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HEREDITARILY STRICTLY CYCLIC OPERATOR ALGEBRAS

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An operator algebra $\mathscr{A}$ on a Hilbert space H is said to *inherit finite strict multiplicity-n* (FSM) [7] if the uniform closure of its restriction to every invariant subspace has finite strict multiplicity-n. $\mathscr{A}$ is said to be *hereditarily strictly cyclic* if the uniform closure of its restriction to every invariant subspace is strictly cyclic [6]. The purpose of this paper is to study the properties of such operator algebras.

Throughout this paper, H denotes a separable (complex) infinite dimensional Hilbert space, and $B(H)$, the algebra of all bounded linear operators on H. By an operator algebra $\mathscr{A}$ on H, we mean a strongly closed subalgebra of $B(H)$containing identity I. If $T \in B(H)$, then $\mathscr{A}(T)$ denotes the algebra generated by T and I. For any subset $\mathscr{B}$ of $B(H)$, Lat $\mathscr{B}$ denotes the lattice of all invariant subspaces of $\mathscr{B}$. An operator algebra $\mathscr{B}$ is said to be *transitive* if Lat $\mathscr{B} = \{\{0\}, H\}$, and unicellular if Lat $\mathscr{B}$ is totally ordered.

An operator algebra $\mathscr{A}$ is said to have *finite strict multiplicity* [3] if there exists a finite subset $\Gamma = \{x_1, x_2, \ldots, x_n\}$ of H such that

$$\mathscr{A}(\Gamma) = \{A_1x_1 + A_2x_2 + \ldots + A_nx_n : A_i \in \mathscr{A}\} = H .$$

The minimum cardinality of all such sets Γ is called *strict multiplicity* of $\mathscr{A}$. If $\mathscr{A}$ has strict multiplicity 1, then $\mathscr{A}$ is said to be *strictly cyclic* [5]. $\mathscr{A}$ is said to satisfy condition-S_n [1] if $A_1x_1 + A_2x_2 + \ldots + A_nx_n = 0$, $A_i \in \mathscr{A}$ implies $A_i = 0$ for all $i = 1, 2, \ldots, n$. A vector x is said to be *separating* [5] for $\mathscr{A}$ if $Ax = 0$, $A \in \mathscr{A}$ implies $A = 0$.

An operator T on H is said to be of *finite strict multiplicity* if $\mathscr{A}(T)$ is so. T is said to *inherit* FSM-n if $\mathscr{A}(T|_M)$ is of FSM-n for every invariant subspace M of T. Operator T is said to be *power bounded* if there exists a positive real number M such that $\|T^n\| < M$ for all $n = 1, 2, 3, \ldots$.

Eric J. Rosenthal [6] has proved that if T is a strictly cyclic operator, and M an invariant subspace of T, then compression of T to $M^\perp$ is strictly cyclic. He also proves that if T is hereditarily strictly cyclic, power bounded with $\sigma(T) = \{\lambda_0\}$ where $|\lambda_0| = 1$, then T acts on a one dimensional space. Our first result carries the later one to operators which inherit FSM.

Theorem 1. *Let T inherit* FSM-n *and be power bounded. Let $\sigma(T) = \{\lambda_0\}$ where $|\lambda_0| = 1$. Then T acts on a space of dimension at most n.*

Proof. Replacing T by $(1/\lambda_0)\,T$, we may assume that $\lambda_0 = 1$. As $\partial\sigma(T) \subseteq \sigma_p(T^*)$ [3], $1 \in \sigma_p(T^*)$. Thus there exists a vector e_1 such that $T^*e_1 = e_1$. Let the decomposition of T relative to the decomposition of the space $H = V\{e_1\} \oplus \{e_1\}^\perp$ be

$$T = \begin{bmatrix} 1 & 0 \\ A & B \end{bmatrix}$$

If $\{e_1\}^\perp \neq 0$ then $T_1 = T|_{\{e_1\}^\perp}$ has FSM-n and $\sigma(T_1) = \{1\}$. So there exists a unit vector $e_2 \perp e_1$ with $T^*e_2 = e_2$. Let the decomposition of T relative to $H = V\{e_1, e_2\} \oplus \{e_1, e_2\}^\perp$ be

$$T = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} & 0 \\ C & D \end{bmatrix}$$

Now

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ n\lambda & 1 \end{bmatrix}$$

This implies that $\lambda = 0$. Hence

$$T = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \\ C & D \end{bmatrix} = \begin{bmatrix} I_2 & 0 \\ C & D \end{bmatrix}$$

