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PRINCIPAL TWO-SIDED IDEALS IN THE DIRECT PRODUCT OF TWO SEMIGROUPS

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It is known that the direct product of two ideals (both two-sided and one-sided) in the direct product of two semigroups is an ideal in the direct product of two semigroups. However, if we add some other condition, e.g. for maximal ideals or principal ideals, it need not be so (see [6]). Principal left ideals in the direct product of two semigroups are studied in [2].

The aim of the paper is to find conditions, under which $J(a, b) = J(a) \times J(b)$ in $S_1 \times S_2$, for $(a, b) \in S_1 \times S_2$, where

(1) $$J(a, b) = (a, b) \cup (S_1 a \times S_2 b) \cup (aS_1 \times bS_2) \cup (S_1 a S_1 \times S_2 b S_2)$$

is a principal two-sided ideal, generated by $(a, b) \in S_1 \times S_2$, and

(2) $$\begin{aligned} J(a) \times J(b) &= (a \cup S_1 a \cup aS_1 \cup S_1 a S_1) \times (b \cup S_2 b \cup bS_2 \cup \\ &\cup S_2 b S_2) = (a, b) \cup (a \times S_2 b) \cup (a \times bS_2) \cup (a \times S_2 b S_2) \cup \\ &\cup (S_1 a \times b) \cup (S_1 a \times S_2 b) \cup (S_1 a \times bS_2) \cup (S_1 a \times S_2 b S_2) \cup \\ &\cup (aS_1 \times b) \cup (aS_1 \times S_2 b) \cup (aS_1 \times bS_2) \cup (aS_1 \times S_2 b S_2) \cup \\ &\cup (S_1 a S_1 \times b) \cup (S_1 a S_1 \times S_2 b) \cup (S_1 a S_1 \times bS_2) \cup (S_1 a S_1 \times S_2 b S_2) \end{aligned}$$

From the relation (1) and (2) we get

Lemma 1. *Let $a \in S_1$, $b \in S_2$. Then*

$$J(a, b) \subseteq J(a) \times J(b) .$$

In some cases $J(a, b) \subset J(a) \times J(b)$, in some other cases $J(a, b) = J(a) \times J(b)$.

Example 1. If $z_1 \in S_1$ is zero in S_1, $z_2 \in S_2$ is zero in S_2, then for any $a \in S_1$, $b \in S_2$,

$$J(z_1, b) = J(z_1) \times J(b) \quad \text{and} \quad J(a, z_2) = J(a) \times J(z_2) .$$

It immediately implies from (1) and (2).

Lemma 2. *Let $a \in S_1$, $b \in S_2$ be nonzero elements and at least for one of them $a \notin (S_1 a \cup aS_1 \cup S_1 a S_1)$, $b \notin (S_2 b \cup bS_2 \cup S_2 b S_2)$. Then*

$$J(a, b) \subset J(a) \times J(b) .$$

Proof. Let for $a \in S_1$ be $a \notin (S_1 a \cup aS_1 \cup S_1 aS_1)$. Then in $J(a, b)$ there is the only element having the first component a, namely (a, b). However, in $J(a) \times J(b)$ there are certainly more of such elements, namely $(a \times S_2 b), (a \times bS_2), (a \times S_2 bS_2)$ (as b is nonzero element).

Corollary. *Let $a \in S_1$, $b \in S_2$. A necessary condition for that $J(a, b) = J(a) \times J(b)$ in $S_1 \times S_2$ is that at least one of the folowing conditions holds:*

1. $S_1 a = aS_1 = \{a\}$
2. $S_2 b = bS_2 = \{b\}$
3. $a \in (S_1 a \cup aS_1 \cup S_1 aS_1)$ *and* $b \in (S_2 b \cup bS_2 \cup S_2 bS_2)$.

Remark 1. If at least one of $a \in S_1$, $b \in S_2$ is zero, then

$$J(a, b) = J(a) \times J(b),$$

as we have seen in Example 1.

If we want to find all cases, when $J(a, b) = J(a) \times J(b)$ in $S_1 \times S_2$, then it is necessary to consider all possibilities in which the condition 3 of Corollary holds.

The investigation will be devided into two cases:

I. $a \in (S_1 a \cup aS_1 \cup S_1 aS_1)$ and $b \in (S_2 b \cup bS_2 \cup S_2 bS_2)$, but $(a, b) \notin \{(S_1 a \times S_2 b) \cup (aS_1 \times bS_2) \cup (S_1 aS_1 \times S_2 bS_2)\}$

II. $(a, b) \in \{(S_1 a \times S_2 b) \cup (aS_1 \times bS_2) \cup (S_1 aS_1 \times S_2 bS_2)\}$.

Example 2. Let $S_1 = \{a_1, a_2, a_3, a_4\}$, $S_2 = \{b_1, b_2, b_3, b_4\}$. The associative binary operations in S_1 and in S_2 are given by means of the following tables:

	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_2
a_3	a_1	a_1	a_1	a_3
a_4	a_1	a_1	a_1	a_4

	b_1	b_2	b_3	b_4
b_1	b_1	b_1	b_1	b_1
b_2	b_1	b_1	b_1	b_2
b_3	b_1	b_1	b_1	b_1
b_4	b_1	b_1	b_3	b_4

$J(a_3) = \{a_1, a_3\}$, $J(b_3) = \{b_1, b_3\}$, $a_3 \in (S_1 a_3 \cup a_3 S_1 \cup S_1 a_3 S_1)$ and $b_3 \in (S_2 b_3 \cup b_3 S_2 \cup S_2 b_3 S_2)$, but $(a_3, b_3) \notin \{(S_1 a_3 \times S_2 b_3) \cup (a_3 S_1 \times b_3 S_2) \cup (S_1 a_3 S_1 \times S_2 b_3 S_2)\}$. However, $J(a_3, b_3) = J(a_3) \times J(b_3)$.

