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ON THE EXISTENCE OF NON-TRIVIAL TOLERANCES IN PERMUTABLE ALGEBRAS

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Recall that an algebra A is *permutable* if $\Theta \circ \Phi = \Phi \circ \Theta$ for each $\Theta, \Phi \in \operatorname{Con} A$; a variety $\mathscr{V}$ is *permutable* if each $A \in \mathscr{V}$ has this property.

By a *tolerance* on an algebra A we mean a reflexive and symmetric binary relation on A compatible with all operations of A (i.e., a subalgebra of the direct product $A \times A$). Evidently, every congruence on A is a tolerance on A but not vice versa in the general case, see e.g. [1], [2]. An algebra A is called *tolerance trivial* if each tolerance on A is a congruence; a variety $\mathscr{V}$ is *tolerance trivial* if each $A \in \mathscr{V}$ has this property. The following result was proved in [1] (see also [4]):

Proposition. *For a variety $\mathscr{V}$ the following conditions are equivalent:*
(1) *$\mathscr{V}$ is tolerance trivial;*
(2) *$\mathscr{V}$ is permutable.*

It is a natural question whether the foregoing assertion is true also for a single algebra. The negative answer was given in [2] by constructing examples of permutable algebras, namely lattices, with tolerances which are not congruences. However, all these examples are either simple lattices or direct products of simple lattice. This result was presented at the Summer School of Algebra and Ordered Sets 1988; E. T. Schmidt posed the following problem:

Problem. *Does there exist a permutable algebra A such that A is not a direct product of (simple) algebras but there exists a tolerance on A which is not a congruence?*

The aim of this short note is to answer this Problem in the affirmative:

Theorem. *There exists a (26 element) algebra A with one binary operation satisfying the following conditions:*
(1) *A is permutable;*
(2) *A is subdirectly irreducible;*
(3) *A is not simple;*
(4) *there exists a tolerance T on A which is not a congruence.*

Proof. Let $(Q, \cdot)$ be a five element loop with the multiplication table

	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	b
b	b	c	d	1	a
c	c	d	a	b	1
d	d	b	1	a	c

Let S be the direct product $Q \times Q$ and put $A = \{x\} \cup S$. Define a binary operation $\circ$ on A by setting:

(i) if $p, q \in S$, then $p = (p_1, p_2)$, $q = (q_1, q_2)$ and

$$p \circ q = (p_1 \cdot q_1, p_2 \cdot q_2) ;$$

(ii) $x \circ x = e$, $x \circ e = e \circ x = x$ and $x \circ p = p \circ x = p$ for each $p \in S$, $p \neq e$, where e is the unit element of the loop S, i.e. $e = (1, 1)$.

By [3], a quasigroup S has exactly 5 congruences forming Con S isomorphic with M_3, see Fig. 1, where

$$(x, y)\, \Theta_1(z, v) \quad \text{if and only if} \quad x = z ,$$
$$(x, y)\, \Theta_2(z, v) \quad \text{if and only if} \quad y = v ,$$
$$(x, y)\, \Theta_3(z, v) \quad \text{if and only if} \quad x \,.\, y = z \,.\, v ;$$

moreover, S is permutable. Now, investigate congruences on $(A, \circ)$. Clearly, $\theta(x, e) = \omega \cup \{\langle x, e\rangle, \langle e, x\rangle\}$. Further, let $\Theta \in \text{Con}\, A$ and $\omega \neq \Theta \neq \theta(x, e)$. Then the restriction $\Theta|_S = \Theta \cap (S \times S)$ is a congruence on S which is not the identity on S. Hence $\Theta|_S \supseteq \Theta_i$ for some $i = 1, 2, 3$. Thus $\langle p, e\rangle \in \Theta|_S$ for some $p \in S$, $p \neq e$. Since $\langle x, x\rangle \in \Theta$ and $p \circ x = p$, $e \circ x = x$, we also have $\langle p, x\rangle \in \Theta$; the transitivity implies also $\langle x, e\rangle \in \Theta$. Hence $\theta(x, e) \subseteq \Theta$ for each $\Theta \in \text{Con}\, A$, $\Theta \neq \omega$.

Define $\Psi_i = \{\langle x, p\rangle, \langle p, x\rangle;\ \langle p, e\rangle \in \Theta_i\}$ and put $\Phi_i = \theta(x, e) \cup \Theta_i \cup \Psi_i$ for $i = 1, 2, 3$. It is easy to verify that Φ_i are congruences on A and, moreover, A contains exactly 6 elements forming a lattice in Fig. 2. Hence A is subdirectly irreducible and it is not a simple algebra. Since $\omega, \iota, \Theta_1, \Theta_2, \Theta_3$ are permuting congruences on S (as was shown in [3]), also $\omega, \iota, \Phi_1, \Phi_2, \Phi_3$ are permutable on A. The inclusion $\theta(x, e) \subseteq \Phi_i$ $(i = 1, 2, 3)$ implies that also $\theta(x, e)$ permutes with all congruences on A, thus A is permutable.

It remains to prove that there exists a tolerance on A which is not a congruence. Define T on A by setting:

$$\langle p, q\rangle \in T \quad \text{if and only if} \quad p = q \quad \text{or} \quad p, q \in S \quad \text{or} \quad p, q \in \{x, e\} .$$

Then T is a reflexive and symmetric binary relation on A with exactly two blocks: S and $\{x, e\}$. It is easy to verify that T is compactible with $\circ$, thus T is a tolerance on A. However, T is not transitive since $\langle x, e\rangle \in T$, $\langle e, p\rangle \in T$ for each $p \in S$, $p \neq e$ but $\langle x, p\rangle \notin T$, i.e. T is not a congruence on A.

Remark. An arbitrary loop Q which is not a group can be used in the foregoing proof. By [3], it works in the same way but the algebra A should have more than 26 elements.

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