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A FUNDAMENTAL ESTIMATE AND AN INEQUALITY  
OF CACCIOPPOLI'S TYPE FOR NONLINEAR PARABOLIC  
SYSTEMS OF HIGHER ORDER

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1. INTRODUCTION

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^n$ , with  $n \geq 1$  and boundary sufficiently smooth; let  $x$  be a point of  $\mathbb{R}^n$ ,  $t \in \mathbb{R}$  and  $X = (x, t)$  a point of  $\mathbb{R}^n \times \mathbb{R}$ . Let  $N$  be an integer  $> 1$ ,  $(\cdot)_k$  and  $\|\cdot\|$  the scalar product and the norm in  $\mathbb{R}^k$  respectively.

We set  $Q = \Omega \times (-T, 0)$ , with  $T > 0$ , and  $X_0 = (x^0, t_0)$ .

We define

$$B(x^0, \sigma) = \{x \in \mathbb{R}^n: \|x - x^0\| < \sigma\},$$

$$Q(X_0, \sigma) = B(x^0, \sigma) \times (t_0 - \sigma^{2m}, t_0)$$

and we say that  $Q(X_0, \sigma) \in Q$  if

$$B(x^0, \sigma) \in \Omega \quad \text{and} \quad \sigma^{2m} < t_0 + T \leq T.$$

Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$  be a multi-index and  $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ ; we denote by  $\mathcal{R}$ ,  $\mathcal{R}^*$  and  $\mathcal{R}'$  respectively the cartesian products  $\prod_{|\alpha| \leq m} \mathbb{R}_\alpha^N$ ,  $\prod_{|\alpha| \leq m-1} \mathbb{R}_\alpha^N$  and  $\prod_{|\alpha|=m} \mathbb{R}_\alpha^N$ , while  $p = \{p^\alpha\}_{|\alpha| \leq m}$ ,  $p^* = \{p^\alpha\}_{|\alpha| \leq m-1}$  and  $p' = \{p^\alpha\}_{|\alpha|=m}$ ,  $p^\alpha \in \mathbb{R}^N$ , are respectively points of  $\mathcal{R}$ ,  $\mathcal{R}^*$  and  $\mathcal{R}'$ .

If  $u: Q \rightarrow \mathbb{R}^N$ , we set

$$Du = \{D^\alpha u\}_{|\alpha| \leq m}, \quad \delta u = \{D^\alpha u\}_{|\alpha| \leq m-1},$$

$$D^{(k)}u = \{D^\alpha u\}_{|\alpha|=k}, \quad k = 1, 2, \dots, m.$$

Let  $A^\alpha(X, p)$ ,  $|\alpha| = m$ , be vectors of  $\mathbb{R}^N$  defined in  $Q \times \mathcal{R}$ , measurable in  $X$  and continuous in  $p$  and such that

$$A^\alpha(X, p^*, 0) = 0, \quad |\alpha| = m.$$

Let us consider the nonlinear differential operator

$$(1.1) \quad E_0 u = (-1)^m \sum_{|\alpha|=m} D^\alpha A^\alpha(X, Du) + \frac{\partial u}{\partial t}.$$

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Let us suppose that the vectors  $A^\alpha(X, p)$  are differentiable with respect to  $p'$  with derivatives  $\partial A^\alpha / \partial p_k^\beta$ ,  $|\alpha| = |\beta| = m$ ,  $k = 1, 2, \dots, N$ , measurable in  $X$  continuous in  $p$  and bounded in  $Q \times \mathcal{R}$

$$(1.2) \quad \left\{ \sum_{h,k=1}^N \sum_{|\alpha|=m} \sum_{|\beta|=m} \left| \frac{\partial A_h^\alpha}{\partial p_k^\beta} \right|^2 \right\}^{1/2} \leq M, \quad \forall (X, p) \in Q \times \mathcal{R}.$$