This example indicates that $J(a, b) = J(a) \times J(b)$ may occur even in the case if $(a, b) \notin \{(S_1 a \times S_2 b) \cup (aS_1 \times bS_2) \cup (S_1 aS_1 \times S_2 bS_2)\}$.

Considering all possibilities which can occur in the case I. it can be verified that the following statement is true.

Lemma 3. *Let $a \in (S_1 a \cup aS_1 \cup S_1 aS_1)$ and $b \in (S_2 b \cup bS_2 \cup S_2 bS_2)$. Then $(a, b) \notin \{(S_1 a \times S_2 b) \cup (aS_1 \times bS_2) \cup (S_1 aS_1 \times S_2 bS_2)\}$ iff, any of the following conditions is satisfied:*

1. $[a \in S_1 a \wedge a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2 b \cup S_2 bS_2)]$
2. $[a \in aS_1 \wedge a \notin (S_1 a \cup S_1 aS_1)] \wedge [b \in S_2 b \wedge b \notin (bS_2 \cup S_2 bS_2)]$

3. $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in S_2bS_2 \wedge b \notin (S_2b \cup bS_2)]$

4. $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in S_2bS_2 \wedge b \notin (S_2b \cup bS_2)[$

5. $[a \in S_1aS_1 \wedge a \notin (S_1a \cup aS_1)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$

6. $[a \in S_1aS_1 \wedge a \notin (S_1a \cup aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$

7. $[a \in (S_1a \cap S_1aS_1) \wedge a \notin aS_1] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$

8. $[a \in (aS_1 \cap S_1aS_1) \wedge a \notin S_1a] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$

9. $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in (S_2b \cap S_2bS_2) \wedge b \notin bS_2]$

10. $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in (bS_2 \cap S_2bS_2) \wedge b \notin S_2b]$

I.

In this part we shall investigate the cases 1. – 10. and we shall show in which of them the equality: $J(a, b) = J(a) \times J(b)$ may occur and under which conditions.

Remark 2. It can be verified that if L is a left ideal of S, R is a right ideal of S, then $L \cap R \neq \emptyset$.

And since S_1a is a left ideal of S_1, aS_1 is a right ideal of S_1, then $S_1a \cap aS_1 \neq \emptyset$ and equaly $S_2b \cap bS_2 \neq \emptyset$. For our purposes denote: $S_1a \cap aS_1 = P_1$, $S_2b \cap bS_2 = P_2$.

Theorem 1. *Let* $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in bS_2 \wedge b \notin (S_2b \cup S_2bS_2)]$. *Then* $J(a, b) = J(a) \times J(b)$ *iff*

$$(1) \qquad [(aS_1 = S_1aS_1) \wedge S_1a = P_1 \cup \{a\}] \wedge [(S_2b = S_2bS_2) \wedge \wedge (bS_2 = P_2 \cup \{b\})].$$

Proof. a) $a \in S_1a$ implies $aS_1 \subseteq S_1aS_1$. Denote by $U = S_1aS_1 - aS_1$, $U' = S_1a \cap U$, $U'' = S_1aS_1 - (S_1a \cup aS_1)$.

$b \in bS_2$ implies $S_2b \subseteq S_2bS_2$. Denote by $V = S_2bS_2 - S_2b$, $V' = bS_2 \cap V$, $V'' = S_2bS_2 - (S_2b \cup bS_2)$.

$$S_1a = P_1 \cup U' \cup U_1, \quad aS_1 = P_1 \cup U_2,$$
$$S_1aS_1 = P_1 \cup U_2 \cup U' \cup U''.$$
$$S_2b = P_2 \cup V_1, \quad bS_2 = P_2 \cup V' \cup V_2,$$
$$S_2bS_2 = P_2 \cup V_1 \cup V' \cup V''.$$

Then

$$J(a) = S_1a \cup S_1aS_1 = P_1 \cup U_1 \cup U_2 \cup U' \cup U'',$$
$$J(b) = bS_2 \cup S_2bS_2 = P_2 \cup V_1 \cup V_2 \cup V' \cup V'',$$

and all subsets, expressing both $J(a)$ and $J(b)$ are mutualy disjoint and $a \in U_1$ and

