triangle with sides w and y and let $Z(\vartheta) = \vartheta \vee \varrho_1 \vee \varrho_2 = \{s, t, u, v, w, y\}$ and $z(\vartheta) = \{v, y\} \cup (\{t\} \setminus \varrho_1) \cup (\{s\} \setminus \varrho_2)$. Observe that

(A1) $Z(\vartheta) = z(\vartheta) \vee \varrho_1 \vee \varrho_2$, $\operatorname{Reg}(\vartheta) = \operatorname{Reg}(\varrho_1) \cup \operatorname{Reg}(\varrho_2) \cup \operatorname{Reg}(z(\vartheta))$ and $\operatorname{Reg}(\varrho_1)$, $\operatorname{Reg}(\varrho_2)$ and $\operatorname{Reg}(z(\vartheta))$ are pair-wise disjoint.

(A2) $q(z(\vartheta)) = q(\vartheta)$ and $p(z(\vartheta)) = p(\vartheta) - q(\vartheta)$.

B. Let $2p(\vartheta) \geqq q(\vartheta) > p(\vartheta)$. Define a segment w to be parallel to v and with extreme points in $s \cap u$ and $w \cap t$. Denote by ϱ_1 the triangle with sides u and w. Define a segment y to be parallel to u and with extreme points in $s \cap v$ and $y \cap w$. Denote by ϱ_2 the triangle with sides s and y and let $Z(\vartheta) = \vartheta \vee \varrho_1 \vee \varrho_2 = \{s, t, u, v, w, y\}$ and $z(\vartheta) = \{y, v\} \cup (\{t\} \setminus \varrho_1) \cup (\{w\} \setminus \varrho_2)$. Observe that

(B1) $Z(\vartheta) = z(\vartheta) \vee \varrho_1 \vee \varrho_2$, $\operatorname{Reg}(\vartheta) = \operatorname{Reg}(\varrho_1) \cup \operatorname{Reg}(\varrho_2) \cup \operatorname{Reg}(z(\vartheta))$, and $\operatorname{Reg}(\varrho_1)$, $\operatorname{Reg}(\varrho_2)$ and $\operatorname{Reg}(z(\vartheta))$ are pair-wise disjoint.

(B2) $q(z(\vartheta)) = p(\vartheta)$ and $p(z(\vartheta)) = q(\vartheta) - p(\vartheta)$.

C. Let $q(\vartheta) > 2p(\vartheta)$. Define segments w and y so that they meet at a common extreme point, w is parallel to v, y is parallel to u and the other extreme point of w (or y) is in $u \cap s$ (or $v \cap s$). Denote by ϱ_1 the triangle with sides y and w, and ϱ_2, ϱ_3, respectively, the other triangle with the side w, y, respectively. Define a segment m to be parallel to s and t, and such that $m \cap y \cap w \neq \emptyset$ and the extreme points of m are in $m \cap u$ and $m \cap v$. Let $Z(\vartheta) = \vartheta \vee \varrho_1 \vee \varrho_2 \vee \varrho_3 = \vartheta \vee \varrho_2 \vee \varrho_3 = \{s, t, u, v, w, y, m\}$ and $z(\vartheta) = \{t, m\} \cup (\{u\} \setminus \varrho_2) \cup (\{v\} \setminus \varrho_3)$. Observe that $m \cap y \cap w = \{m(z(\vartheta))\}$ and

(C1) $Z(\vartheta) = z(\vartheta) \vee \varrho_1 \vee \varrho_2 \vee \varrho_3$, $\operatorname{Reg}(\vartheta) = \operatorname{Reg}(\varrho_1) \cup \operatorname{Reg}(\varrho_2) \cup \operatorname{Reg}(\varrho_3) \cup \operatorname{Reg}(z(\vartheta))$, and $\operatorname{Reg}(\varrho_1)$, $\operatorname{Reg}(\varrho_2)$, $\operatorname{Reg}(\varrho_3)$ and $\operatorname{Reg}(z(\vartheta))$ are pair-wise disjoint.

