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ON INTEGRATION IN BANACH SPACES, XII (INTEGRATION WITH RESPECT TO POLYMEASURES)

IVAN DOBRAKOV, Bratislava

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INTRODUCTION

Shortly to the context of this part (we shall use freely the notation and concepts of the previous parts, treated as chapters):

Theorems 1, 2 and 3 give new results concerning $\mathscr{P}$-measurable functions $f: T \to X$, $\mathscr{P} \subset 2^T$ being a δ-ring. Particularly we obtain that each $\mathscr{P}$-measurable $f: T \to X$ is a $\mathscr{P}$-measurable function as $f: T \to X_f = \operatorname{sp}\{f(T)\}$. Using this fact, in Theorem 4 we indicate some improvements of our previous results from parts IX and X. In Theorem 5 we show that $\mathscr{I}(\Gamma) = \mathscr{I}_2(\Gamma)$, provided $c_0 \not\subset Y$.

In section 2 a treatment of weak (also called scalar) and weak* integrability is given. For $d = 1$ see also [25] and [26]. There are at least two reasons to do it carefully: a) General multilinear operators $U: \mathsf{X} C_0(T_i, X_i) \to Y$ (even linear $U: C_0(T) \to Y$) are representable by polymeasures $\Gamma: \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to L^{(d)}(X_i; Y^{**})$ such that only $\Gamma(\ldots)(x_i)(y^*): \mathsf{X}\sigma(\mathscr{B}_{0,i}) \to K =$ the scalars, is separately countably additive for each $(x_i) \in \mathsf{X} X_i$ and each $y^* \in Y^*$, see [17], [24], [18] and [19], and b) In both weak and weak* integrability we have the good situation $\Gamma^*: \mathsf{X}\mathscr{P}_i \to L^{(d)}(X_i; K)$, i.e., that $Y = K \not\supset c_0$. Let us also note that in proving the measurability of the partial integral, see section 2 in part III and section 3 in part VII, first we proved its weak measurability in Theorem III.10.

The contents of the last section 3 is given by its title. The proof of finiteness of the L_1-gauge $\hat{\Gamma}[(\cdot), (T_i)]$ on $\mathscr{L}_1(\Gamma)$ is postponed to Theorem XIII.12.

1. PRELIMINARIES

In the first three theorems $\mathscr{P}$ is a δ-rings of subsets of a non empty set T, and X is a Banach space.

A direct consequence of Generalized Egoroff-Lusin Theorem, i.e., of Theorem X.2, is the following

Theorem 1. *Let $\mu: \sigma(\mathscr{P}) \to [0, +\infty]$ be either a σ-finite measure, or a submeasure* *in the sense of Definition* 1 *in* [20]. *Let* $f_n\colon T \to X$, $n = 0, 1, 2, \ldots$ *be* $\mathscr{P}$*-measurable functions and let* $f_n(t) \to f_0(t)$ *for* μ – *a.e.* $t \in T$. *Put* $F = \bigcup_{n=0}^{\infty} \{t \in T, f_n(t) \neq 0\} \in \sigma(\mathscr{P})$. *Then there are* $N \in F \cap \sigma(\mathscr{P})$ *and* $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ *such that* $\mu(N) = 0$, $F_k \nearrow F - N$, *on each* F_k, $k = 1, 2, \ldots$ *the sequence* f_n, $n = 1, 2, \ldots$ *convergences uniformly to the function* f_0, *and the set* $\bigcup_{n=0}^{\infty} f_n(F_k) \subset X$ *is relatively compact for each* $k = 1, 2, \ldots$. *Hence* $\bigcup_{n=0}^{\infty} f_n(F - N) \subset X$ *is relatively* σ*-compact.*

We shall need the following improvement of Theorem X.1.

Theorem 2. *Let* $f\colon T \to X$ *be a* $\mathscr{P}$*-measurable function and let* $f(T) \subset X$ *be relatively* σ*-compact. Then there are* $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ *such that* $F_k \nearrow F = \{t \in T, f(t) \neq 0\} \in \sigma(\mathscr{P})$, $k^{-1} \leqq |f(t)| \leqq k$ *for each* $t \in F_k$, *and each* $k = 1, 2, \ldots$, *and such that for any sequence* $\varepsilon_k \searrow 0$, $k = 1, 2, \ldots$ *there are*:

1) *a sequence of finite* $\mathscr{P}$*-partitions* $\pi_{\varepsilon_k}(F_k) = (F_{k,j})_{j=1}^{r_k}$ *such that* $(F_{k+1,j} \cap F_k)_{j=1}^{r_{k+1}} \geqq \pi_{\varepsilon_k}(F_k)$ *in the sense of refinements for each* $k = 1, 2, \ldots$, *and for arbitrary fixed* $k \in \{1, 2, \ldots\}$, *for any points* $t_{k,j} \in F_{k,j}$, $j = 1, \ldots, r_k$, *the inequality*

$$|f(t)\, \chi_{F_k}(t) - \sum_{j=1}^{r_k} f(t_{k,j})\, \chi_{F_{k,j}}(t)| \leqq \varepsilon_k(1 \wedge |f(t)|)$$

holds for each $t \in T$, *and*

2) *a sequence* $f_k \in S(F_k \cap \mathscr{P}, X_f)$, $k = 1, 2, \ldots$, *where* $X_f = \overline{\text{sp}}\,\{f(T)\}$, *such that* $f_k = f_k \chi_{F_k}$ *for each* $k = 1, 2, \ldots$, $f_k \to f$, $|f_k| \nearrow |f|$, *and the inequality*

$$|f(t)\, \chi_{F_k}(t) - f_k(t)| \leqq \varepsilon_k(1 \wedge |f(t)|)$$

holds for each $t \in T$ *and each* $k = 1, 2, \ldots$.

Proof. 1) easily follows from Theorem X.1.

2) Without loss of generality we may suppose that $\varepsilon_1 \leqq 1$. Let us apply 1) for the sequence $\varepsilon'_k = 3^{-1} \varepsilon_k$, $k = 1, 2, \ldots$, and put $f'_k = \sum_{j=1}^{r_k'} f(t'_{k,j})\, \chi_{F'_{k,j}}$, $k = 1, 2, \ldots$, where $\pi_{\varepsilon_k'}(F_k) = (F'_{k,j})_{j=1}^{r_k'}$, $k = 1, 2, \ldots$ satisfy the requirements of 1) and $t'_{k,j} \in F'_{k,j}$. For each $k = 1, 2, \ldots$ let n_k be the whole part of $k\varepsilon_k^{-1}$, and put

$$\varphi_k = \sum_{j=1}^{\infty} \frac{j-1}{3^{n_k}}\, \chi_{D_k} \quad \text{where} \quad D_k = \left\{t \in T, \frac{j-1}{3^{n_k}} \leqq |f(t)| < \frac{j}{3^{n_k}}\right\}.$$

Put finally $f_k = (f'_k/|f'_k|)\, \varphi_k$ for $k = 1, 2, \ldots$. Then clearly $f_k \in S(F_k \cap \mathscr{P}, X_f)$ and $f_k = f_k \chi_{F_k}$ for each $k = 1, 2, \ldots$. It is easy to see that $f_k \to f$ and that $|f_k| \nearrow |f|$. Finally

$$|f(t)\, \chi_{F_k}(t) - f_k(t)| \leqq |f(t)\, \chi_{F_k}(t) - f'_k(t)| + \\ + \left|f'_k(t) - \frac{f'_k(t)}{|f'_k(t)|}\, \varphi_k(t)\right| \leqq 3^{-1}\, \varepsilon_k(1 \wedge |f(t)|) +$$

$$+ |f_k'(t) - \varphi_k(t)\,\chi_{F_k}(t)| \leqq 3^{-1}\,\varepsilon_k(1 \wedge |f(t)|) +$$
$$+ |f_k'(t) - f(t)\,\chi_{F_k}(t)| + |f(t)\,\chi_{F_k}(t) - \varphi_k(t)\,\chi_{F_k}(t)| \leqq$$
$$2\,.\,3^{-1}\,\varepsilon_k(1 \wedge |f(t)|) + k^{-1}\,.\,3^{-1}\,\varepsilon_k\,\chi_{F_k}(t) \leqq \varepsilon_k(1 \wedge |f(t)|)$$

for each $t \in T$, since $k^{-1} \leqq |f(t)|$ for each $t \in F_k$, and each $k = 1, 2, \ldots$.