$a \notin (U_2 \cup U' \cup U'')$, $b \in V_2$ and $b \notin (V_1 \cup V' \cup V'')$,

$$J(a) \times J(b) = (P_1 \cup U_1 \cup U_2 \cup U' \cup U'') \times$$
$$\times (P_2 \cup V_1 \cup V_2 \cup V' \cup V'') =$$
$$= (P_1 \times P_2) \cup (P_1 \times V_1) \cup (P_1 \times V_2) \cup (P_1 \times V') \cup (P_1 \times V'') \cup$$
$$\cup (U_1 \times P_2) \cup (U_1 \times V_1) \cup (U_1 \times V_2) \cup (U_1 \times V') \cup (U_1 \times V'') \cup$$
$$\cup (U_2 \times P_2) \cup (U_2 \times V_1) \cup (U_2 \times V_2) \cup (U_2 \times V') \cup (U_2 \times V'') \cup$$
$$\cup (U' \times P_2) \cup (U' \times V_1) \cup (U' \times V_2) \cup (U' \times V') \cup (U' \times V'') \cup$$
$$\cup (U'' \times P_2) \cup (U'' \times V_1) \cup (U'' \times V_2) \cup (U'' \times V') \cup (U'' \times V'') .$$

$$J(a, b) = (a, b) \cup (P_1 \cup U_1 \cup U') \times (P_2 \cup V_1) \cup (P_1 \cup U_2) \times$$
$$\times (P_2 \cup V_2 \cup V') \cup (P_1 \cup U_2 \cup U' \cup U'') \times (P_2 \cup V_1 \cup V' \cup V'') =$$
$$= (a, b) \cup (P_1 \times P_2) \cup (P_1 \times V_1) \cup (U_1 \times P_2) \cup$$
$$\cup (U_1 \times V_1) \cup (U' \times P_2) \cup (U' \times V_1) \cup (P_1 \times V_2) \cup (P_1 \times V') \cup$$
$$\cup (U_2 \times P_2) \cup (U_2 \times V_2) \cup (U_2 \times V') \cup (P_1 \times V'') \cup (U_2 \times V_1) \cup$$
$$\cup (U_2 \times V'') \cup (U' \times V') \cup (U' \times V'') \cup (U'' \times P_2) \cup (U'' \times V_1) \cup$$
$$\cup (U'' \times V') \cup (U'' \times V'') .$$

If $J(a, b) = J(a) \times J(b)$, then $(U_1 \times V_2) = (a, b)$ and

$$(U_1 \times V') \cup (U_1 \times V'') = [U_1 \times (V' \cup V'')] = \emptyset$$

and

$$(U' \times V_2) \cup (U'' \times V_2) = [(U' \cup U'') \times V_2] = \emptyset$$

As $U_1 \neq \emptyset$ and $V_2 \neq \emptyset$, then $V' \cup V'' = \emptyset$ and $U' \cup U'' = \emptyset$, hence $U' = U'' = \emptyset$ and $V' = V'' = \emptyset$. Then $U_1 = \{a\}$, $V_2 = \{b\}$, and $[(aS_1 = S_1aS_1) \wedge S_1a = P_1 \cup \{a\}] \wedge [(S_2b = S_2bS_2) \wedge bS_2 = P_2 \cup \{b\}]$.

b) Let (1) be satisfied. Then $a \in S_1a$, $b \in bS_2$ implies:

$$aS_1 \subseteq S_1aS_1 , \quad S_2b \subseteq S_2bS_2 .$$
$$J(a) = S_1a \cup S_1aS_1 = S_1a \cup aS_1 ,$$
$$J(b) = bS_2 \cup S_2bS_2 = S_2b \cup bS_2 .$$

Then

$$J(a) \times J(b) = (S_1a \times S_2b) \cup (S_1a \times bS_2) \cup (aS_1 \times S_2b) \cup$$
$$\cup (aS_1 \times bS_2) = [(P_1 \cup \{a\}) \times S_2b] \cup [(P_1 \cup \{a\}) \times (P_2 \cup \{b\})] \cup$$
$$\cup (aS_1 \times S_2b) \cup [aS_1 \times (P_2 \cup \{b\})] = (P_1 \times S_2b) \cup (\{a\} \times S_2b) \cup$$
$$\cup (P_1 \times P_2) \cup (P_1 \times \{b\}) \cup (\{a\} \times P_2) \cup (a, b) \cup (aS_1 \times S_2b) \cup$$
$$\cup (aS_1 \times P_2) \cup (aS_1 \times \{b\}) = (aS_1 \times S_2b) \cup (\{a\} \times S_2b) \cup$$
$$\cup (P_1 \times \{b\}) \cup (\{a\} \times P_2) \cup (aS_1 \times \{b\}) \cup (a, b) =$$
$$= (a, b) \cup (\{a\} \times S_2b) \cup (aS_1 \times \{b\}) \cup (aS_1 \times S_2b) .$$

$$J(a, b) = (a, b) \cup (S_1a \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2) =$$
$$= (a, b) \cup [(P_1 \cup \{a\}) \times S_2b] \cup [aS_1 \times (P_2 \cup \{b\}) \cup (aS_1 \times S_2b) =$$
$$= (a, b) \cup (P_1 \times S_2b) \cup (\{a\} \times S_2b) \cup (aS_1 \times P_2) \cup (aS_1 \times \{b\}) \cup$$
$$\cup (aS_1 \times S_2b) = (a, b) \cup (\{a\} \times S_2b) \cup (aS_1 \times \{b\}) \cup (aS_1 \times S_2b) =$$
$$= J(a) \times J(b) .$$

Theorem 2. *Let* $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$. *Then* $J(a, b) = J(a) \times J(b)$ *iff*

$$[(S_1a = S_1aS_1) \wedge (aS_1 = P_1 \cup \{a\})] \wedge \wedge [(bS_2 = S_2bS_2) \wedge (S_2b = P_2 \cup \{b\})] . \tag{2}$$

Proof. The proof is analogous to that of Theorem 1.