(C2) $q(z(\vartheta)) = q(\vartheta) - p(\vartheta)$ and $p(z(\vartheta)) = 2p(\vartheta)$.

Now we will list several properties of the Z-extension, all of which may be checked easily:

3.1. Lemma. *Let ϑ be a trapezoid with $q(\vartheta) \neq p(\vartheta)$.*

(i) *If x is a vertex of $Z(\vartheta)$ and t' such a segment that $x \in t'$ and $t \subseteq t'$ for a segment $t \in Z(\vartheta)$, then $x \in t$.*

(ii) *If x is an obtuse of $z(\vartheta)$ and $x \in t \in Z(\vartheta)$, then there is $s \subseteq t$ such that $x \in s$ and s is a side of $z(\vartheta)$.*

(iii) *If x is an obtuse vertex of* $z(\vartheta)$ *and x is also a vertex of ϑ, then x is an obtuse vertex of ϑ. Moreover, if then s is a side of ϑ with $x \in s$, then we have $x \in s' \subseteq \subseteq s$ for a side s' of* $z(\vartheta)$.

(iv) *If x is a vertex of ϑ and $x \in [z(\vartheta)]$, then x is a vertex of* $z(\vartheta)$.

(v) *If x is a vertex of* $Z(\vartheta)$ *and x is neither a vertex of ϑ, nor a vertex of* $z(\vartheta)$, *then* $x = m(z(\vartheta))$ *and* $q(\vartheta) > 2p(\vartheta)$.

3.2. Lemma. *Let ϑ be a trapezoid with* $q(\vartheta) \neq p(\vartheta)$. *Then* $Z(\vartheta)$ *is a binary configuration.*

In the rest of this section we shall deal with trapezoids ϑ such that $2q(\vartheta) > p(\vartheta)$. They satisfy some additional properties:

3.3. Lemma. *Let ϑ be a trapezoid with* $q(\vartheta) \neq p(\vartheta)$ *and* $2q(\vartheta) > p(\vartheta)$ *and let* $\eta = z(\vartheta)$. *Then* $2q(\eta) > p(\eta)$. *Moreover, from* $2p(\vartheta) \geqq q(\vartheta)$ *it follows that* $q(\eta) \geqq \geqq p(\eta)$.

Proof. Check directly the individual cases A, B and C.

3.4. Lemma. *Let ϑ be a trapezoid with* $p(\vartheta) \neq q(\vartheta)$ *and* $2q(\vartheta) > p(\vartheta)$. *Then* $m(\vartheta) \notin [z(\vartheta)]$ *and* $m(\vartheta)$ *is never a vertex of* $Z(\vartheta)$.

Proof. Consider the set $s(\vartheta) \cap [z(\vartheta)]$. By the definition of $Z(\vartheta)$ it is empty in the case C and contains only one point in the case B – this point is a common vertex of ϑ and $z(\vartheta)$. We have $s(z(\vartheta)) \subseteq s(\vartheta)$ in the case A and $s(z(\vartheta))$ and $s(\vartheta)$ have a common extreme point. Moreover, $\Delta(s(z(\vartheta)) = p(z(\vartheta)) = p(\vartheta) - q(\vartheta) < p(\vartheta)/2 = = \Delta(s(\vartheta))/2$, which implies $m(\vartheta) \notin [z(\vartheta)]$.