If $\{e_1, e_2\}^\perp \neq \{0\}$, we can repeat the process to get $e_3 \perp \{e_1, e_2\}$ such that $T_2^* e_3 = e_3$ where $T_2 = T|_{\{e_1, e_2\}^\perp}$. Decomposition of T relative to the decomposition $H = V\{e_1, e_2, e_3\} \oplus \{e_1, e_2, e_3\}^\perp$ gives

$$T = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_1 & \lambda_2 & 1 \end{bmatrix} & 0 \\ E & F \end{bmatrix}$$

As T is power bounded, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_1 & \lambda_2 & 1 \end{bmatrix}$ is also so. Again

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda_1 & \lambda_2 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n\lambda_1 & n\lambda_2 & 1 \end{bmatrix}$$

This implies that $\lambda_1 = 0$, $\lambda_2 = 0$. Hence

$$T = \begin{bmatrix} I_3 & 0 \\ E & F \end{bmatrix}$$

We claim that the process must terminate after n steps. For, if $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$

are mutually perpendicular unit vectors such that

$T^*e_1 = e_1$, $(T|_{\{e_1\}^\perp})^* e_2 = e_2, \ldots, (T|_{\{e_1,e_2,\ldots,e_n\}^\perp})^* e_{n+1} = e_{n+1}$ then we can write

$$T = \begin{bmatrix} I_{n+1} & 0 \\ P & Q \end{bmatrix}$$

As $V\{e_1, e_2, \ldots, e_{n+1}\}^\perp \in \mathrm{Lat}\, T$, $T|_{V\{e_1,e_2,\ldots,e_{n+1}\}}$ has FSM at most n [7]. But the identity operator is of FSM-n only on a space of dimension-n. Thus $\{e_1, e_2, \ldots, e_n\}^\perp = \{0\}$. Hence H has dimension at the most 'n'.

The proof of the following theorem follows using Theorem 1, [2, Theorem 1.3] and the techniques developed by E. J. Rosenthal in [6, Theorem 2]. Hence we omit the proof.

Theorem 2. *A power bounded operator which inherits* FSM, *is similar to a contraction.*

The following is an easy consequence of Theorem 2.

Corollary 3. *A power bounded operator with is the direct sum of a finite number of operators that inherit* FSM, *is similar to a contraction.*

By $H^{(n)}$, we mean the direct sum of n copies of H. For T in $B(H)$, $T^{(n)}$ is the operator on $H^{(n)}$ defined by

$$T^{(n)}(x_1, x_2, \ldots, x_n) = (Tx_1, Tx_2, \ldots, Tx_n) .$$

For a subset $\mathscr{B}$ of $B(H)$, let $\mathscr{B}^{(n)} = \{T^{(n)}: T \in \mathscr{B}\}$. If M is a subspace of $H^{(n)}$, then ith kernel of M is the collection of all vectors in M whose ith coordinate is zero. If $M \in \mathrm{Lat}\, T^{(n)}$, then ith kernel of M is invariant under $T^{(n)}$, and is isomorphic to an element of Lat $T^{(n-1)}$. If $\mathscr{B}^{(n)} \subseteq B(H^{(n)})$ and $M \in \mathrm{Lat}\, \mathscr{B}^{(n)}$ then M is an invariant graph subspace of $\mathscr{B}^{(n)}$ on the ith co-ordinate if M has the form

$$M = \{(T_1x, T_2x, \ldots, T_{i-1}x, x, T_{i+1}x, \ldots, T_nx): x \in D\}$$

for some linear manifold D of H, and for all linear transformations T_i with domain D and range contained in H. The T_i's are called *graph transformations* for $\mathscr{B}$. If M is an invariant subspace of $\mathscr{B}^{(n)}$, then M is a graph subspace on the ith co-ordinate if and only if its ith kernel is $\{0\}$; equivalently, if and only if the ith co-ordinate of a vector determines the vector. Also the domain of a graph transformation for $\mathscr{B}$ is invariant under $\mathscr{B}$ and the transformation commutes with every operator in $\mathscr{B}$. In particular, if T is a graph transformation, then so is $T - \lambda I$ for every scalar λ.

The following theorem is an extension of [8, Theorem 1].

Theorem 4. *Let $\mathscr{A}$ be a unicellular operator algebra which inherits* FSM-n *together with condition*-S_n. *Then* Lat $\mathscr{A}^{(m)}$ *can be expressed as a span of at the most m invariant graph subspaces whose domains are in* Lat $\mathscr{A}$.