Lemma 4. *Let* $[a \in aS_1 \wedge a \notin (S_1a \cup S_1aS_1)] \wedge [b \in S_2bS_2 \wedge b \notin (S_2b \cup bS_2)]$. *If* $J(a, b) = J(a) \times J(b)$, *then* $bS_2 \cup \{b\} = S_2bS_2$.

Proof. $a \in aS_1$ implies $S_1a \subseteq S_1aS_1$. Debote by $U = S_1aS_1 - S_1a$, $U' = aS_1 \cap U$, $U'' = S_1aS_1 - (S_1a \cup aS_1)$.

$b \in S_2bS_2$ implies $S_2b \subset S_2bS_2$, $bS_2 \subset S_2bS_2$, $S_2b \cup bS_2 \subset S_2bS_2$. Denote by $V_1 = S_2b - P_2$, $V_2 = bS_2 - P_2$, $V = S_2bS_2 - (S_2b \cup bS_2)$; $S_1a = P_1 \cup U_1$, $aS_1 = P_1 \cup U_2 \cup U'$, $S_1aS_1 = P_1 \cup U_1 \cup U' \cup U''$. $J(a) = aS_1 \cup S_1aS_1 = P_1 \cup U_1 \cup U_2 \cup U' \cup U''$, $J(b) = S_2bS_2 = P_2 \cup V_1 \cup V_2 \cup V$. P_1, P_2 denote the same, as it was said in Remark 2, $a \in U_2$, $b \in V$. Recall that P_1, U_1, U_2, U', U'' and P_2, V_1, V_2, V are mutualy disjoint. Then

$$J(a) \times J(b) = (P_1 \cup U_1 \cup U_2 \cup U' \cup U'') \times (P_2 \cup V_1 \cup V_2 \cup V) =$$
$$= (P_1 \times P_2) \cup (P_1 \times V_1) \cup (P_1 \times V_2) \cup (P_1 \times V) \cup (U_1 \times P_2) \cup$$
$$\cup (U_1 \times V_1) \cup (U_1 \times V_2) \cup (U_1 \times V) \cup (U_2 \times P_2) \cup (U_2 \times V_1) \cup$$
$$\cup (U_2 \times V_2) \cup (U_2 \times V) \cup (U' \times P_2) \cup (U' \times V_1) \cup (U' \times V_2) \cup$$
$$\cup (U' \times V) \cup (U'' \times P_2) \cup (U'' \times V_1) \cup (U'' \times V_2) \cup (U'' \times V) .$$

$$J(a, b) = (a, b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2) =$$
$$= (a, b) \cup (P_1 \cup U_2 \cup U') \times (P_2 \cup V_2) \cup (P_1 \cup U_1 \cup U' \cup U'') \times$$
$$\times (P_2 \cup V_1 \cup V_2 \cup V) = (a, b) \cup (P_1 \times P_2) \cup (P_1 \times V_2) \cup$$
$$\cup (U_2 \times P_2) \cup (U_2 \times V_2) \cup (U' \times P_2) \cup (U' \times V_2) \cup (P_1 \times V_1) \cup$$
$$\cup (P_1 \times V) \cup (U_1 \times P_2) \cup (U_1 \times V_1) \cup (U_1 \times V_2) \cup (U_1 \times V) \cup$$
$$\cup (U' \times V_1) \cup (U' \times V) \cup (U'' \times P_2) \cup (U'' \times V_1) \cup$$
$$\cup (U'' \times V_2) \cup (U'' \times V) .$$

If $J(a, b) = J(a) \times J(b)$, then $(U_2 \times V) = (a, b)$ and $(U_2 \times V_1) = \emptyset$, hence $U_2 = \{a\}$, $V = \{b\}$. However $U_2 \neq \emptyset$, then $V_1 = \emptyset$. Then

$$bS_2 \cup \{b\} = S_2bS_2$$

Theorem 3. *Let* $[a \in aS_1 \wedge a \notin (S_1 a \cup S_1 a S_1)] \wedge [b \in S_2 b S_2 \wedge b \notin (S_2 b \cup b S_2)]$. *Then*

$$J(a, b) \subset J(a) \times J(b).$$

Proof. First we show that $\mathscr{J}$-class $J_b = \{x \in S \mid J(b) = J(x)\}$ contains more than one element. If $b \in S_2 b S_2$, but $b \notin (S_2 b \cup b S_2)$, then there is $c \in S_2$ such that either $c \in S_2 b$ and $b \in c S_2$, or $c \in b S_2$ and $b \in S_2 c$. It is evident that $c \neq b$. Let $c \in S_2 b$ and $b \in c S_2$, then $S_2 c S_2 \subseteq S_2 b S_2$ and $S_2 b S_2 \subseteq S_2 c S_2$. It implies $S_2 c S_2 = S_2 b S_2$. So, $J(b) = S_2 b S_2 = S_2 c S_2 = J(c)$, therefore $c \in J_b$, $c \neq b$. From Lemma 4 we have that if $J(a, b) = J(a) \times J(b)$, then necessarily: $b S_2 \cup \{b\} = S_2 b S_2$. From there we have: for $c \in J_b$, $J(c) = c S_2 \cup \{c\} = S_2 c S_2 = S_2 b S_2 = b S_2 \cup \{b\} = J(b)$. It implies $c S_2 \cup \{c\} = b S_2 \cup \{b\}$. The last equality may occur if

1. $c = b$, or
2. $c \neq b$. Then $c \in b S_2$ and $b \in c S_2$. It implies $c S_2 \subseteq b S_2 \wedge b S_2 \subseteq c S_2$. Then $c S_2 = b S_2$.