We shall construct now a t-configuration $T(\vartheta)$ for any trapezoid ϑ with $2q(\vartheta) > > p(\vartheta)$. Suppose that σ and τ are the triangles with $\mathrm{Reg}(\tau) \setminus \mathrm{Reg}(\sigma) = \mathrm{Reg}(\vartheta)$. Put $\vartheta_0 = \vartheta$ and for $i \geqq 0$ define $\vartheta_{i+1} = z(\vartheta_i)$ whenever $q(\vartheta_i) \neq p(\vartheta_i)$. Then $p(\vartheta_i) + + q(\vartheta_i) > p(\vartheta_{i+1}) + q(\vartheta_{i+1}) > 1$ and therefore $p(\vartheta_k) = q(\vartheta_k)$ for some $k \geqq 0$. Let s_1, s_2, s_3 and s_4 be the segments of ϑ_k, suppose that $s_1 = s(\vartheta_k)$ and s_4 is parallel to s_1. Then $\Delta(s_1) = \Delta(s_2) = \Delta(s_3) = q(\vartheta_k) = p(\vartheta_k)$ and $\Delta(s_4) = 2q(\vartheta_k)$. Let ϱ_j, $1 \leqq j \leqq 3$ be the triangle with one side equal to s_j and $\mathrm{Reg}(\varrho_j) \subseteq \mathrm{Reg}(\vartheta_k)$. We have

(D1) $\vartheta_k \leqq \bigvee \varrho_j$, $\mathrm{Reg}(\vartheta_k) = \bigcup \mathrm{Reg}(\varrho_j)$ and $\mathrm{Reg}(\varrho_j)$ are pair-wise disjoint, $1 \leqq \leqq j \leqq 3$.

(D2) $q(\vartheta_k) = p(\vartheta_k) = \Delta(\varrho_j)$, $1 \leqq j \leqq 3$.

For any $0 \leqq j < k$ we put $Z_j = Z(\vartheta_j)$ and $Z_k = \bigvee \varrho_i$, $1 \leqq i \leqq 3$. We define $T(\vartheta) = (\bigvee Z_j) \vee \tau$, where $0 \leqq j \leqq k$, $T_0 = \tau \vee \vartheta$ and $T_{j+1} = (\bigvee Z_i) \vee \tau$, $0 \leqq \leqq i \leqq j \leqq k$.

3.5. Lemma. *The following conditions hold for any* $0 \leqq j \leqq k$.

(i) $\mathrm{Reg}(\tau) \setminus \mathrm{Reg}(\vartheta_j)$ *is triangulated by T_j and $T_j \setminus \vartheta_j$ is covered by* $\mathrm{Reg}(\tau) \setminus \mathrm{Reg}(\vartheta_j)$.

(ii) $[T_j] \cap [Z_j] = [\vartheta_j]$ *and* $T_j \wedge Z_j = \vartheta_j$.

(iii) *If x is a vertex of T_{j+1} which is not a vertex of T_j, then x is a vertex of Z_j.*

Proof. (i) For $j = 0$ we have $\operatorname{Reg}(\tau) \setminus \operatorname{Reg}(\vartheta_0) = \operatorname{Reg}(\sigma)$ and $T_0 \setminus \vartheta_0 \leqq \sigma$. We shall use induction for $k \geqq j > 0$. By the definition of Z_{j-1}, T_{j-1} and by the induction hypothesis we have $\vartheta_{j-1} \leqq T_{j-1} \wedge Z_{j-1} \leqq (\operatorname{Compl}(\operatorname{Reg}(\tau) \setminus \operatorname{Reg}(\vartheta_{j-1})) \vee \vee\, \vartheta_{j-1}) \wedge \operatorname{Compl}(\operatorname{Reg}(\vartheta_{j-1})) = \vartheta_{j-1}$. Hence $T_{j-1} \setminus Z_{j-1} = T_{j-1} \setminus \vartheta_{j-1}$ and $Z_{j-1} \setminus T_{j-1} = Z_{j-1} \setminus \vartheta_{j-1}$. $\operatorname{Reg}(\vartheta_{j-1}) \setminus \operatorname{Reg}(\vartheta_j)$ is by (A1), (B1) and (C1) triangulated by Z_{j-1} and it covers $Z_{j-1} \setminus \vartheta_{j-1}$. Using the induction hypothesis we get from 1.5 that $\operatorname{Reg}(\tau) \setminus \operatorname{Reg}(\vartheta_j) = (\operatorname{Reg}(\tau) \setminus \operatorname{Reg}(\vartheta_{j-1})) \cup (\operatorname{Reg}(\vartheta_{j-1}) \setminus \operatorname{Reg}(\vartheta_j))$ is triangulated by $T_j = T_{j-1} \vee Z_{j-1}$ and $T_j \setminus \vartheta_{j-1} = (T_{j-1} \setminus Z_{j-1}) \vee (Z_{j-1} \setminus T_{j-1})$ is covered by $\operatorname{Reg}(\tau) \setminus \operatorname{Reg}(\vartheta_j)$. It follows from (A1), (B1) and (C1) that $\vartheta_{j-1} \setminus \vartheta_j$ is covered by $\operatorname{Reg}(\vartheta_{j-1}) \setminus \operatorname{Reg}(\vartheta_j)$.