A function $f\colon T \to X$ is called *$\mathscr{P}$-elementary* if it is of the form $f = \sum_{j=1}^{\infty} x_j \chi_{A_j}$, where $x_j \in X$, $A_j \in \mathscr{P}_j$, $j = 1, 2, \ldots$, and A_j, $j = 1, 2, \ldots$, are pairwise disjoint. Obviously each $\sigma(\mathscr{P})$-elementary function is $\mathscr{P}$-elementary. We denote by $E(\mathscr{P}, X)$ the linear space of all $\mathscr{P}$-elementary functions $f\colon T \to X$. Evidently $\varphi f \in E(\mathscr{P}, X)$ whenever $\varphi \in E(\mathscr{P}, K)$ and $f \in E(\mathscr{P}, X)$. It is well known, see [22], that each $\mathscr{P}$-measurable function $f\colon T \to X$ is a uniform limit of a sequence from $E(\mathscr{P}, X)$. Using this fact we prove

Theorem 3. *Let $f\colon T \to X$ be a $\mathscr{P}$-measurable function, let $F = \{t \in T, f(t) \neq 0\} \in \sigma(\mathscr{P})$, and let $X_f = \overline{\mathrm{sp}}\,\{f(T)\}$. Then:*

1) *for each $\varepsilon > 0$ there is a countable $\mathscr{P}$-partition $\pi_\varepsilon^*(F) = (F_j)_{j\in J}$ such that for any points $t_j \in F_j$, $j \in J$, the inequality*

$$|f(t) - \sum_{j\in J}^{\infty} f(t_j)\,\chi_{F_j}(t)| \leqq \varepsilon(1 \wedge |f(t)|)$$

holds for each $t \in T$,

2) *$f\colon T \to X_f$ is a $\mathscr{P}$-measurable function, i.e., there are $u_n \in S(F \cap \mathscr{P}, X_f)$, $n = 1, 2, \ldots$ such that $u_n \to f$ and $|u_n| \nearrow |f|$, and*

3) *there is a sequence $f_n \in E(F \cap \mathscr{P}, X_f)$, $n = 1, 2, \ldots$ such that $|f_n| \nearrow |f|$, and*

$$|f(t) - f_n(t)| \leqq (1/n)\,(1 \wedge |f(t)|)$$

for each $t \in T$ and each $n = 1, 2, \ldots$.

Proof. 1) Let $\varepsilon > 0$. Put $A_1 = \{t \in T,\ |f(t)| \geqq 1\}$, and $A_k = \{t \in T,\ k^{-1} \leqq |f(t)| < (k-1)^{-1}\}$ for $k = 2, 3, \ldots$. Then $A_k \in F \cap \sigma(\mathscr{P})$, $k = 1, 2, \ldots$ are pairwise disjoint and $\bigcup_{k=1}^{\infty} A_k = F$. Hence it is enough to prove 1) for each function $f\chi_{A_k}\colon A_k \to X$, $k = 1, 2, \ldots$ Let k be fixed. Take $h_k \in E(F \cap \mathscr{P}, X)$ such that $|f(t)\,\chi_{A_k}(t) - h_k(t)| < \varepsilon/2k = \frac{1}{2}\varepsilon(1 \wedge |f(t)|)$ for each $t \in A_k$. Let $h_k = \sum_{j=1}^{\infty} x_{k,j}\chi_{A_{k,j}}$, where $x_{k,j} \in X$, $A_{k,j} \in A_k \cap \mathscr{P}$, $j = 1, 2, \ldots$, and $A_{k,j}$, $j = 1, 2, \ldots$ are pairwise disjoint. Put $A_{k,0} = A_k - \bigcup_{j=1}^{\infty} A_{k,j}$. Then there are pairwise disjoint $A_{k,0,j} \in A_k \cap \mathscr{P}$, $j = 1, 2, \ldots$ such that $A_{k,0} = \bigcup_{j=1}^{\infty} A_{k,0,j}$. Now for any points $t_{k,j} \in A_{k,j}$ and $t_{k,0,j} \in A_{k,0,j}$, $j = 1, 2, \ldots$ we have the inequalities

$$|f(t)\,\chi_{A_k}(t) - \sum_{j=1}^{\infty} f(t_{k,j})\,\chi_{A_{k,j}}(t) - \sum_{j=1}^{\infty} f(t_{k,0,j})\,\chi_{A_{k,0,j}}(t)| =$$

$$= |f(t)\,\chi_{A_k}(t) - h_k(t)| + |h_k(t) - h'_k(t)| \leqq 2\,.\,\|f\chi_{A_k} - h_k\|_{A_k} \leqq$$
$$\leqq \tfrac{1}{2}\varepsilon \leqq \varepsilon(1 \wedge |f(t)|)\,,$$
where $h'_k(t) = \sum_{j=1}^{\infty} f(t_{k,j})\,\chi_{A_{k,j}}(t) + \sum_{j=1}^{\infty} f(t_{k,0,j})\,\chi_{A_{k,0,j}}(t)\,,$

for each $t \in A_k$, which we wanted to show.

2) Immediately follows from 1) since the $F \cap \mathscr{P}$-measurable functions $f: T \to X_f$ are closed under the formation of pointwise limits of sequences, and each $f \in E(F \cap \mathscr{P}, X_f)$ is evidently a pointwise limit of a sequence from $S(F \cap \mathscr{P}, X_f)$.

3) For $k = 2, 3, \ldots$ let A_k be as in the proof of 1), and put $B_k = \{t \in F, k \leqq |f(t)| < k + 1\} \in F \cap \sigma(\mathscr{P})$ for $k = 1, 2, \ldots$. Then $\pi^*(F) = \{(A_k)_{k=2}^{\infty}, (B_k)_{k=1}^{\infty}\}$ is a countable $\sigma(\mathscr{P})$-partition of F. Obviously it is enough to prove 3) for each A_k, $k = 2, 3, \ldots$, and each B_k, $k = 1, 2, \ldots$.

For $n = 1, 2, \ldots$ put

$$\varphi_n = \sum_{j=1}^{\infty} \frac{j-1}{2^n}\,\chi_{D_n} \quad \text{where} \quad D_n = \left\{t \in T, \frac{j-1}{2^n} \leqq |f(t)| < \frac{j}{2^n}\right\}.$$

Then φ_n, $n = 1, 2, \ldots$ are $\sigma(\mathscr{P})$-elementary, hence also $\mathscr{P}$-elementary functions, $0 \leqq \varphi_n(t) \nearrow |f(t)|$ for each $t \in T$, and $|f(t)| - \varphi_n(t)| < 2^{-n}$ for each $t \in T$ and each $n = 1, 2, \ldots$.

Consider first a fixed B_k, $k \in \{1, 2, \ldots\}$. By 1) for each $n = 1, 2, \ldots$ there is a $\mathscr{P}$-elementary function $f'_{k,n}: B_k \to f(B_k)$ such that

$$|f(t) - f'_{k,n}(t)| \leqq \frac{1}{4n(k+1)}$$

for each $t \in B_k$. For $n = 1, 2, \ldots$ put

$$f_{k,n}(t) = \frac{\varphi_{n+3}(t)}{|f'_{k,n}(t)|}\,f'_{k,n}(t)$$

if $t \in B_k$, and put $f_{k,n}(t) = 0$ if $t \in T - B_k$. Then $f_{k,n}: B_k \to X_f$ is obviously a $B_k \cap \mathscr{P}$-elementary function for each $n = 1, 2, \ldots$, and $|f_{k,n}(t)| \nearrow |f(t)|$ for each $t \in B_k$. Since for each $t \in B_k$ and each $n = 1, 2, \ldots$ the following inequalities hold:

$$\frac{\varphi_{n+3}(t)}{|f'_{k,n}(t)|} \leqq \frac{|f(t)|}{|f(t)| - \dfrac{1}{4n(k+1)}} < 1 + \frac{1}{4n(k+1)}\,\frac{1}{1 - \frac{1}{8}} < 1 + \frac{1}{3n(k+1)}\,,$$

and

$$\frac{\varphi_{n+3}(t)}{|f'_{k,n}(t)|} \geqq \frac{|f(t)| - \dfrac{1}{2^{n+3}}}{|f(t)| + \dfrac{1}{4n(k+1)}} > 1 - \frac{1}{4n(k+1)}\left(\frac{1}{1 + \frac{1}{8}} + \frac{4n(k+1)}{2^{n+3}\,.\,k}\right) >$$
$$> 1 - \frac{1}{2n(k+1)}\,,$$

we obtain that

$$|f_{k,n}(t) - f(t)| \leqq |f_{k,n}(t) - f'_{k,n}(t)| + |f'_{k,n}(t) - f(t)| \leqq$$

$$\leqq \left|\frac{\varphi_{n+3}(t)}{|f'_{k,n}(t)|} - 1\right| |f'_{k,n}(t)| + \frac{1}{4n(k+1)} < \left|\frac{\varphi_{n+3}(t)}{|f'_{k,n}(t)|} - 1\right| \cdot$$

$$\cdot (k + 1 + \tfrac{1}{8}) + \frac{1}{4n(k+1)} < \frac{1}{n} = \frac{1}{n}(1 \wedge |f(t)|)$$

for each $t \in B_k$ and each $n = 1, 2, \ldots$.