Since $b \in c S_2 = b S_2$, it implies $b \in b S_2$, but it is contradicting with the hypothesis, as $b \notin (b S_2 \cup S_2 b)$.

Remark 3. Similarly, as we have shown in Lemma 4 what is a necessary condition for that $J(a, b) = J(a) \times J(b)$ in the case 3 of Lemma 3, then in a similar way we can show that a necessary condition for that $J(a, b) = J(a) \times J(b)$ in cases 4, 5, 6, of Lemma 3 is that the following conditions are satisfied:

$$S_2 b \cup \{b\} = S_2 b S_2,$$

$$S_1 a \cup \{a\} = S_1 a S_1,$$

$$a S_1 \cup \{a\} = S_1 a S_1.$$

And similarly as in Theorem 3 we can prove that in the cases 4, 5, 6, of Lemma 3

$$J(a, b) \subset J(a) \times J(b).$$

Lemma 5. *Let* $[a \in (S_1 a \cap S_1 a S_1) \wedge a \notin a S_1] \wedge [b \in b S_2 \wedge b \notin (S_2 b \cup S_2 b S_2)]$. *If* $J(a, b) = J(a) \times J(b)$ *then*

$$a S_1 \cup \{a\} = S_1 a S_1.$$

Proof. The proof is similar to that of Lemma 4.

Theorem 4. *Let* $[a \in (S_1 a \cap S_1 a S_1) \wedge a \notin a S_1] \wedge [b \in b S_2 \wedge b \notin (S_2 b \cup S_2 b S_2)]$. *Then*

$$J(a, b) \subset J(a) \times J(b).$$

Proof. The proof is similar to that of Theorem 3.

Remark 4. And similarly as in Lemma 4 we can show that a necessary condition for that $J(a, b) = J(a) \times J(b)$ in the cases 8, 9, 10, is that the the following conditions are satisfied:

$$S_1a \cup \{a\} = S_1aS_1 ,$$
$$bS_2 \cup \{b\} = S_2bS_2 ,$$
$$S_2b \cup \{b\} = S_2bS_2 .$$

And similarly, as in Theorem 3 we can prove that in the cases 8, 9, 10, of Lemma 3

$$J(a, b) \subset J(a) \times J(b) .$$

II.

In this part we shall consider the case that $(a, b) \in S_1 \times S_2$ satisfies the condition: $(a, b) \in [(S_1s \times S_2b) \cup (aS_1 \times bS_2) \cup (S_1aS_1 \times S_2bS_2)]$. It includes several possibilities:

a) (a, b) is contained in any component

b) (a, b) is contained just in two components

c) (a, b) is contained just in one component.

Lemma 6. *If* $(a, b) \in (S_1a \times S_2b) \cap (aS_1 \times bS_2)$ *then* $(a, b) \in (S_1aS_1 \times S_2bS_2)$.

Proof. Let $(a, b) \in (S_1a \times S_2b) \cap (aS_1 \times bS_2)$. Then $(a \in S_1a \wedge a \in aS_1) \wedge$ $\wedge (b \in S_2b \wedge b \in bS_2)$. It implies $aS_1 \subseteq S_1aS_1$ and $bS_2 \subseteq S_2bS_2$. And because $a \in aS_1$, $b \in bS_2$ we have $a \in S_1aS_1$, $b \in S_2bS_2$, therefore $(a, b) \in (S_1aS_1 \times S_2bS_2)$.

Theorem 5. *If* $(a, b) \in (S_1aS_1 \times S_2bS_2)$, *then* $J(a, b) = J(a) \times J(b)$.

Proof. If $(a, b) \in (S_1aS_1 \times S_2bS_2)$, then $J(a, b) = (S_1aS_1 \times S_2bS_2)$ and at the same time $a \in S_1aS_1$, $b \in S_2bS_2$. It implies $J(a) = S_1aS_1$ and $J(b) = S_2bS_2$, hence

$$J(a, b) = (S_1aS_1 \times S_2bS_2) = J(a) \times J(b) .$$

Before we shall proceed, let us consider one example more.

Example 3. Let $S_1 = \{a_1, a_2, a_3, a_4\}$, $S_2 = \{b_1, b_2, b_3, b_4\}$ be two semigroups, whose an associative operation is given by means of the following tables:

	a_1	a_2	a_3	a_4
a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_1	a_1	a_1
a_3	a_1	a_2	a_3	a_4
a_4	a_1	a_2	a_3	a_4

	b_1	b_2	b_3	b_4
b_1	b_1	b_2	b_1	b_1
b_2	b_1	b_2	b_1	b_1
b_3	b_1	b_2	b_3	b_4
b_4	b_1	b_2	b_4	b_4

$a_2 \in S_1a_2$, $b_2 \in S_2b_2$, so $(a_2, b_2) \in (S_1a_2 \times S_2b_2)$, $(a_2, b_2) \notin (a_2S_1 \times b_2S_2)$, $(a_2, b_2) \notin$ $\notin (S_1a_2S_1 \times S_2b_2S_2)$. $J(a_2) = \{a_1, a_2\}$, $J(b_2) = \{b_1, b_2)$. $J(a_2) \times J(b_2) =$ $= \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2)\}$. $J(a_2, b_2) = (S_1a_2 \times S_2b_2) \cup$ $\cup (S_1a_2S_1 \times S_2b_2S_2) = \{a_1, a_2\} \times \{b_2\} \cup \{a_1\} \times \{b_1, b_2\} =$ $= \{(a_1, b_1), (a_1, b_2), (a_2, b_2)\} \subset J(a_2) \times J(b_2)$.