(ii) Obviously $\vartheta_j \leqq T_j \wedge Z_j$ and $Z_j \leqq \operatorname{Compl}(\operatorname{Reg}(\vartheta_j))$ for any $0 \leqq j \leqq k$. By (i) we have $T_j \setminus \vartheta_j \leqq \operatorname{Compl}(\operatorname{Reg}(\tau) \setminus \operatorname{Reg}(\vartheta_j))$ and the rest follows from $[\operatorname{Compl}(\operatorname{Reg}(\vartheta_j))] \cap [\operatorname{Compl}(\operatorname{Reg}(\tau) \setminus \operatorname{Reg}(\vartheta_j))] = [\vartheta_j]$.

(iii) $[T_{j+1}] = [T_j] \cup [Z_j]$ as $T_{j+1} = T_j \vee Z_j$. If x is neither a vertex of T_j, nor a vertex of Z_j, then $x \in [T_j] \cap [Z_j] = [\vartheta_j]$. If x is not a vertex of $[\vartheta_j]$, then there exists a segment s such that $x \in s$, $\{s\} \wedge \vartheta_j = \emptyset$ and either $\{s\} \leqq T_j$ or $\{s\} \leqq Z_j$. However, then $x \in s \cap t$ for a side t of $\vartheta_j \leqq T_j \wedge Z_j$.

Note that up to now we have not used the assumed fact that $2\mathrm{q}(\vartheta) > \mathrm{p}(\vartheta)$.

3.6. Lemma. *The following conditions hold for any* $0 \leqq j \leqq k$.

(i) $2\mathrm{q}(\vartheta_j) > \mathrm{p}(\vartheta_j)$,

(ii) *if x is a vertex of T_j and $x \in [\vartheta_j]$, then either x is a vertex of ϑ_j, or* $x = \mathrm{m}(\vartheta_j)$.

Proof. (i) follows from 3.3. To prove (ii) we use induction – the condition obviously holds for $j = 0$, let it hold for some $k > j \geqq 0$. If x is a vertex of T_{j+1}, then x is a vertex of Z_j or x is a vertex of T_j by 3.5 (iii). If $x \in [\vartheta_{j+1}]$ and x is a vertex of Z_j, then x is either a vertex of ϑ_{j+1} or $x = \mathrm{m}(\vartheta_{j+1})$ by 3.1 (iv) and (v). If $x \in [\vartheta_{j+1}]$ is not a vertex of Z_j, then x is a vertex of T_j and by 3.5 (ii) we have $x \in [\vartheta_j]$, as $[\vartheta_{j+1}] \subseteq \subseteq [Z_j]$. By the induction hypothesis x is now either a vertex of ϑ_j, or $x = \mathrm{m}(\vartheta_j)$. In the former case x is a vertex of ϑ_{j+1} by 3.1 (iv). In the latter case $x \notin [\vartheta_{j+1}]$ by 3.4 – a contradiction.