Proof. Let $M \in \mathrm{Lat}\, \mathscr{A}^{(m)}$. Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_m$. Then each $\overline{M}_i \in \mathrm{Lat}\, \mathscr{A}$. As Lat $\mathscr{A}$ is totally ordered, we can choose i_0 such that $M_i \subseteq \overline{M}_{i_0}$ for all

$i = 1, 2, \ldots, m$. Let $\tilde{\mathscr{A}} = \bar{\mathscr{A}}|_{M_{i_0}}$. $\tilde{\mathscr{A}}$ has FSM-n. Also

$$\tilde{\mathscr{A}}^{(m)} = \{T^{(m)}: T \in \tilde{\mathscr{A}}\} = \{T^{(m)}: T \in \bar{\mathscr{A}}|_{\bar{M}_{i_0}}\} = \mathscr{A}^{(m)}|_{\bar{M}_{i_0}{}^m}.$$

Thus $\tilde{\mathscr{A}}^{(m)} = \mathscr{A}^{(m)}|_N$ where $N = \bar{M}_{i_0}^m$. This implies that M is in Lat $\tilde{\mathscr{A}}^{(m)}$. Hence each M_i, and in particular M_{i_0} is invariant under $\tilde{\mathscr{A}}$. By [4], $\bar{M}_{i_0} = M_{i_0}$. Thus M_{i_0} is closed.

Let $\{x_1, x_2, \ldots, x_n\}$ be a subset of M_{i_0} such that $(\mathscr{A}, \{x_i\}_{i=1}^n)$ is an algebra of FSM satisfying condition-S_n. Let $f_1, f_2, \ldots, f_n$ be vectors in M having i_0^{th} co-ordinates as $x_1, x_2, \ldots, x_n$ respectively. Let

$$G_0 = \mathscr{A}^{(m)}[f_1, f_2, \ldots, f_n] = \{A_1^m f_1 + A_2^{(m)} f_2 + \ldots + A_n^{(m)} f_n : A_i \in \mathscr{A}\}$$

and

$$G = \tilde{\mathscr{A}}^{(m)}[f_1, f_2, \ldots, f_n] = \{\bar{A}_1^{(m)} f_1 + \bar{A}_2^m f_2 + \ldots + \bar{A}_n^{(m)} f_n : \bar{A}_i \in \tilde{\mathscr{A}}\}$$

As $\tilde{\mathscr{A}} = \bar{\mathscr{A}}|_{M_{i_0}}$, every element of G is a limit of elements of G_0. Therefore $G = \bar{G}_0$. Thus G is invariant under $\mathscr{A}^{(m)}$.

We claim that G is graph on the i_0th co-ordinate. To prove this, it is enough to show that i_0th kernel of G is zero. Let $(y_1, y_2, \ldots, y_n)$ in G be such that $y_{i_0} = 0$. There exist $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ in $\tilde{\mathscr{A}}$ such that

$$\bar{A}_1^{(m)} f_1 + \bar{A}_2^{(m)} f_2 + \ldots + \bar{A}_n^{(m)} f_n = (y_1, y_2, \ldots, y_n)$$

Comparing i_0th co-ordinates on both the sides, we get

$$A_1 x_1 + A_2 x_2 + \ldots + A_n x_n = y_{i_0} = 0$$

Using condition-S_n on $\tilde{\mathscr{A}}$, we get that $(y_1, y_2, \ldots, y_n) = 0$. Hence G is a graph subspace on the i_0th co-ordinate and domain of G is M_{i_0} which is in Lat $\mathscr{A}$. Thus graph transformations are densely defined and commute with $\tilde{\mathscr{A}}$, an algebra of finite strict multiplicity. By [3], graph transformations are bounded. This implies that G is closed.

Let K be the i_0th kernel of M. Then K is invariant under $\mathscr{A}^{(m)}$. Let $x \in M$. There exists y in G such that $P_{i_0}(x) = P_{i_0}(y)$. This implies that $(x - y) \in K = i_0$th kernel of M. Thus $x = y + (x - y) \in G \vee K$. Hence $M = G \vee K$.

We can perform the same procedure on K. As $P_{i_0}K = \{0\}$, the index chosen will be different from i_0, and next kernel chosen will have at least two co-ordinate projections which are $\{0\}$. We continue the process getting n invariant graph subspaces of $\mathscr{A}^{(m)}$. As the number of zero co-ordinate projections increases at each step, the process must terminate. M is the span of these subspaces. This completes the proof of the theorem.

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