It remains only to find conditions, under which $J(a, b) = J(a) \times J(b)$, if

(i) $(a, b) \in (S_1 a \times S_2 b) \wedge (a, b) \notin [(aS_1 \times bS_2) \cup (S_1 aS_1 \times S_2 bS_2)]$,

or

$$\text{(ii)}\ (a, b) \in (aS_1 \times bS_2) \wedge (a, b) \notin [(S_1 a \times S_2 b) \cup (S_1 aS_1 \times S_2 bS_2)] .$$

Let (i) hold. The relation: $(a, b) \notin [(aS_1 \times bS_2) \cup (S_1 aS_1 \times S_2 bS_2)]$ includes the following possibilities:

1. $[a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \in (bS_2 \cap S_2 bS_2)]$
2. $[a \in (aS_1 \cap S_1 aS_1)] \wedge [b \notin (bS_2 \cup S_2 bS_2)]$
3. $[a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \notin (bS_2 \cup S_2 bS_2)]$
4. $[a \in aS_1 \wedge a \notin S_1 aS_1] \wedge [b \notin bS_2 \wedge b \in S_2 bS_2]$
5. $[a \notin aS_1 \wedge a \in S_1 aS_1] \wedge [b \in bS_2 \wedge b \notin S_2 bS_2]$

4. and 5. cannot occur, as $(a, b) \in (S_1 a \times S_2 b)$ and 4.: $a \in aS_1$ imply $S_1 a \subseteq S_1 aS_1$. However, $a \in S_1 a \subseteq S_1 aS_1$ implies $a \in S_1 aS_1$ and it is contradicting with $a \notin S_1 aS_1$. Similarly 5 can be verified.

Combining $(a, b) \in (S_1 a \times S_2 b)$ with each of 1., 2., 3., we get the following three possibilities:

(α) $[a \in S_1 a \wedge a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \in (S_2 b \cap bS_2 \cap S_2 bS_2)]$
(β) $[a \in (S_1 a \cap aS_1 \cap S_1 aS_1)] \wedge [b \in S_2 b \wedge b \notin (bS_2 \cup S_2 bS_2)]$
(γ) $[a \in S_1 a \wedge a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \in S_2 b \wedge b \notin (bS_2 \cup S_2 bS_2)]$

Theorem 6. *Let* $[a \in S_1 a \wedge a \notin (aS_1 \cup S_1 aS_1)] \wedge [b \in (S_2 b \cap bS_2 \cap S_2 bS_2)]$. *Then*

$$J(a, b) = J(a) \times J(b) \quad \textit{iff} \quad S_2 b = S_2 bS_2 .$$

Proof. a) P_1, P_2 denote the same as it was said in Remark 2. $a \in S_1 a$ implies $aS_1 \subseteq S_1 aS_1$. Denote by $U_2 = aS_1 - P_1$, $S_1 aS_1 - aS_1 = U$, $S_1 a \cap U = U'$, $S_1 a - (P_1 \cup U') = U_1$, $S_1 aS_1 - (aS_1 \cup U') = U''$. And $V_1 = S_2 b - P_2$, $V_2 = bS_2 - P_2$, $V = S_2 bS_2 - (S_2 b \cup bS_2)$. $a \in U_1$, $b \in P_2$.

$$J(a) = S_1 a \cup S_1 aS_1 = P_1 \cup U_1 \cup U_2 \cup U' \cup U'' ,$$

$$J(b) = S_2 bS_2 = P_2 \cup V_1 \cup V_2 \cup V .$$

$$J(a) \times J(b) = (P_1 \cup U_1 \cup U_2 \cup U' \cup U'') \times (P_2 \cup V_1 \cup V_2 \cup V) =$$
$$= (P_1 \times P_2) \cup (P_1 \times V_1) \cup (P_1 \times V) \cup (U_1 \times P_2) \cup (P_1 \times V_2) \cup$$
$$\cup (U_1 \times V_1) \cup (U_1 \times V_2) \cup (U_1 \times V) \cup (U_2 \times P_2) \cup (U_2 \times V_1) \cup$$
$$\cup (U_2 \times V_2) \cup (U_2 \times V) \cup (U' \times P_2) \cup (U' \times V_1) \cup (U' \times V_2) \cup$$
$$\cup (U' \times V) \cup (U'' \times P_2) \cup (U'' \times V_1) \cup (U'' \times V_2) \cup (U'' \times V) .$$

$$J(a, b) = (S_1 a \times S_2 b) \cup (S_1 aS_1 \times S_2 bS_2) = (P_1 \cup U_1 \cup U') \times$$
$$\times (P_2 \cup V_1) \cup (P_1 \cup U_2 \cup U' \cup U'') \times (P_2 \cup V_1 \cup V_2 \cup V) =$$
$$= (P_1 \times P_2) \cup (P_1 \times V_1) \cup (U_1 \times P_2) \cup (U_1 \times V_1) \cup (U' \times P_2) \cup$$
$$\cup (U' \times V_1) \cup (P_1 \times V_2) \cup (P_1 \times V) \cup (U_2 \times P_2) \cup (U_2 \times V_1) \cup$$

$$\cup (U_2 \times V_2) \cup (U_2 \times V) \cup (U' \times V_2) \cup (U' \times V) \cup (U'' \times P_2) \cup$$
$$\cup (U'' \times V_1) \cup (U'' \times V_2) \cup (U'' \times V).$$