3.7. Corollary. *If x is a vertex of both T_j and Z_j, $0 \leqq j \leqq k$, then it is a vertex of ϑ_j.*

Proof. From 3.5 (ii) we obtain $x \in [\vartheta_j]$ and by 3.6 (ii) the only case to be considered is $x = \mathrm{m}(\vartheta_j)$. As $Z_j = \mathrm{Z}(\vartheta_j)$, it follows from 3.4 (if $j < k$) and from (D1) (if $j = k$) that x is not a vertex of Z_j.

3.8. Lemma. *Let $0 \leqq j \leqq k$. If x is an obtuse vertex of ϑ_j and $x \in t$ for some $t \in T_j$, then there is a side s of ϑ_j with $s \subseteq t$.*

Proof. For $j = 0$ such a side obviously exists, so we may proceed by induction and assume $1 \leqq j \leqq k$. If $\{t\} \wedge Z_{j-1} \neq \emptyset$, then by 3.1 (i) $x \in t_1$ for some segment $t_1 \in Z_{j-1}$, $t_1 \subseteq t$ and 3.1 (ii) may be used. If $\{t\} \wedge Z_{j-1} = \emptyset$, then $t \in T_{j-1}$ and $x \in [Z_{j-1}] \cap [T_{j-1}] = [\vartheta_{j-1}]$. Then $x \in t \cap s_1$ for a side s_1 of ϑ_{j-1} and x is therefore a vertex of T_{j-1}. By 3.7 it is also a vertex of ϑ_{j-1} and by 3.1 (iii) it is an obtuse vertex of ϑ_{j-1}. We obtain the assertion from the induction hypothesis and 3.1 (iii).

3.9. Lemma. *T_j is a binary configuration for any $0 \leqq j \leqq k + 1$.*

Proof. We shall again proceed by induction, the case T_0 being obvious. Let us assume that T_j is binary for some j, $0 \leqq j \leqq k$. Since $T_{j+1} = T_j \vee Z_j$, we may use 1.6, as Z_j is binary by 3.2 for $0 \leqq j < k$ and Z_k is binary too. We have $[T_j] \cap [Z_j] \subseteq [\vartheta_j]$, so by 1.6 only the vertices in $[\vartheta_j]$ need to be considered. Let x be such a vertex and put $L = \bigvee(\{s\};\ x \in s \text{ and } \Delta(s) = 1)$. If x is not a vertex of ϑ_j, it is by 3.5 (iii) and 3.7 either a vertex of T_j, or a vertex of Z_j. We have $T_j \wedge L = T_{j+1} \wedge L$ in the former case and $Z_j \wedge L = T_{j+1} \wedge L$ in the latter case, which implies that T_{j+1} is binary at x in any case. Let now x be a vertex of ϑ_j and s, t the sides of ϑ_j with $x \in s$, $x \in t$ and $s \neq t$. If x is an acute vertex of ϑ_j, then $Z_j \wedge L \leqq T_j \wedge L$ and T_{j+1} is binary at x by the induction hypothesis. For x an obtuse vertex of ϑ_j denote by U and V those halfplanes determined by s and t, respectively, which have $[Z_j] \subset U$ and $[Z_j] \subset V$. By 3.8 and the induction hypothesis we obtain $[T_j \wedge L] \subset U$ or $[T_j \wedge L] \subset V$, and hence $[T_{j+1} \wedge L] \subset U$ or $[T_{j+1} \wedge L] \subset V$.