Consider now a fixed A_k, $k \in \{2, 3, \ldots\}$. By 1) for each $n = 1, 2, \ldots$ there is a $\mathscr{P}$-elementary function $g'_{k,n}\colon A_k \to f(A_k)$ such that $|f(t) - g'_{k,n}(t)| \leqq 1/4nk \,.\, |f(t)|$ for each $t \in A_k$. For $n = 1, 2, \ldots$ put

$$g_{k,n}(t) = \frac{\varphi_{n+k+3}(t)}{|g'_{k,n}(t)|}\, g'_{k,n}(t)$$

if $t \in A_k$, and put $g_{k,n}(t) = 0$ if $t \in T - A_k$. Then $g_{k,n}\colon A_k \to X_f$, $n = 1, 2, \ldots$ are obviously $A_k \cap \mathscr{P}$-elementary functions, and $|g_{k,n}(t)| \nearrow |f(t)|$ for each $t \in A_k$. Since for each $n = 1, 2, \ldots$ and each $t \in A_k$ the following inequalities hold:

$$\frac{\varphi_{n+k+3}(t)}{|g'_{k,n}(t)|} \geqq \frac{|f(t)| - \dfrac{1}{2^{n+k+3}} + \dfrac{1}{4nk}\left(|f(t)| - \dfrac{1}{4nk}|f(t)|\right)}{|f(t)| + \dfrac{1}{4nk}|f(t)|} >$$

$$> 1 - \frac{1}{4nk} - \frac{1}{4n}\,\frac{4n(k+1)}{2^{n+k+3}} > 1 - \frac{1}{4n},$$

and

$$\frac{\varphi_{n+k+3}(t)}{|g'_{k,n}(t)|} \leqq \frac{|f(t)|}{|f(t)| - \dfrac{1}{4nk}|f(t)|} \leqq 1 - \frac{1}{4nk}\,\frac{1}{1 - \dfrac{1}{4nk}} < 1 - \frac{1}{7n},$$

we obtain that

$$|g_{k,n}(t) - f(t)| \leqq |g_{k,n}(t) - g'_{k,n}(t)| + |g'_{k,n}(t) - f(t)| \leqq$$

$$\leqq \left|\frac{\varphi_{n+k+3}(t)}{|g'_{k,n}(t)|} - 1\right| |g'_{k,n}(t)| + \frac{1}{4nk}|f(t)| \leqq \left(\left|\frac{\varphi_{n+k+3}(t)}{|g'_{k,n}(t)|} - 1\right| \cdot\right.$$

$$\left.\cdot \left(1 + \frac{1}{4nk}\right) + \frac{1}{4nk}\right)|f(t)| < \left(\frac{1}{4n}\left(1 + \frac{1}{8n}\right) + \frac{1}{8n}\right)(f(t)| < \frac{1}{n}|f(t)|$$

for each $t \in A_k$ and each $n = 1, 2, \ldots$.

Hence $f_n = \sum_{k=1}^{\infty} f_{k,n} + \sum_{k=2}^{\infty} g_{k,n}$, $n = 1, 2, \ldots$ have the required properties.

For $\mathscr{P}_i$-measurable $f_i\colon T_i \to X_i$, $i = 1, \ldots, d$, put $X_{f_i} = \overline{\mathrm{sp}}\,\{f_i(T_i)\}$ and $\mathscr{P}_{f_i} =$

$= \bigcup_{k=1}^{\infty} \mathscr{P}_i \cap \{t_i \in T_i, \ |f_i(t_i)| > k^{-1}\}$. Then, using assertion 2) of Theorem 3, and Theorems X.5 and XI.5 we immediately obtain the following improvements of results from part X:

Theorem 4. 1) *Let* $(f_i) \in \mathscr{I}(\Gamma)$ *and let* $\Gamma(\ldots)(x_i)\colon \mathsf{X}\mathscr{P}_{f_i} \to Y$ *have locally control d-polymeasure for each* $(x_i) \in \mathsf{X}X_{f_i}$. *Then* $(f_i) \in \mathscr{I}_1(\Gamma)$ *and its indefinite integral* $\int_{(\cdot)} (f_i)\,\mathrm{d}\Gamma\colon \mathsf{X}\sigma(\mathscr{P}_i) \to Y$ *has a control d-polymeasure* (see Theorem X.3 and its Corollary 1).

2) *The assertions of Theorems* X.9, X.13 *and* X.15 *remain to hold if we suppose either that* $\Gamma(\ldots)(x_i)\colon \mathsf{X}\mathscr{P}_{f_i} \to Y$ *has locally a control d-polymeasure for each* $(x_i) \in \mathsf{X}X_{f_i}$, *or that* $f_i(T_i) \subset X_i$ *is relatively σ-compact for each* $i = 1, \ldots, d$. *In Theorem* X.9 – 1) *we may assert that* $(f_{i,n}) \in \mathsf{X}S(\mathscr{P}_{f_i}, X_{f_i})$ *for each* $n = 1, 2, \ldots$.

3) *The assertion of Theorem* X.13 *remains to hold if* $c_0 \not\subset Y$.

Using, among others, Theorems XI.5 and XI.10 we now prove

Theorem 5. 1) *Let* $(f_i) \in \mathscr{I}(\Gamma)$ *and let there be* $(g_{i,n}) \in \mathscr{L}_1(\Gamma)$, $n = 1, 2, \ldots$, *such that* $g_{i,n} \to f_i$ *for each* $i = 1, \ldots, d$. *Then* $(f_i) \in \mathscr{I}_2(\Gamma)$.

2) *Let* $c_0 \not\subset Y$. *Then* $\mathscr{I}(\Gamma) = \mathscr{I}_2(\Gamma)$.

Proof. 1) Put $F_i = \{t_i \in T_i, \ f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, and $H_{i,n} = \{t_i \in F_i, \ |f_i(t_i)| \leqq 2|g_{i,n}(t_i)|\} \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, and $n = 1, 2, \ldots$. Then $H_{i,n} \to F_i$ for each $i = 1, \ldots, d$. Hence

$$\int_{(A_i)} (f_i)\,\mathrm{d}\Gamma = \lim_{n\to\infty} \int_{(A_i \cap H_{i,n})} (f_i)\,\mathrm{d}\Gamma = \lim_{n\to\infty} \int_{(A_i)} (f_i \chi_{H_{i,n}})\,\mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ by Theorems IX.4 and VIII.1. Since $|f_i \chi_{H_{i,n}}| \leqq 2|g_{i,n}|$ for each $i = 1, \ldots, d$ and each $n = 1, 2, \ldots$, and since $(2g_{i,n}) \in \mathscr{L}_1(\Gamma)$ for each $n = 1, 2, \ldots$, $(f_i \chi_{H_{i,n}}) \in \mathscr{L}_1(\Gamma)$ for each $n = 1, 2, \ldots$. Take $(f_{i,k}) \in \mathsf{X}S(\mathscr{P}_i, X_i) = \mathscr{I}_0(\Gamma)$, $k = 1, 2, \ldots$ such that $f_{i,k} \to f_i$ and $|f_{i,k}| \nearrow |f_i|$ for each $i = 1, \ldots, d$. Then

$$\int_{(A_i)} (f_i \chi_{H_{i,n}})\,\mathrm{d}\Gamma = \lim_{k\to\infty} \int_{(A_i)} (f_{i,k} \chi_{H_{i,n}})\,\mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each $n = 1, 2, \ldots$ by Lebesgue Dominated Convergence Theorem in $\mathscr{L}_1(\Gamma)$, i.e., by Theorem XI.10.