We denote by  $A_{\alpha\beta} N \times N$  matrices defined in  $Q$  setting

$$A_{\alpha\beta} = \{A_{\alpha\beta}^{hk}\} \quad \text{with} \quad A_{\alpha\beta}^{hk}(X, p) = \int_0^1 \frac{\partial A_h^\alpha(X, p^*, \tau p')}{\partial p_k^\beta} d\tau$$

so that, thank to the fact that  $A^\alpha(X, p^*, 0) = 0$ , we have

$$\begin{aligned} \sum_{k=1}^N \sum_{|\beta|=m} A_{\alpha\beta}^{hk}(X, p) p_k^\beta &= \int_0^1 \sum_{k=1}^N \sum_{|\beta|=m} \frac{\partial A_h^\alpha(X, p^*, \tau p')}{\partial p_k^\beta} p_k^\beta d\tau = \\ &= \int_0^1 \frac{\partial A_h^\alpha(X, p^*, \tau p')}{\partial \tau} d\tau = A_h^\alpha(X, p) \end{aligned}$$

from which it results

$$A^\alpha(X, p) = \sum_{|\beta|=m} A_{\alpha\beta}(X, p) p^\beta$$

and

$$E_0 u = (-1)^m \sum_{|\alpha|=m} \sum_{|\beta|=m} D^\alpha [A_{\alpha\beta}(X, Du) D^\beta u] + \frac{\partial u}{\partial t}.$$

We also suppose that the operator  $E_0$  is strongly parabolic in the following sense: there exists  $\nu > 0$  such that

$$(1.4) \quad \sum_{h,k=1}^N \sum_{|\alpha|=m} \sum_{|\beta|=m} \frac{\partial A_h^\alpha}{\partial p_k^\beta} \xi_h^\alpha \xi_k^\beta \geq \nu \sum_{|\alpha|=m} \|\xi^\alpha\|^2$$

for every  $(X, p) \in Q \times \mathcal{R}$  and for any system  $(\xi^\alpha)_{|\alpha|=m}$  of vectors of  $\mathbb{R}^N$ .  $H^{k,p}(\Omega, \mathbb{R}^N)$  and  $H_0^{k,p}(\Omega, \mathbb{R}^N)$ ,  $k$  integer  $\geq 0$ , and  $p \geq 1$ , are the usual Sobolev space of the vectors  $u: \Omega \rightarrow \mathbb{R}^N$ , that, if  $p = 2$ , we shall simply write with  $H^k$  and  $H_0^k$  respectively.

If  $u \in H^{k,p}(\Omega, \mathbb{R}^N)$ ,  $1 \leq p < +\infty$ , we define

$$\begin{aligned} |u|_{k,p,\Omega} &= \left\{ \int_\Omega \left( \sum_{|\alpha|=k} \|D^\alpha u\|^2 \right)^{p/2} dx \right\}^{1/p}, \\ \|u\|_{k,p,\Omega} &= \left\{ \int_\Omega \left( \sum_{|\alpha| \leq k} \|D^\alpha u\|^2 \right)^{p/2} dx \right\}^{1/p} \end{aligned}$$

that, if  $p = 2$ , we shall simply write  $|\cdot|$  and  $\|\cdot\|$  respectively.

The object of the present work is to prove a fundamental inequality [see 2.3] together with an inequality of Caccioppoli's type; these results can be useful to guarantee, as in the case  $m = 1$ , (see [2]), a fundamental estimate for solutions of the basic systems (see [1]). Therefore the results of this work can constitute a first

step in order to obtain, at least for small  $n$ , the partial Holder continuity of the solutions to nonlinear parabolic systems of higher order with strictly controlled growth together with an evaluation of Hausdorff measure of singular set; however there are some open problems which we have in mind to examine in a next paper.