3.10. Proposition. *Let ϑ be a trapezoid with $2\mathrm{q}(\vartheta) > \mathrm{p}(\vartheta)$. Then $\mathrm{T}(\vartheta)$ is a t-configuration.*

Proof. $\mathrm{T}(\vartheta) = T_{k+1}$ is binary by 3.9. From 3.5 (i) and (D1) we obtain that $\mathrm{Reg}(\tau) \setminus \mathrm{Reg}(\vartheta_k)$ is triangulated by T_k, $\mathrm{Reg}(\vartheta_k)$ is triangulated by Z_k, $T_k \setminus Z_k = T_k \setminus \vartheta_k$ is covered by $\mathrm{Reg}(\tau) \setminus \mathrm{Reg}(\vartheta_k)$ and Z_k is covered by $\mathrm{Reg}(\vartheta_k)$. It follows then from 1.5 that $\mathrm{Reg}(\tau)$ is triangulated by $\mathrm{T}(\vartheta)$, and as $\mathrm{Reg}(\tau)$ obviously covers $\mathrm{T}(\vartheta)$, we conclude the proof.

3.11. Corollary. *Let $k \geqq 0$ and let p_i, q_i, a_i be positive integers, $0 \leqq i \leqq k$, such that the following conditions are satisfied.*

(i) *$2q_i > p_i$ for any $k \geqq i \geqq 0$.*
(ii) *$q_i \neq p_i$ for any $k > i \geqq 0$ and $q_k = p_k$.*
(iii) *$a_0 = 1$.*
(iv) *If $q_i < p_i$, then $q_{i+1} = q_i$, $p_{i+1} = p_i - q_i$ and $a_i = 2$.*
(v) *If $2p_i \geqq q_i > p_i$, then $q_{i+1} = p_i$, $p_{i+1} = q_i - p_i$ and $a_i = 2$.*
(vi) *If $q_i > 2p_i$, then $q_{i+1} = q_i - p_i$, $p_{i+1} = 2p_i$ and $a_i = 3$.*

Then there exists a quasigroup Q on $\mathbf{Z}_n$, $n = p_0 + q_0$, such that $\mathrm{dist}(Q, \mathbf{Z}_n) = 3 + \sum a_i$, $0 \leqq i \leqq k$. Moreover, for any positive integers p, q with $2q > p$ there exist uniquely determined sequences p_i, q_i, a_i satisfying (i–vi) and $p_0 = p$, $q_0 = q$.

Proof. Consider a trapezoid ϑ with $\mathrm{p}(\vartheta) = p_0$ and $\mathrm{q}(\vartheta) = q_0$. Then $\mathrm{T}(\vartheta)$ is a t-configuration by 3.10 and 2.5 may be used. Hence $p_i = \mathrm{p}(\vartheta_i)$ and $q_i = \mathrm{q}(\vartheta_i)$. The sequence a_i expresses how many triangles of the final triangulation have been constructed at a given step – cf. (A1), (B1) and (C1). As Z_k by (D1) adds three further triangles, we have $\deg(\mathrm{T}(\vartheta)) = 3 + \sum a_i$, $0 \leqq i \leqq k$.

Let p_i, q_i, a_i, $0 \leqq i \leqq k$ be positive integers satisfying conditions (i–vi) of 3.11 and define $u_i = p_i + q_i$ and $n = u_0 = p_0 + q_0$.

4.1. Lemma. *For any* $0 \leqq i < k$ *the following implications hold*:

(i) *if* $q_i < p_i$, *then* $3u_{i+1} < 2u_i$,

(ii) *if* $2p_i \geqq q_i > p_i$, *then* $3u_{i+1} \leqq 2u_i$ *and*

(iii) *if* $q_i > 2p_i$, *then* $u_{i+1} = u_i$.

Proof. (iii) is obvious, (i) follows from $2q_i > p_i$ and (ii) from $2p_i \geqq q_i$.

4.2. Lemma. *Let* $0 \leqq i_1 < i_2 \leqq k + 1$ *be such that* $q_j < 2p_j$ *for any* $i_1 \leqq j < i_2$. *Then* $i_2 - i_1 < \log_2 u$, *where* $u = u_{i_1} = u_j$.

Proof. Let $r = p_{i_1}$, $s = p_{i_2}$ and $m = i_2 - i_1$. Because $s = 2^m r$ and $s/r < s + 1 \leqq u$, we have $m = \log_2 (s/r) < \log_2 u$.