2) Let $(f_i) \in \mathscr{I}(\Gamma)$. By assumed local σ-finiteness of the semivariation $\hat{\Gamma}$ on $\mathsf{X}\sigma(\mathscr{P}_i)$, see the beginning of Part IX, there are $(F'_{i,k}) \in \mathsf{X}\mathscr{P}_i$, $k = 1, 2, \ldots$ such that $F'_{i,k} \nearrow F_i = \{t_i \in T_i, f_i(t_i) \neq 0\}$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}(F'_{i,k}) < \infty$ for each $k = 1, 2, \ldots$. For $i = 1, \ldots, d$ and $k = 1, 2, \ldots$ put $F_{i,k} = F'_{i,k} \cap \{t_i \in T_i, \ |f_i(t_i)| \leqq k\} \in \mathscr{P}_i$. Then $\hat{\Gamma}[(f_i \chi_{F_{i,k}}), (T_i)] < +\infty$ for each $k = 1, 2, \ldots$. Hence $(f_i \chi_{F_{i,k}}) \in \mathscr{L}_1(\Gamma)$ for each $k = 1, 2, \ldots$ by Theorem XI.5. It remains to apply the last consideration of 1).

Definition 1. Let $f_i\colon T_i \to X_i$ be $\mathscr{P}_i$-measurable, $i = 1, \ldots, d$. We say that (f_i) is a *weakly Γ-integrable d-tuple*, and write $(f_i) \in w\mathscr{I}(\Gamma)$ if $(f_i) \in \mathscr{I}(y^*\Gamma)$ for each $y^* \in Y^*$. For $(f_i) \in w\mathscr{I}(\Gamma)$, $(A_i) \in \mathrm{X}\sigma(\mathscr{P}_i)$ and $y^* \in Y^*$ we put

$$\left(w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\right)(y^*) = \int_{(A_i)} (f_i)\, \mathrm{d}(y^*\Gamma)\,.$$

Let $(f_i) \in w\mathscr{I}(\Gamma)$. We write $(f_i) \in (w\mathscr{I})_1\,(\Gamma)$, and say that (f_i) belongs to the first weak integrable class, if there are $(f_{i,n}) \in (w\mathscr{I})_0\,(\Gamma) = \mathrm{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ such that $f_{i,n} \to f_i$ for each $i = 1, \ldots, d$, and

$$\left(w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\right)(y^*) = \lim_{n\to\infty} \left(w \int_{(A_i)} (f_{i,n})\, \mathrm{d}\Gamma\right)(y^*)$$

for each $(A_i) \in \mathrm{X}\sigma(\mathscr{P}_i)$ and each $y^* \in Y^*$.

Similarly, starting from $(w\mathscr{I})_1\,(\Gamma)$, we define the second weak integrable class $(w\mathscr{I})_2\,(\Gamma)$.

The basic properties of the weak integral are given by

Theorem 6. 1) *$\mathscr{I}(\Gamma) \subset w\mathscr{I}(\Gamma)$, $\mathscr{I}_1(\Gamma) \subset (w\mathscr{I})_1\,(\Gamma)$ and $w\mathscr{I}(\Gamma) = (w\mathscr{I})_2\,(\Gamma)$.*

2) *Let $(f_i) \in w\mathscr{I}(\Gamma)$. Then $w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma \in Y^{**}$ for each $(A_i) \in \mathrm{X}\sigma(\mathscr{P}_i)$, and $w \int_{(\cdot)} (f_i)\, \mathrm{d}\Gamma\colon \mathrm{X}\sigma(\mathscr{P}_i) \to Y^{**}$ is separately w^*-countably additive.*

3) *If $(f_i) \in \mathscr{I}(\Gamma)$, then $w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \dot{\int}_{(A_i)} (f_i)\, \mathrm{d}\Gamma$ for each $(A_i) \in \mathrm{X}\sigma(\mathscr{P}_i)$, where $\dot{y} \in Y^{**}$ is the image of $y \in Y$ under the natural embedding of Y in Y^{**}.*

4) *If $c_0 \not\subset Y$, then $w\mathscr{I}(\Gamma) = \mathscr{I}(\Gamma)$ and the integrals coincide.*

Proof. 1) and 2). The inclusions $\mathscr{I}(\Gamma) \subset w\mathscr{I}(\Gamma)$ and $\mathscr{I}_1(\Gamma) \subset (w\mathscr{I})_1\,(\Gamma)$ follow from assertion 3) of Theorem IX.4.

Let $(f_i) \in w\mathscr{I}(\Gamma)$, i.e., let $(f_i) \in \mathscr{I}(y^*\Gamma)$ for each $y^* \in Y^*$. Take $(f_{i,n}) \in \mathrm{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ such that $f_{i,n} \to f_i$ and $|f_{i,n}| \nearrow |f_i|$ for each $i = 1, \ldots, d$. Let us use the notation from the proof of 2) of Theorem 5. Then by Theorems XI.5 and XI.10 we obtain the equalities

$$\left(w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\right)(y^*) = \int_{(A_i)} (f_i)\, \mathrm{d}(y^*\Gamma) = \lim_{k\to\infty} \int_{(A_i \cap F_{i,k})} (f_i)\, \mathrm{d}(y^*\Gamma) =$$
$$= \lim_{k\to\infty} \lim_{n\to\infty} \int_{(A_i \cap F_{i,k})} (f_{i,n})\, \mathrm{d}(y^*\Gamma)$$

for each $(A_i) \in \mathrm{X}\sigma(\mathscr{P}_i)$ and each $y^* \in Y^*$. Hence $(f_i) \in (w\mathscr{I})_2\,(\Gamma)$, $w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma \in Y^{**}$ for each $(A_i) \in \mathrm{X}\sigma(\mathscr{P}_i)$ using Banach-Steinhaus theorem, and $w \int_{(\cdot)} (f_i)\, \mathrm{d}\Gamma\colon \mathrm{X}\sigma(\mathscr{P}_i) \to$ $\to Y^{**}$ is separately w^*-countably additive using (VHSN)-theorem for vector d-polymeasures, see the beginning of part VIII.

3) If $(f_i) \in \mathscr{I}(\Gamma)$, then

$$y^*\left(\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\right) = \int_{(A_i)} (f_i)\, \mathrm{d}(y^*\Gamma) = \left(w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\right)(y^*)$$

for each $(A_i) \in \mathrm{X}\sigma(\mathscr{P}_i)$ and each $y^* \in Y^*$ by assertion 3) of Theorem IX.4. It remains to apply Hahn-Banach theorem.

4) Let $(f_i) \in w\mathscr{I}(\Gamma)$. Since $(f_i \chi_{F_{i,k}}) \in \mathscr{I}_1(\Gamma)$ for each $k = 1, 2, \dots$ in the notation of the proof of 2) of Theorem 5, analogously as in the proof of Theorem X.14 it follows that $(f_i) \in \mathscr{I}_2(\Gamma)$.

Definition 2. Let $g_i\colon T_i \to X_i$ be $\mathscr{P}_i$-measurable, $i = 1, \dots, d$. We write $(g_i) \in \mathscr{L}_1\mathscr{M}(\Gamma)$ if $\hat{\Gamma}[(g_i), (T_i)] < +\infty$. Put $\mathscr{L}_1\mathscr{I}(\Gamma) = \mathscr{L}_1\mathscr{M}(\Gamma) \cap \mathscr{I}(\Gamma)$.

We write $(g_i) \in w\mathscr{L}_1(\Gamma)$ if $(f_i) \in w\mathscr{I}(\Gamma)$ whenever $f_i\colon T_i \to X_i$ is $\mathscr{P}_i$-measurable and $|f_i| \leqq |g_i|$ for each $i = 1, \dots, d$.

Theorem 7. 1) $(g_i) \in \mathscr{L}_1\mathscr{M}(\Gamma)$ *if and only if* $(g_i) \in \mathscr{L}_1(y^*\Gamma)$ *for each* $y^* \in Y^*$.

2) $\mathscr{L}_1\mathscr{M}(\Gamma) = w\mathscr{L}_1(\Gamma) \subset (w\mathscr{I})_1(\Gamma)$.

3) *If* $c_0 \not\subset Y$, *then* $w\mathscr{L}_1(\Gamma) = \mathscr{L}_1\mathscr{M}(\Gamma) = \mathscr{L}_1(\Gamma)$.

4) *For the weak integral in* w^**-convergence the analogues of Theorems* XI.6 (Fubini), XI.9, XI.10 (LDCT) *and their corollaries hold in* $\mathscr{L}_1\mathscr{M}(\Gamma)$.

Proof. 1) If $(g_i) \in \mathscr{L}_1\mathscr{M}(\Gamma)$, then $(g_i) \in \mathscr{L}_1(y^*\Gamma)$ for each $y^* \in Y^*$ by Theorem XI.5. If $(g_i) \in \mathscr{L}_1(y^*\Gamma)$ for each $y^* \in Y^*$, then $y^*\Gamma[(g_i), (T_i)] < +\infty$ by Theorem XIII.6 for each $y^* \in Y^*$, hence $\hat{\Gamma}[(g_i), (T_i)] = \sup_{|y^*| \leq 1} y^*\Gamma[(g_i), (T_i)] < +\infty$ by uniform boundedness principle.