The type of solution that we consider is the following:

set

$$a(u, \varphi) = \int_Q \sum_{|\alpha|=m} (A^\alpha(X, Du) | D^\alpha \varphi) - \left( u \left| \frac{\partial \varphi}{\partial t} \right. \right) dX,$$

$$(1.5) \quad W(Q) = L^2(-T, 0, H^m(\Omega, \mathbb{R}^N)) \cap H^1(-T, 0, L^2(\Omega, \mathbb{R}^N)),$$

for solution of the system

$$E_0 u = 0 \quad \text{in } Q$$

we mean a vector

$$u \in L^2(-T, 0, H^m(\Omega, \mathbb{R}^N)) \cap L^\infty(-T, 0, L^2(\Omega, \mathbb{R}^N))$$

so that

$$a(u, \varphi) = 0$$

$$\forall \varphi \in W(Q): \varphi(x, -T) = \varphi(x, 0) = 0, \quad \text{in } \Omega$$

or, which is the same, so that

$$\int_Q \left\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(X, Du) D^\alpha u | D^\beta \varphi) - \left( u \left| \frac{\partial \varphi}{\partial t} \right. \right) \right\} dx = 0$$

$$\forall \varphi \in W(Q): \varphi(x, -T) = \varphi(x, 0) = 0, \quad \text{in } \Omega.$$

## 2. A FUNDAMENTAL ESTIMATE FOR NONLINEAR PARABOLIC SYSTEMS WITH DISCONTINUOUS COEFFICIENTS OF HIGHER ORDER

In this section for the sake of brevity we set  $Q(X_0, \sigma) = Q(\sigma)$ ,  $Q(X_0, \sigma)$  being such that:  $X_0 \in Q$ ,  $Q(X_0, \sigma) \Subset Q$  and we show that the following theorem holds:

**Theorem 2.1.** *If  $u \in L^2(t_0 - \sigma^2, t_0, H^m(B(\sigma)))$  is solution in  $Q(X_0, \sigma)$  of the system*

$$\int_{Q(\sigma)} \left\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(X, Du) D^\alpha u | D^\beta \varphi) - \left( u \left| \frac{\partial \varphi}{\partial t} \right. \right) \right\} dx = 0$$

$$\forall \varphi \in C_0^\infty(Q(\sigma)),$$

then it exists  $\varepsilon(v, M) \in (0, 1)$  such that  $\forall \lambda \in (0, 1)$

$$(2.1) \quad \int_{Q(\sigma)} \|D^{(m)}u\|^2 dX \leq c(v, M) \lambda^{\varepsilon(n+2m)} \int_{Q(\sigma)} \|D^{(m)}u\| dX.$$

Fixed  $\mu = (M^2 - v^2)/v$ , we decompose  $u$  as  $u = v + w$  where

$w \in L^2(t_0 - \sigma^2, t_0, H_0^m(B(\sigma)))$  is the solution of C.D. problem

$$(2.2) \quad \int_{Q(\sigma)} (M + \mu) \sum_{|\alpha|=m} (D^\alpha w | D^\alpha \varphi) - \left( w | \frac{\partial \varphi}{\partial t} \right) dX = \\ = \int_{Q(\sigma)} \sum_{|\alpha|=m} ((M + \mu) D^\alpha u - \sum_{|\beta|=m} A_{\alpha\beta} D^\beta u | D^\alpha \varphi) dX = 0, \\ \forall \varphi \in W(Q(\sigma)): \varphi(x, t_0) = 0 \quad \text{in } B(\sigma)$$

while  $v \in L^2(t_0 - \sigma^2, t_0, H^m(B(\sigma)))$  is the solution of the system

$$\int_{Q(\sigma)} (M + \mu) \sum_{|\alpha|=m} (D^\alpha v | D^\alpha \varphi) - \left( v | \frac{\partial \varphi}{\partial t} \right) dX = 0 \\ \forall \varphi \in C_0^\infty(Q(\sigma)).$$