If $J(a) \times J(b) = J(a, b)$, then $(U_1 \times V_2) \cup (U_1 \times V) = U_1 \times (V_2 \cup V) = \emptyset$. As $U_1 \neq \emptyset$, then $V_2 \cup V = \emptyset$, hence $V_2 = V = \emptyset$. Then from the expression: $J(b) = S_2bS_2 = P_2 \cup V_1 \cup V_2 \cup V$ we get $S_2bS_2 = P_2 \cup V_1 = S_2b$, therefore

$$S_2bS_2 = S_2b.$$

b) Let (α) be satisfied and $S_2bS_2 = S_2b$. Then $a \in S_1a$ implies $aS_1 \subseteq S_1aS_1$, $b \in S_2b$ implies $bS_2 \subseteq S_2bS_2 = S_2b$. Then $J(a) = S_1a \cup S_1aS_1$, $J(b) = S_2b$, and

$$J(a) \times J(b) = (S_1a \cup S_1aS_1) \times S_2b = (S_1a \times S_2b) \cup (S_1aS_1 \times S_2b),$$
$$J(a, b) = (S_1a \times S_2b) \cup (S_1aS_1 \times S_2bS_2) =$$
$$= (S_1a \times S_2b) \cup (S_1aS_1 \times S_2b) = J(a) \times J(b).$$

Theorem 7. *Let* $[a \in (S_1a \cap aS_1 \cap S_1aS_1)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$. *Then*

$$J(a, b) = J(a) \times J(b) \quad \textit{iff} \quad S_1a = S_1aS_1.$$

Proof. The proof is similar to that of Theorem 6.

Theorem 8. *Let* $[a \in S_1a \wedge a \notin (aS_1 \cup S_1aS_1)] \wedge [b \in S_2b \wedge b \notin (bS_2 \cup S_2bS_2)]$. *Then*

$$J(a, b) = J(a) \times J(b) \quad \textit{iff}$$
$$(aS_1 \subseteq S_1aS_1 \subset S_1a) \wedge (bS_2 \subseteq S_2bS_2 \subset S_2b).$$

Proof. a) $a \in S_1a$ implies $aS_1 \subseteq S_1aS_1$ and $J(a) = S_1a \cup S_1aS_1$ $b \in S_2b$ implies $bS_2 \subseteq S_2bS_2$ and $J(b) = S_2b \cup S_2bS_2$. Denote by $U_2 = aS_1 - P_1$, $U = S_1aS_1 - aS_1$, $U' = S_1a \cap U$, $U_1 = S_1a - (P_1 \cup U')$, $U'' = S_1aS_1 - (aS_1 \cup U')$.

$$V_2 = bS_2 - P_2, \quad V = S_2bS_2 - bS_2, \quad V' = S_2b \cap V,$$
$$V_1 = S_2b - (P_2 \cup V')),$$
$$V'' = S_2bS_2 - (bS_2 \cup V').$$

Then $J(a) = P_1 \cup U_1 \cup U_2 \cup U' \cup U''$, $J(b) = P_2 \cup V_1 \cup V_2 \cup V' \cup V''$, and

$$J(a) \times J(b) = (P_1 \cup U_1 \cup U_2 \cup U' \cup U'') \times$$
$$\times (P_2 \cup V_1 \cup V_2 \cup V' \cup V'') =$$
$$= (P_1 \times P_2) \cup (P_1 \times V_1) \cup (P_1 \times V_2) \cup (P_1 \times V') \cup (P_1 \times V'') \cup$$
$$\cup (U_1 \times P_2) \cup (U_1 \times V_1) \cup (U_1 \times V_2) \cup (U_1 \times V') \cup (U_1 \times V'') \cup$$
$$\cup (U_2 \times P_2) \cup (U_2 \times V_1) \cup (U_2 \times V_2) \cup (U_2 \times V') \cup (U_2 \times V'') \cup$$
$$\cup (U' \times P_2) \cup (U' \times V_1) \cup (U' \times V_2) \cup (U' \times V') \cup (U' \times V'') \cup$$
$$\cup (U'' \times P_2) \cup (U'' \times V_1) \cup (U'' \times V_2) \cup (U'' \times V') \cup (U'' \times V'').$$

$$J(a, b) = (S_1 a \times S_2 b) \cup (S_1 a S_1 \times S_2 b S_2) = (P_1 \cup U_1 \cup U') \times$$
$$\times (P_2 \cup V_1 \cup V') \cup (P_1 \cup U_2 \cup U' \cup U'') \times (P_2 \cup V_2 \cup V' \cup V'') =$$
$$= (P_1 \times P_2) \cup (P_1 \times V_1) \cup (P_1 \times V') \cup (U_1 \times P_2) \cup (U_1 \times V_1) \cup$$
$$\cup (U_1 \times V') \cup (U' \times P_2) \cup (U' \times V_1) \cup (U' \times V') \cup (P_1 \times V_2) \cup$$
$$\cup (P_1 \times V'') \cup (U_2 \times P_2) \cup (U_2 \times V_2) \cup (U_2 \times V') \cup$$
$$\cup (U_2 \times V'') \cup (U' \times V_2) \cup (U' \times V'') \cup (U'' \times P_2) \cup (U'' \times V_2) \cup$$
$$\cup (U'' \times V') \cup (U'' \times V'') .$$