2) The inclusions $\mathscr{L}_1\mathscr{M}(\Gamma) \subset w\mathscr{L}_1(\Gamma)$ and $w\mathscr{L}_1(\Gamma) \subset (w\mathscr{I})_1(\Gamma)$ are consequences of Theorem XI.5 and Corollary 1 of Theorem XI.10 respectively. If $(g_i) \in w\mathscr{L}_1(\Gamma)$, then $(g_i) \in \mathscr{L}_1(y^*\Gamma)$ for each $y^* \in Y^*$, hence $(g_i) \in \mathscr{L}_1\mathscr{M}(\Gamma)$ by 1).

3) If $c_0 \not\subset Y$, then $\mathscr{L}_1\mathscr{M}(\Gamma) \subset \mathscr{L}_1(\Gamma)$ by Theorem XI.5, while $\mathscr{L}_1(\Gamma) \subset \mathscr{L}_1\mathscr{M}(\Gamma)$ by Theorem XIII.12.

4) is evident.

Let $Y = Z^*$ = the dual of Z, where Z is a Banach space. Then we may suppose that only $\Gamma(\dots)(x_i)\, z\colon \mathsf{X}\mathscr{P}_i \to K$ is separately countably additive for each $(x_i) \in \mathsf{X}X_i$ and each $z \in Z$. We keep however, the assumption that the semivariation $\hat{\Gamma}\colon \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ is locally σ-finite. It is important to remind that by the deep result of J. Diestel and B. Faires, see 1.2 in [3] or Theorem I.4 2 in [2] if $l_\infty \not\subset Z^*$ (equivalently, if $c_0 \not\subset Z^*$), then a w^*-separately countably additive $\gamma\colon \mathsf{X}\mathscr{P}_i \to Z^*$ is norm separately countably additive.

Definition 3. Let $Y = Z^*$, where Z is a Banach space and let Γ be as described above. Let $f_i\colon T_i \to X_i$ be $\mathscr{P}_i$-measurable functions, $i = 1, \dots, d$. We say that (f_i) is *a weak** Γ*-integrable* d*-tuple*, and write $(f_i) \in w^*\mathscr{I}(\Gamma)$ if $(f_i) \in \mathscr{I}(\Gamma(\dots)\, z)$ for each $z \in Z$. For $(f_i) \in w^*\mathscr{I}(\Gamma)$, $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and $z \in Z$ we put

$$\left(w^* \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\right)(z) = \int_{(A_i)} (f_i)\, \mathrm{d}(\Gamma(\dots)\, z)\,.$$

We define $(w^*\mathscr{I})_1(\Gamma)$, $(w^*\mathscr{I})_2(\Gamma)$ and $w^*\mathscr{L}_1(\Gamma)$ analogously as their w counterparts in Definition 1.

The basic properties of the weak* integral are given by

Theorem 8. *Let* $Y = Z^*$, *where* Z *is a Banach space, and let* Γ *be as described before Definition* 3. *Then*:

1) $w^*\mathcal{I}(\Gamma) = (w^*\mathcal{I})_2(\Gamma)$.

2) *Let* $(f_i) \in w^*\mathcal{I}(\Gamma)$. *Then* $w^* \int_{(A_i)} (f_i)\,\mathrm{d}\Gamma \in Z^*$ *for each* $(A_i) \in \mathsf{X}\sigma(\mathcal{P}_i)$, *and* $w^* \int_{(\cdot)} (f_i)\,\mathrm{d}\Gamma \colon \mathsf{X}\sigma(\mathcal{P}_i) \to Z^*$ *is separately w*-countably additive.*

3) $(g_i) \in \mathcal{L}_1\mathcal{M}(\Gamma)$ *if and only if* $(g_i) \in \mathcal{L}_1(\Gamma(\ldots)\,z)$ *for each* $z \in Z$.

4) $\mathcal{L}_1\mathcal{M}(\Gamma) = w^*\mathcal{L}_1(\Gamma) \subset (w^*\mathcal{I})_1(\Gamma)$.

5) *For the weak* integral in w*-convergence the analogues of Theorems* XI.6 (Fubini), XI.9, XI.10 (LDCT), *and their corollaries hold.*

6) *If* $\Gamma(\ldots)(x_i) \colon \mathsf{X}\mathcal{P}_i \to Z^* = Y$ *is separately countably additive for each* $(x_i) \in \mathsf{X}X_i$, *then* $\mathcal{I}(\Gamma) \subset w\mathcal{I}(\Gamma) \subset w^*\mathcal{I}(\Gamma)$, $\mathcal{I}_1(\Gamma) \subset (w\mathcal{I})_1(\Gamma) \subset (w^*\mathcal{I})_1(\Gamma)$. *Further,*

$$\int_{(A_i)} (f_i)\,\mathrm{d}\Gamma = w \int_{(A_i)} (f_i)\,\mathrm{d}\Gamma = w^* \int_{(A_i)} (f_i)\,\mathrm{d}\Gamma$$

for each $(f_i) \in \mathcal{I}(\Gamma)$ *and each* $(A_i) \in \mathsf{X}\sigma(\mathcal{P}_i)$.

7) *If* Z *is a Grothendieck space, i.e., if w* and weak convergence of sequences in* Z^* *coincide*, see p. 179 in [1], *then* $w^*\mathcal{I}(\Gamma) = w\mathcal{I}(\Gamma) = \mathcal{I}(\Gamma)$ *and the integrals coincide.*

Proof. Assertions 1)–6) follow similarly as their weak analogues in Theorems 6 and 7.

7) Let $(f_i) \in w^*\mathcal{I}(\Gamma)$. Since the weak* and weak convergence of sequences coincide in Z^*, the proof of assertion 2) of Theorem 6 works for both weak* and weak integration. Hence $(f_i) \in w\mathcal{I}(\Gamma)$ and $w^* \int_{(\cdot)} (f_i)\,\mathrm{d}\Gamma = w \int_{(\cdot)} (f_i)\,\mathrm{d}\Gamma \colon \mathsf{X}\sigma(\mathcal{P}_i) \to Z^*$, and it is separately countably additive in the weak topology of Z^*. But then $w \int_{(\cdot)} (f_i)\,\mathrm{d}\Gamma \colon \mathsf{X}\sigma(\mathcal{P}_i) \to Z^*$ is separately countably additive in the norm of Z^* by the Orlicz-Pettis theorem. Since $(f_i\chi_{F_{i,k}}) \in \mathcal{I}_1(\Gamma)$ for each $k = 1, 2, \ldots$, see the proof of 2) of Theorem 5, $(f_i) \in \mathcal{I}_2(\Gamma)$ by Corollary 2 of Theorem IX.4. (we used the fact that $c_0 \not\subset Z^*$).

The just proved assertion is a particular case of assertion 1) of the next

Theorem 9. *Let* $(f_i) \in w\mathcal{I}(\Gamma)$ *and let* $w \int_{(\cdot)} (f_i)\,\mathrm{d}\Gamma \colon \mathsf{X}\sigma(\mathcal{P}_i) \to \dot{Y} =$ *the image of* Y *in* Y^{**} *under natural embedding. Then*:

1) $(f_i) \in \mathcal{I}(\Gamma)$ *if and only if there are* $(F_{i,k}) \in \mathsf{X}\mathcal{P}_i$, $k = 1, 2, \ldots$ *such that* $F_{i,k} \nearrow F_i = \{t_i \in T_i, f_i(t_i) \neq 0\}$ *for each* $i = 1, \ldots, d$, *and* $(f_i\chi_{F_{i,k}}) \in \mathcal{I}(\Gamma)$ *for each* $k = 1, 2, \ldots$.

2) $(f_i) \in \mathcal{I}_1(\Gamma)$ *provided* $\Gamma(\ldots)(x_i) \colon \mathsf{X}(F_i \cap \mathcal{P}_i) \to Y$ *has a control d-polymeasure for each* $(x_i) \in \mathsf{X}X_{f_i}$, *where* $X_{f_i} = \overline{\mathrm{sp}}\,\{f_i(T_i)\}$, $i = 1, \ldots, d$. *Particularly this is true if each* $\mathcal{P}_i$, $i = 1, \ldots, d$, *is generated by a countable family of sets, see Corollary of Theorem* VIII.11.

3) $(f_i) \in \mathcal{I}_1(\Gamma)$ *provided* $f_i(T_i) \subset X_i$ *is relatively σ-compact for each* $i = 1, \ldots, d$. *Particularly this happens if each* X_i, $i = 1, \ldots, d$, *is finite dimensional, and if* $(f_i) \in \mathsf{X}E(\mathcal{P}_i, X_i)$.