From the linear theory it is known (see Lemma 1.1. p. 68 of [3]) that for any solution  $w \in L^2(t_0 - \sigma^2, t_0, H_0^m(B(\sigma)))$  of the system (2.2) we have

$$(2.3) \quad \int_{Q(\sigma)} \|D^{(m)} w\|^2 dX \leq K^2(\mu) \int_{Q(\sigma)} \|D^{(m)} u\|^2 dX$$

where

$$K(\mu) = \frac{M - \nu + \sqrt{(\mu^2 + M_-^2)}}{M + \mu} \quad \text{with } M_- = \sup \left\{ \sum_{|\alpha|=m} \sum_{|\beta|=m} \|A_{\alpha\beta}^-\|^2 \right\}^{1/2}$$

$A_{\alpha\beta}^- = \frac{1}{2}(A_{\alpha\beta} - A_{\alpha\beta}^*)$ ,  $A_{\alpha\beta}^*$  adjoint of  $A_{\alpha\beta}$ .

Moreover it is known that for the function  $v$  the following fundamental estimate holds:

$\forall \lambda \in (0, 1)$

$$(2.4) \quad \int_{Q(\lambda\sigma)} \sum_{|\alpha|=m} \|D^\alpha v\|^2 dX \leq c(v, M) \lambda^{n+2m} \int_{Q(\sigma)} \sum_{|\alpha|=m} \|D^\alpha v\|^2 dX;$$

then from estimates (2.3), (2.4) it follows

$$\begin{aligned} |D^{(m)} u|_{0, Q(\lambda\sigma)} &\leq |D^{(m)} w|_{0, Q(\lambda\sigma)} + |D^{(m)} v|_{0, Q(\lambda\sigma)} \leq \\ &\leq K(\mu) |D^{(m)} u|_{0, Q(\sigma)} + c^{1/2}(v, M) \lambda^{(n+2m)/2} |D^{(m)} v|_{0, Q(\sigma)} \leq \\ &\leq K(\mu) |D^{(m)} u|_{0, Q(\sigma)} + c^{1/2}(v, M) \lambda^{(n+2m)/2} |D^{(m)} u|_{0, Q(\sigma)} + \\ &+ c^{1/2}(v, M) \lambda^{(n+2m)/2} |D^{(m)} w|_{0, Q(\sigma)}^2 \leq K(\mu) |D^{(m)} u|_{0, Q(\sigma)} + \\ &+ c^{1/2}(v, M) \lambda^{(n+2m)/2} |D^{(m)} u|_{0, Q(\sigma)} + \\ &+ c^{1/2}(v, M) \lambda^{(n+2m)/2} K(\mu) |D^{(m)} u|_{0, Q(\sigma)} = \\ &= \{c^{1/2}(v, M) (1 + K(\mu)) \lambda^{(n+2m)/2} + K(\mu)\} |D^{(m)} u|_{0, Q(\sigma)}. \end{aligned}$$

Since  $K(\mu)$ , according to the choice we have done for  $\mu$ , results  $< 1$  we may apply the lemma 1.V, p. 12 of [Q] for which it exists  $\varepsilon(v, M) \in (0, 1)$  such that (2.1) holds; then the proof of Theorem 2.1 is achieved.

### 3. CACCIOPPOLI'S TYPE INEQUALITY

In this section we prove the following Caccioppoli's type inequality:

**Theorem 3.1.** *If  $u \in L^2(-T, 0, H^m(\Omega))$  is solution in  $Q$  of the system*

$$\int_Q \sum_{|\alpha|=m} \sum_{|\beta|=m} (A_{\alpha\beta}(X, Du) D^\alpha u | D^\beta \varphi) - \left( u \left| \frac{\partial \varphi}{\partial t} \right. \right) dx = 0$$

$$\forall \varphi \in C_0^\infty(Q)$$

*then,  $\forall B(x^0, 2\sigma) \in \Omega, \forall 2a \in (0, T)$ , and for every vector polynomial  $\eta(x)$  of degree at most  $m - 1$  it result:*

$$(3.2) \quad \int_{-a}^0 dt \int_{B(\sigma)} \|D^{(m)}u\|^2 dx \leq \frac{c(v)}{a} \int_{-2a}^0 dt \int_{B(2\sigma)} \|u - \eta(x)\|^2 dx +$$

$$+ \sum_{|\beta|=m} \sum_{\gamma < \beta} \frac{c(v, M)}{\sigma^{2(m-|\gamma|)}} \int_{-2a}^0 dt \int_{B(2\sigma)} \|D^\gamma(u - \eta(x))\|^2 dx.$$