Put now $j_0 = 0$ and suppose that j_i is defined for any $0 \leqq i \leqq s$ and $0 \leqq j_i \leqq k$. If $u_{js} > u_k$, let $j_{s+1} = \min \{0 \leqq j \leqq k;\ u_j < u_{j_s}\}$. If $u_{j_s} = u_k$, j_s is the last member of the sequence (j_i) and we put $r = s$. Further, we put $v_i = u_{j_i}$ for any $0 \leqq i \leqq r$. Obviously, v_i is a subsequence of u_j and for any $0 \leqq j \leqq k$ there is exactly one $0 \leqq i \leqq r$ with $v_i = u_j$. Finally, for any $0 \leqq i \leqq r$ let $g_i = \sum a_h$, $h \in \{h \in \mathbf{Z};\ 0 \leqq h \leqq k$ and $u_h = v_i\}$.

Using 4.1 we now list several properties of j_i, v_i and g_i.

(1) $0 \leqq j_i < j_h \leqq k$ and $n = u_0 = v_0 \geqq v_i > v_h \geqq u_k = v_r$ whenever $0 \leqq i < h \leqq r$.

(2) $v_{i+1}/v_i \leqq 2/3$ for any $0 \leqq i < r$.

(3) For $0 \leqq i \leqq r$ and $0 \leqq j \leqq k$ we have $\sum g_i = \sum a_j$.

It follows from 4.2 that $j_{i+1} - j_i < \log_2 v_i$ for any $0 \leqq i \leqq r$ (set $j_{r+1} = k + 1$ for this case). Hence $g_i < 3(\log_2 v_i - 1) + 2$ for $i > 0$ and $g_0 = 3(\log_2 v_0 - 1) + 1$.

By (3) we now have

(4) $\sum a_j + 3 < 3(\sum \log_2 v_i) - r + 2$, where $0 \leqq j \leqq k$ and $0 \leqq i \leqq r$.

By (2) $v_i \leqq (2/3)^i\, n$, which implies $\sum \log_2 v_i = \log_2 \prod v_i \leqq \log_2 n^{r+1}(2/3)^{r(r+1)/2} = (r + 1)(\log_2 n + (r/2)(1 - \log_2 3))$. Further, $n \geqq (3/2)^r v_r$ and from $v_r \geqq 2$ we get $r \leqq (\log_2 n - 1)/(\log_2 3 - 1)$. Therefore $\sum a_j + 3 < 3(r + 1)(\log_2 n - tr/2) - r + 2 = 3 \log_2 n + 2 + r(3 \log_2 n - 3 + r/2 - 3t/2 - 1)$, where $t = \log_2 3 - 1 > 0$. For $n \geqq 3$ we may choose p_0, q_0 so that $r \geqq 1$, and hence $\sum a_j + 3 < 3 \log_2 n + 2 + r(3 \log_2 n - 3t - 1) \leqq 3 \log_2 n + 2 + (\log_2 n - 1)(3 \log_2 n - 3t - 1)/t = 3 \log_2^2/t - 4 \log_2 n/t + (5t + 1)/t$. With respect to 3.11 we may conclude:

4.3. Theorem. *For any* $n \geqq 3$ *we have* $\operatorname{gdist}(n) \leqq A \log_2^2 n + B \log_2 n + C$, *where* $A = 3/t$, $B = -4/t$, $C = (5t + 1)/t$ *and* $t = \log_2 3 - 1$.

The constants A, B, C can be computed and we obtain $A \approx 5.13$, $B \approx -6.84$,

$C \approx 6.71$. However, the exact values of A, B, C are probably not so important. There are reasons to hope that $\operatorname{gdist}(n) \leqq K \log n$ for some $K > 0$. Note that by [1] $\operatorname{gdist}(n) > \mathrm{e} \log n + 3$ for any odd $n \geqq 3$ and $\operatorname{gdist}(n) = 4$ for any even $n \geqq 2$.

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