Proof. 1) follows by Orlicz-Pettis theorem, which is valid for polymeasures, and by Corollary 2 of Theorem IX.4.

2) According to Theorems VIII.17 and VIII.19 there is a control d-polymeasure, say $\lambda_1 \times \ldots \times \lambda_d$: $\mathsf{X}(F_i \cap \mathscr{P}_i) \to [0, 1]$ for $\Gamma' = \Gamma$: $\mathsf{X}(F_i \cap \mathscr{P}_i) \to L^{(d)}(X_{f_i}; Y)$. Owing to assertion 2) of Theorem 3 for each $i = 1, \ldots, d$ there are $f_{i,n} \in S(F_i \cap \mathscr{P}_i, X_{f_i})$, $n = 1, 2, \ldots$ such that $f_{i,n} \to f_i$ and $|f_{i,n}| \nearrow |f_i|$. Applying coordinatewise Egoroff-Lusin theorem, the σ-finiteness of the semivariation $\hat{\Gamma}'$ on $\mathsf{X}\sigma(F_i \cap \mathscr{P}_i)$, and Corollary 3 of Theorem IX.4 imply the existence of a sequence $(F_{i,k}) \in \mathsf{X}\mathscr{P}_i$, $k = 1, 2, \ldots$ such that $(f_i \chi_{F_{i,k}}) \in \mathscr{I}_1(\Gamma')$ for each $k = 1, 2, \ldots$, and $F_{i,k} \nearrow F_i$ for each $i = 1, \ldots, d$. Hence $(f_i) \in \mathscr{I}(\Gamma')$ by 1). Clearly $\mathscr{I}(\Gamma') \subset \mathscr{I}(\Gamma)$. $\mathscr{I}(\Gamma') = \mathscr{I}_1(\Gamma')$ by Theorem X.3.

3) Follows similarly as 2), using Theorem X.1 instead of Egoroff-Lusin theorem, and Theorem X.5 instead of Theorem X.3.

From this theorem and from assertions 3) of Theorem 6 and 4) of Theorem 7 we immediately obtain the following

Corollary. *Let* $(g_i) \in \mathscr{L}_1\mathscr{I}(\Gamma) = \mathscr{L}_1\mathscr{M}(\Gamma) \cap \mathscr{I}(\Gamma)$ *and suppose either that* $\Gamma(\ldots)(x_i)$: $\mathsf{X}\mathscr{P}_{g_i} \to Y$ *has locally a control d-polymeasure for each* $(x_i) \in \mathsf{X}X_{g_i}$, *or that* $g_i(T_i) \subset X_i$ *is relatively* σ*-compact for each* $i = 1, \ldots, d$. *Then the assertions of Theorem* XI.6 (Fubini), *its Corollary* 1 *and* a) $\Rightarrow$ b) *of its Corollary* 2 *hold for* (g_i).

The next theorem is in the spirit of Theorems X.3 and X.5.

Theorem 10. *Let* f_i: $T_i \to X_i$ *be* $\mathscr{P}_i$*-measurable,* $i = 1, \ldots, d$, *and let either* $\Gamma(\ldots)(x_i)$: $\mathsf{X}(F_i \cap \mathscr{P}_i) \to Y$ *have a control d-polymeasure for each* $(x_i) \in \mathsf{X}X_{f_i}$, *where* $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\}$, *and* $X_{f_i} = \overline{\text{sp}}\,\{f_i(T_i)\}$, *or that* $f_i(T_i) \subset X_i$ *be relatively* σ*-compact for each* $i = 1, \ldots, d$. *Then*:

1) *If* $(f_i) \in w\mathscr{I}(\Gamma)$, *then* $(f_i) \in (w\mathscr{I})_1(\Gamma)$ *and there are* $(f_{i,k}) \in \mathsf{X}S(F_i \cap \mathscr{P}_i, X_{f_i})$, $k = 1, 2, \ldots$ *such that* $f_{i,k} \to f_i$ *and* $|f_{i,k}| \nearrow |f_i|$ *for each* $i = 1, \ldots, d$, *and*

$$\lim_{k\to\infty} y^*\left(\int_{(A_i)} (f_{i,k})\, \mathrm{d}\Gamma\right) = \int_{(A_i)} (f_i)\, \mathrm{d}(y^*\Gamma) = \left(w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\right)(y^*)$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *and each* $y^* \in Y^*$.

2) *If* $Y = Z^*$, *where* Z *is a Banach space, and if* $(f_i) \in w^*\mathscr{I}(\Gamma)$, *then* $(f_i) \in (w^*\mathscr{I})_1(\Gamma)$ *and there are* $(f_{i,k}) \in \mathsf{X}S(F_i \cap \mathscr{P}_i, X_{f_i})$, $k = 1, 2, \ldots$ *such that* $f_{i,k} \to f_i$ *and* $|f_{i,k}| \nearrow |f_i|$ *for each* $i = 1, \ldots, d$, *and*

$$\lim_{k\to\infty} \left(\int_{(A_i)} (f_{i,k})\, \mathrm{d}\Gamma\right)(z) = \int_{(A_i)} (f_i)\, \mathrm{d}(\Gamma(\cdot)\, z) = \left(w^* \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma\right)(z)$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each $z \in Z$.

Proof. Using a control d-polymeasure for $\Gamma' = \Gamma$: $\mathsf{X}(F_i \cap \mathscr{P}_i) \to L^{(d)}(X_{f_i}, Y)$, see Theorems VIII.17 and VIII.19, throught Theorem 1 the case of the first alternative assumption can be reduced to the case of the second assumption. Hence let $f_i(T_i) \subset X_i$ be relatively σ-compact for each $i = 1, \ldots, d$. Since by assumption the semivariation $\hat{\Gamma}$ is σ-finite, there are $(F'_{i,k}) \in \mathsf{X}(F_i \cap \mathscr{P}_i)$, $k = 1, 2, \ldots$ such that $F'_{i,k} \nearrow F_i$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}(F'_{i,k}) < +\infty$ for each $k = 1, 2, \ldots$. For $k = 1, 2, \ldots$ put $\varepsilon_k = [k^d d(\hat{\Gamma}(F'_{i,k}) + 1)]^{-1}$, and for each $i = 1, \ldots, d$ take $F_{i,k} \in F_i \cap \mathscr{P}_i$ and $f_{i,k} \in$

$\in S(F_i \cap \mathscr{P}_i, X_{f_i})$ $k = 1, 2, \ldots$ in accordance with assertion 2) of Theorem 2. For $i = 1, \ldots, d$ and $k = 1, 2, \ldots$ put $F''_{i,k} = F'_{i,k} \cap F_{i,k}$ and $f''_{i,k} = f_{i,k}\chi_{F''_{i,k}}$. Then $F''_{i,k} \nearrow F_i$, $f''_{i,k} \to f_i$ and $|f''_{i,k}| \nearrow |f_i|$ for each $i = 1, \ldots, d$. Further $(f_i\chi_{F''_{i,k}}) \in \mathscr{I}_1(\Gamma)$ for each $k = 1, 2, \ldots$, and

$$\left|\int_{(A_i)} (f_i\chi_{F''_{i,k}})\,\mathrm{d}\Gamma - \int_{(A_i)} (f''_{i,k})\,\mathrm{d}\Gamma\right| < 1/k$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each $k = 1, 2, \ldots$.

1) If now $(f_i) \in w\mathscr{I}(\Gamma)$, then

$$\int_{(A_i)} (f_i)\,\mathrm{d}(y^*\Gamma) = \lim_{k\to\infty} \int_{(A_i \cap F''_{i,k})} (f_i)\,\mathrm{d}(y^*\Gamma) =$$
$$= (\text{by Theorem 6 } -, 3)) = \lim_{k\to\infty} y^*\left(\int_{(A_i)} (f_i\chi_{F''_{i,k}})\,\mathrm{d}\Gamma\right) =$$
$$= \lim_{k\to\infty} y^*\left(\int_{(A_i)} (f''_{i,k})\,\mathrm{d}\Gamma\right)$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each $y^* \in Y^*$, which we wanted to show.

2) Follows similarly as 1).