*Proof.* Let  $\vartheta(x) \in C_0^\infty(\mathbb{R}^n)$  be a function with the following properties:

$$(3.3) \quad 0 \leq \vartheta \leq 1, \quad \vartheta = 1 \text{ in } B(\sigma), \quad \vartheta = 0 \text{ in } \mathbb{R}^n \setminus B(\frac{3}{2}\sigma)$$

$$\|D^\gamma \vartheta\| \leq c\sigma^{-|\gamma|}, \quad \forall \gamma.$$

Let  $\varrho_p(t)$ , with  $p$  integer  $> 2/a$ , ( $2a \in (0, T)$ ), be a function defined on  $\mathbb{R}$  as follows

$$(3.4) \quad \begin{cases} \varrho_p(t) = 1 & \text{if } -a \leq t \leq -2/p \\ \varrho_p(t) = 0 & \text{if } t > -1/p \text{ or } t < -2a \\ \varrho_p(t) = t/a + 2 & \text{if } -2a \leq t \leq -a \\ \varrho_p(t) = -(p \cdot t + 1) & \text{if } -2/p \leq t \leq -1/p \end{cases}$$

Let, at last,  $\{g_s(t)\}$  be a sequence of symmetric mollifying functions

$$(3.5) \quad \begin{cases} g_s(t) \in C(\mathbb{R}), \quad g_s(t) \geq 0, \quad g_s(t) = g_s(-t) \\ \text{supp } g_s(t) \subset [-1/s, 1/s] \\ \int_{\mathbb{R}} g_s(t) dt = 1. \end{cases}$$

Since (3.1) is true for any  $\varphi \in W(Q)$ :  $\varphi(x, -T) = \varphi(x, 0) = 0$  in  $\Omega$ , if  $s > \max\{p, 1/(T - 2a)\}$  we can assume in (3.1)

$$(3.6) \quad \varphi(x) = \vartheta^{2m} \varrho_p[\varrho_p(u - \eta(x)) * g_s].$$

Taking into account that

$$D^\beta \varphi(x) = \vartheta^{2m} \varrho_p[\varrho_p D^\beta u * g_s] + \vartheta^m \varrho_p[\sum_{\gamma < \beta} \varrho_p(C_{\beta\gamma}(\vartheta) D^\gamma(u - \eta(x)) * g_s)]$$

with

$$(3.7) \quad |C_{\beta\gamma}(\vartheta)| \leq c\sigma^{-|\gamma|}, \quad |\gamma| < m$$

we obtain

$$\begin{aligned}
 (3.8) \quad & \int_{\mathcal{Q}} (\vartheta^{2m} \varrho_p \sum_{|\alpha|=m} \sum_{|\beta|=m} A_{\alpha\beta} D^\alpha u \mid (D^\beta u \varrho_p * g_s)) \, dX + \\
 & + \int_{\mathcal{Q}} (\vartheta^{2m} \varrho_p \sum_{|\alpha|=m} \sum_{|\beta|=m} A_{\alpha\beta} D^\alpha u \mid \sum_{\gamma < \beta} \varrho_p (C_{\beta\gamma}(\vartheta) D^\gamma (u - \eta) * g_s)) \, dX - \\
 & - \int_{\mathcal{Q}} (u - \eta) \mid \vartheta^{2m} \varrho'_p [(\varrho_p (u - \eta(x)) * g_s)] \, dX = \\
 & = \int_{\mathcal{Q}} (u - \eta) \mid \vartheta^{2m} \varrho_p [(\varrho_p (u - \eta(x)) * g_s)]' \, dX .
 \end{aligned}$$