If $J(a) \times J(b) = J(a, b)$, then

$$(U_1 \times V_2) \cup (U_1 \times V'') = U_1 \times (V_2 \cup V'') = \emptyset, \quad \text{and}$$
$$(U_2 \times V_1) \cup (U'' \times V_1) = (U_2 \cup U'') \times V_1 = \emptyset .$$

As $a \in U_1 \neq \emptyset$ and $b \in V_1 \neq \emptyset$, then $U_2 \cup U'' = \emptyset$, and $V_2 \cup V'' = \emptyset$, so, $U_2 = U'' = \emptyset$, and $V_2 = V'' = \emptyset$. Then $S_1 a = P_1 \cup U_1 \cup U'$, $S_1 a S_1 = P_1 \cup U'$, $aS_1 = P_1$, $S_2 b = P_2 \cup V_1 \cup V'$, $S_2 b S_2 = P_2 \cup V'$, $bS_2 = P_2$. Therefore

$$(aS_1 \subseteq S_1 a S_1 \subset S_1 a) \wedge (bS_2 \subseteq S_2 b S_2 \subset S_2 b) .$$

b) Let (γ) be satisfied and $(aS_1 \subseteq S_1 a S_1 \subset S_1 a) \wedge (bS_2 \subseteq S_2 b S_2 \subset S_2 b)$. Then $J(a) = S_1 a$, $J(b) = S_2 b$, $J(a) \times J(b) = (S_1 a \times S_2 b)$.
$J(a, b) = (S_1 a \times S_2 b) = J(a) \times J(b)$.

Let (ii) hold: $(a, b) \in (aS_1 \times bS_2) \wedge (a, b) \notin [(S_1 a \times S_2 b) \cup (S_1 a S_1 \times S_2 b S_2)]$. The relation $(a, b) \notin [(S_1 a \times S_2 b) \cup (S_1 a S_1 \times S_2 b S_2)]$ includes the following possibilities:

1. $[a \notin (S_1 a \cup S_1 a S_1)] \wedge [b \in (S_2 b \cap S_2 b S_2)]$
2. $[a \in (S_1 a \cap S_1 a S_1)] \wedge [b \notin (S_2 b \cup S_2 b S_2)]$
3. $[a \notin (S_1 a \cup S_1 a S_1)] \wedge [b \notin (S_2 b \cup S_2 b S_2)]$
4. $[a \in S_1 a \wedge a \notin S_1 a S_1] \wedge [b \notin S_2 b \wedge b \in S_2 b S_2]$
5. $[a \notin S_1 a \wedge a \in S_1 a S] \wedge [b \in S_2 b \wedge b \notin S_2 b S_2]$

And similarly as in (i) we can show that 4. and 5. cannot occur.

Combining $(a, b) \in (aS_1 \times bS_2)$ with each of 1., 2., 3., we get these possibilities in the following form:

(α') $[a \in aS_1 \wedge a \notin (S_1 a \cup S_1 a S_1)] \wedge [b \in (S_2 b \cap bS_2 \cap S_2 b S_2)]$

(β') $[a \in (S_1 a \cap aS_1 \cap S_1 a S_1)] \wedge [b \in bS_2 \wedge b \notin (S_2 b \cup S_2 b S_2)]$

(γ') $[a \in aS_1 \wedge a \notin (S_1 a \cup S_1 a S_1)] \wedge [b \in bS_2 \wedge b \notin (S_2 b \cup S_2 b S_2)]$.

As (α'), (β'), (γ') are similar to (α), (β), (γ), we can state the corresponding statements.

Theorem 9. *Let* $[a \in aS_1 \wedge a \notin (S_1 a \cup S_1 a S_1)] \wedge [b \in (S_2 b \cap bS_2 \cap S_2 b S_2)]$. *Then*

$$J(a, b) = J(a) \times J(b) \quad \textit{iff} \quad bS_2 = S_2 b S_2 .$$

Proof. The proof is similar to that of Theorem 6.

Theorem 10. *Let* $[a \in (S_1 a \cap aS_1 \cap S_1 a S_1)] \wedge [b \in bS_2 \wedge b \notin (S_2 b \cup S_2 b S_2)]$. *Then*

$$J(a, b) = J(a) \times J(b) \quad \textit{iff} \quad aS_1 = S_1 a S_1 .$$

Theorem 11. *Let* $[a \in aS_1 \wedge a \notin (S_1 a \cup S_1 a S_1)] \wedge [b \in bS_2 \wedge b \notin (S_2 b \cup S_2 b S_2)]$. *Then*

$$J(a, b) = J(a) \times J(b) \quad \textit{iff}$$
$$(S_1 a \subseteq S_1 a S_1 \subset aS_1) \wedge (S_2 b \subseteq S_2 b S_2 \subset bS_2) .$$

Proof. The proof is similar to that of Theorem 8.

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