Let $d = 1$ and $m = \Gamma$. Since each vector measure $m(\cdot)\,x : \mathscr{P} \to Y$, $x \in X$, has locally a control measure, we have the following two consequences:

Corollary 1. *Let $d = 1$. Then $w\mathscr{I}(m) = (w\mathscr{I})_1(m)$ and for each $f \in w\mathscr{I}(m)$ there is a sequence $f_k \in S(F \cap \mathscr{P}, X_f)$, $k = 1, 2, \ldots$ such that $f_k \to f$, $|f_k| \nearrow |f|$ and*

$$\lim_{k\to\infty} y^*\left(\int_E f_k\,\mathrm{d}m\right) = \int_E f\,\mathrm{d}(y^*m)$$

uniformly with respect to $E \in \sigma(\mathscr{P})$, for each $y^* \in Y^*$.

Corollary 2. *Let $d = 1$, let $Y = Z^*$, where Z is a separable Banach space, let $m(\cdot)\,xz : \mathscr{P} \to Y$ be countably additive for each $x \in X$ and each $z \in Z$, and let the semivariation $\hat{m}$ be σ-finite on $\mathscr{P}$. Then $w^*\mathscr{I}(m) = (w^*\mathscr{I})_1(m)$, and for each $f \in w^*\mathscr{I}(m)$ there is a sequence $f_k \in S(F \cap \mathscr{P}, X_f)$, $k = 1, 2, \ldots$ such that $f_k \to f$, $|f_k| \nearrow |f|$, and*

$$\lim_{k\to\infty} \left(\int_E f_k\,\mathrm{d}m\right)(z) = \int_E f\,\mathrm{d}(m(\cdot)\,z)$$

uniformly with respect to $E \in \sigma(\mathscr{P})$, for each $z \in Z$.

Since each scalar bimeasure is uniform, see (Y) at the beginning of part VIII, using Corollary 1 of Theorem VIII.16 and Theorem X.9 we also have

Corollary 3. *Let $d = 2$ and suppose either Y has a countable norming set, or that X_1 and X_2 are finite dimensional. Then $w\mathscr{I}(\Gamma) = (w\mathscr{I})_1(\Gamma)$ and for each $(f_1, f_2) \in w\mathscr{I}(\Gamma)$ there are $(f_{1,k}, f_{2,k}) \in S(F_1 \cap \mathscr{P}_1, X_{f_1}) \times S(F_2 \cap \mathscr{P}_2, X_{f_2})$, $k = 1, 2, \ldots$ such that $f_{i,k} \to f_i$, $|f_{i,k}| \nearrow |f_i|$, $i = 1, 2$, and*

$$\lim_{k\to\infty} y^*\left(\int_{(A_1,A_2)} (f_{1,k}, f_{2,k})\,\mathrm{d}\Gamma\right) = \int_{(A_1,A_2)} (f_1, f_2)\,\mathrm{d}(y^*\Gamma)$$

uniformly with respect to $(A_1, A_2) \in \sigma(\mathscr{P}_1) \times \sigma(\mathscr{P}_2)$ for each $y^* \in Y^*$.

Corollary 4. *Let $d = 2$, let $Y = Z^*$, where Z is a Banach space, and suppose either Z is separable, or that X_1 and X_2 are finite dimensional. Then $w^*\mathscr{I}(\Gamma) = (w^*\mathscr{I})_1(\Gamma)$ and for each $(f_1, f_2) \in w^*\mathscr{I}(\Gamma)$ there are $(f_{1,k}, f_{2,k}) \in S(F_1 \cap \mathscr{P}_1, X_{f_1}) \times S(F_2 \cap \mathscr{P}_2, X_{f_2})$, $k = 1, 2, \ldots$ such that $f_{i,k} \to f_i$, $|f_{i,k}| \nearrow |f_i|$, $i = 1, 2$, and*

$$\lim_{k\to\infty} \left(\int_{(A_1,A_2)} (f_{1,k}, f_{2,k})\, d\Gamma\right)(z) = \int_{(A_1,A_2)} (f_1, f_2)\, d(\Gamma(\cdot)\, z)$$

uniformly with respect to $(A_1, A_2) \in \sigma(\mathscr{P}_1) \times \sigma(\mathscr{P}_2)$ for each $z \in Z$.

Our polymeasure Γ induces by the equality $\Gamma^o(A_i)(k_i) = \prod_{i=1}^{d} k_i\, \Gamma(A_i)$, $(A_i) \in \mathsf{X}\mathscr{P}_i$, $(k_i) \in \mathsf{X}K_i$, $K_i = K =$ the space of scalars for each $i = 1, \ldots, d$, the polymeasure $\Gamma^o\colon \mathsf{X}\mathscr{P}_i \to L^{(d)}(K_i\colon L^{(d)}(X_i; Y))$. Conversely, each such Γ^o induces Γ by the equality $\Gamma(A_i) = \Gamma^o(A_i)(1_i)$. Now we make no requirements on σ-finiteness of the semivariation $\hat{\Gamma}$, since the semivariation $\hat{\Gamma}^o = \|\Gamma^o\| = \|\Gamma\| =$ the scalar semivariation of Γ is finite valued on $\mathsf{X}\mathscr{P}_i$, see Corollary 1 of Theorem VIII.2, Definition VIII.3 and assertion 4) of Theorem VIII.3.

Definition 4. *Let $f_i\colon T_i \to K$* be *$\mathscr{P}_i$*-measurable, *$i = 1, \ldots, d$.* We say that (f_i) is a *Γ^o-integrable d-tuple*, and write $(f_i) \in \mathscr{I}(\Gamma^o)$, if $(f_i) \in \mathscr{I}(\Gamma(\ldots)(x_i)) = \mathscr{I}_1(\Gamma(\ldots) . (x_i))$ for each $(x_i) \in \mathsf{X}X_i$, see Theorem X.5. For $(f_i) \in \mathscr{I}(\Gamma^o)$ and $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ we put

$$\left(\int_{(A_i)} (f_i)\, d\Gamma^o\right)(x_i) = \int_{(A_i)} (f_i)\, d(\Gamma(\ldots)(x_i))\,.$$

We write $(g_i) \in \mathscr{L}_1(\Gamma^o)$ if $g_i\colon T_i \to K$ is $\mathscr{P}_i$-measurable for each $i = 1, \ldots, d$ and $(f_i) \in \mathscr{I}(\Gamma^o)$ whenever $f_i\colon T_i \to K$ is $\mathscr{P}_i$-measurable and $|f_i| \leqq |g_i|$ for each $i = 1, \ldots, d$.

Applying Theorem 2, and using the finiteness of the semivariation $\hat{\Gamma}^o = \|\Gamma\|$, similarly as the preceding theorem one can easily obtain the following

Theorem 11. *Let $(f_i) \in \mathscr{I}(\Gamma^o)$. Then:*

1) *There is a sequence $(F_{i,k}) \in \mathsf{X}\mathscr{P}_i$, $k = 1, 2, \ldots$ such that $F_{i,k} \nearrow F_i = \{t_i \in T_i,\ f_i(t_i) \neq 0\}$, $k^{-1} \leqq |f_i(t_i)| \leqq k$ for each $t_i \in F_{i,k}$, $k = 1, 2, \ldots$, $i = 1, \ldots, d$,* and sequences of finite $\mathscr{P}_i$-partitions $\pi_{i,k}(F_{i,k}) = (F_{i,k,j})_{j=1}^{r_{i,k}}$, $k = 1, 2, \ldots$, $i = 1, \ldots, d$, such that $(F_{i,k+1,j} \cap F_{i,k})_{j=1}^{r_{i,k+1}} \geqq \pi_{i,k}(F_{i,k})$ in the sense of refinements, for each $k = 1, 2, \ldots$, $i = 1, \ldots, d$, and for any points $t_{i,k,j} \in F_{i,k,j}$ the inequality

$$\left|\int_{(A_i \cap F_{i,k})} (f_i)\, d\Gamma^o - \sum_{j=1}^{r_{i,k}} \prod_{i=1}^{d} f(t_{i,k,j})\, \Gamma(A_i \cap F_{i,k})\right| < k^{-1}$$

holds for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each $k = 1, 2, \ldots$.

2) There are $(f_{i,k}) \in \mathsf{X}S(\mathscr{P}_i, K)$, $k = 1, 2, \ldots$ such that $f_{i,k} \to f_i$, $|f_{i,k}| \nearrow |f_i|$ for each $i = 1, \ldots, d$, and

$$\lim_{k\to\infty} \left(\int_{(A_i)} (f_{i,k})\, d\Gamma^o\right)(x_i) = \lim_{k\to\infty} \int_{(A_i)} (f_{i,k})\, d(\Gamma(\ldots)(x_i)) =$$
$$= \int_{(A_i)} (f_i)\, d(\Gamma(\ldots)(x_i)) = \left(\int_{(A_i)} (f_i)\, d\Gamma^o\right)(x_i)$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and each $(x_i) \in \mathsf{X}X_i$.