For  $g_s(t)$  symmetry the integral at the second member is equal to zero; furthermore when  $s \rightarrow +\infty$

$$[\varrho(u - \eta(x))] * g \rightarrow \varrho_p(u - \eta) \quad \text{in } L^2(-T, 0, H^m(\Omega)).$$

So that, from (3.8), taking the limit for  $s \rightarrow +\infty$  we obtain

$$\begin{aligned}
 (3.9) \quad & \int_{\mathcal{Q}} (\vartheta^{2m} \varrho_p^2 \sum_{|\alpha|=m} \sum_{|\beta|=m} A_{\alpha\beta} D^\alpha u \mid D^\beta u) \, dX = \\
 & = \int_{\mathcal{Q}} (\varrho_p^2 \vartheta^m \sum_{|\alpha|=m} \sum_{|\beta|=m} A_{\alpha\beta} D^\alpha u \mid \sum_{\gamma < \beta} C_{\beta\gamma}(\vartheta) D^\gamma (u - \eta(x)) * g_s) \, dX - \\
 & - \int_{\mathcal{Q}} \vartheta^{2m} \varrho_p \varrho'_p \|u - \eta(x)\|^2 \, dX .
 \end{aligned}$$

Furthermore, by elliptic condition and Hölder inequality from (3.9) we obtain for every  $\varepsilon > 0$ :

$$\begin{aligned}
 & v \int_{\mathcal{Q}} \vartheta^{2m} \varrho_p^2 \sum_{|\alpha|=m} \|D^\alpha u\| \, dX \leq m \cdot \varepsilon \int_{\mathcal{Q}} \varrho_p^2 \vartheta^{2m} \sum_{|\alpha|=m} \|D^\alpha u\|^2 \, dX + \\
 & + c(\varepsilon, M) \int_{\mathcal{Q}} \varrho_p^2 \sum_{|\beta|=m} \sum_{\gamma < \beta} \|C_{\beta\gamma}(\vartheta)\|^2 \|D^\gamma (u - \eta(x))\|^2 \, dX + \\
 & + \int_{\mathcal{Q}} \vartheta^2 \varrho_p \varrho'_p \|D^\gamma (u - \eta(x))\|^2 \, dX .
 \end{aligned}$$

Since from (3.4) it results that

$$\varrho_p \varrho'_p = \begin{cases} \leq 0 & \text{for } t \geq -2/\varrho \\ = 0 & \text{for } t \leq 2a \quad \text{and for } -a \leq t \leq -2/\varrho \\ \leq 1/a & \text{for } -2a \leq t \leq -a \end{cases}$$

taking into account (3.3), (3.4) and (3.7) and choosing  $\varepsilon = v/m$  we obtain

$$\begin{aligned}
 & \int_{-a}^{-2/p} dt \int_{B(\sigma)} \|D^{(m)} u\|^2 \, dx \leq \frac{c(v)}{a} \int_{-2a}^{-2/p} dt \int_{B(2\sigma)} \|u - \eta(x)\|^2 \, dx + \\
 & + \sum_{|\beta|=m} \sum_{\gamma < \beta} \frac{c(v, M)}{\sigma^{2(m-|\gamma|)}} \int_{-2a}^{-2/p} dt \int_{B(2\sigma)} \|D^\gamma (u - \eta(x))\|^2 \, dx .
 \end{aligned}$$

Taking the limit for  $p \rightarrow +\infty$  we have

$$\int_{-a}^0 dt \int_{B(\sigma)} \|D^{(m)}u\|^2 dx \leq \frac{c(v)}{a} \int_{-2a}^0 dt \int_{B(2\sigma)} \|u - \eta(x)\|^2 dx +$$

$$+ \sum_{|\beta|=m} \sum_{\gamma < \beta} \frac{c(v, M)}{\sigma^{2(m-|\gamma|)}} \int_{-2a}^0 dt \int_{B(2\sigma)} \|D^\gamma(u - \eta(x))\|^2 dx$$

that is the (3.2).

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