3) $\mathscr{I}(\Gamma^o) = \mathscr{I}_1(\Gamma^o)$. For each $(f_i) \in \mathscr{I}(\Gamma^o)$ the indefinite integral $\int_{(\cdot)} (f_i)\, d\Gamma^o$: $X\sigma(\mathscr{P}_i) \to L^{(d)}(X_i; Y)$, it is separately countably additive in the strong operator topology, and its semivariation $\widehat{\int_{(\cdot)} (f_i)\, d\Gamma^o}$ is locally σ-finite provided $\hat{\Gamma}$ is locally σ-finite on $X\sigma(\mathscr{P}_i)$.

If $g_i \in S(\mathscr{P}_i, K)$, f_i: $T_i \to K$ is $\mathscr{P}_i$-measurable and $|f_i| \leqq |g_i|$, then $f_i \in \bar{S}(G_i \cap \mathscr{P}_i, K)$, where $G_i = \{t_i \in T_i, g_i(t_i) \neq 0\} \in \mathscr{P}_i$, $i = 1, \ldots, d$. From this fact and from assertions 4) of Theorem 6 and 3) of Theorem 7 we immediately obtain

Theorem 12 1) $XS(\mathscr{P}_i, K) \subset \mathscr{L}_1(\Gamma^o)$.

2) *If* $c_0 \not\subset Y$, *then* $\mathscr{L}_1(\Gamma^o) = \mathscr{L}_1\mathscr{M}(\Gamma^o)$, *and* $(f_i) \in \mathscr{I}(\Gamma^o)$ *if and only if* $(f_i) \in \mathscr{I}(y^*\Gamma(\ldots)(x_i))$ *for each* $y^* \in Y^*$ *and each* $(x_i) \in XX_i$.

3. MONOTOE CONVERGENCE THEOREM AND CHARACTERIZATIONS OF $\mathscr{L}_1(\Gamma)$ AND OF BEPPO LEVI PROPERTY

Theorem 13 (Monotone Convergence Theorem in $\mathscr{L}_1(\Gamma)$). *Let* $c_0 \not\subset Y$, *let for each* $i = 1, \ldots, d$ *the functions* $f_i, f_{i,n}$: $T_i \to X_i$, $n = 1, 2, \ldots$ *be* $\mathscr{P}_i$-*measurable, and let* $f_{i,n} \to f_i$ *and* $|f_{i,n}| \nearrow |f_i|$, $i = 1, \ldots, d$, Γ-*almost everywhere, see Definition* XI.1. *Then the following conditions are equivalent*:

a) $\lim_{n\to\infty} \hat{\Gamma}[(f_{i,n}), (T_i)] = \hat{\Gamma}[(f_i), (T_i)] < +\infty$, *and*

b) $(f_i) \in \mathscr{L}_1(\Gamma)$,

and if they hold, then $(f_i), (f_{i,n}) \in \mathscr{L}_1(\Gamma) \subset \mathscr{I}_1(\Gamma)$ *for each* $n = 1, 2, \ldots$, *and*

$$(1) \qquad \lim_{n_1,\ldots,n_d\to\infty} \int_{(A_i)} (f_{i,n_i})\, d\Gamma = \int_{(A_i)} (f_i)\, d\Gamma$$

for each $(A_i) \in X\sigma(\mathscr{P}_i)$.

If in each of the d *coordinates* $i = 1, \ldots, d$ *either the convergence* $f_{i,n} \to f_i$ *is uniform, or the multiple* L_1-*gauge* $\hat{\Gamma}[(f_i), (\ldots, T_{i-1}, \cdot, T_{i+1}, \ldots)]$: $\sigma(\mathscr{P}_i) \to [0, +\infty)$ *is continuous on* $\sigma(\mathscr{P}_i)$, *then the limit in* (1) *is uniform with respect to* $(A_i) \in X\sigma(\mathscr{P}_i)$.

Proof. Without loss of generality we may suppose that the convergences $f_{i,n}(t_i) \to f_i(t_i)$ and $|f_{i,n}(t_i)| \nearrow |f_i(t_i)|$ hold for every $t_i \in T_i$, for each $i = 1, \ldots, d$. Then the equality in a) is a consequence of the Fatou property of the L_1-gauge $\hat{\Gamma}[(\cdot), (\cdot)]$, see Theorem VIII.4. a) $\Rightarrow$ b) by Theorem XI.5. b) $\Rightarrow$ a) by Theorem XIII.12. The remaining assertions follow from LDCT in $\mathscr{L}_1(\Gamma)$, i.e., from Theorem XI.10.

Definition 5. We say that the polymeasure Γ: $X\mathscr{P}_i \to L^{(d)}(X_i; Y)$ has the *Beppo Levi property* if a) $\Rightarrow$ b) in the notations of Theorem 13, provided $(f_{i,n}) \in \mathscr{L}_1(\Gamma)$ for each $n = 1, 2, \ldots$. Note that in this case the conclusions of Theorem 13 hold.

The following theorem is related to Theorem VII.2.

Theorem 14. *The following conditions are equivalent*:

a) $\mathscr{L}_1\mathscr{M}(\Gamma) = \mathscr{L}_1\mathscr{I}(\Gamma)$,

b) $\mathscr{L}_1\mathscr{M}(\Gamma) \cap XE(\mathscr{P}_i, X_i) \subset \mathscr{L}_1\mathscr{I}(\Gamma)$,

c) $\mathscr{L}_1\mathscr{M}(\Gamma) = \mathscr{L}_1(\Gamma)$,

d) If: $(f_{i,n}) \in \mathsf{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$, $f_{i,n} \to f_i$ *and* $|f_{i,n}| \nearrow |f_i|$ *for each* $i = 1, \ldots, d$, *and* $\hat{\Gamma}[(f_i), (T_i)] = \lim_{n\to\infty} \hat{\Gamma}[(f_{i,n}), (T_i)] < +\infty$ *imply that* $\lim_{n\to\infty} \int_{(A_i)} (f_{i,n})\, d\Gamma \in Y$ *exists for each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *hence that* $(f_i) \in \mathscr{I}_1(\Gamma)$ *and* $\lim_{n\to\infty} \int_{(A_i)} (f_{i,n})\, d\Gamma = \int_{(A_i)} (f_i)\, d\Gamma$ *for each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *and if they hold, then* Γ *has the Beppo Levi property.*

Conversely, if to each $(f_i) \in \mathsf{X}S(\mathscr{P}_i, X_i)$ *there is a sequence* $(f_{i,n}) \in \mathscr{L}_1(\Gamma)$, $n = 1, 2, \ldots$ *such that* $f_{i,n} \to f_i$ *for each* $i = 1, \ldots, d$, *and if* Γ *has the Beppo Levi property, then* $\mathscr{L}_1\mathscr{M}(\Gamma) = \mathscr{L}_1(\Gamma)$.

Proof. The only non obvious implication b) $\Rightarrow$ c) is the assertion of Corollary of Theorem 17 below.

Concerning the last assertion of the theorem, let $(f_i) \in \mathsf{X}S(\mathscr{P}_i, X_i)$, let $(f_{i,n}) \in \mathscr{L}_1(\Gamma)$, $n = 1, 2, \ldots$, and let $f_{i,n} \to f_i$ for each $i = 1, \ldots, d$. For each considered i take $\varphi_{i,n} \in S(\mathscr{P}_i, [0, +\infty))$, $n = 1, 2, \ldots$ such that $\varphi_{i,n} \nearrow |f_i|$. Put

$$h_{i,n} = \frac{f_{i,n}}{|f_{i,n}|}\, \varphi_{i,n}\chi_{\{t_i \in T_i, |f_i(t_i)| \geqq 1/n\}}$$

for $n = 1, 2, \ldots$ and $i = 1, \ldots, d$. Then $(h_{i,n}) \in \mathscr{L}_1(\Gamma)$ for each $n = 1, 2, \ldots$, see Lemma XI.2, and $h_{i,n} \to f_i$ and $|h_{i,n}| \nearrow |f_i|$ for each $i = 1, \ldots, d$. Hence $(f_i) \in \mathscr{L}_1(\Gamma)$ by Beppo Levi property of Γ. Thus $\mathsf{X}S(\mathscr{P}_i, X_i) \subset \mathscr{L}_1(\Gamma)$. But then $\mathscr{L}_1\mathscr{M}(\Gamma) = \mathscr{L}_1(\Gamma)$ by Beppo Levi property of Γ.

We now prove the following characterization of elements of $\mathscr{L}_1(\Gamma)$.

Theorem 15. *Let* $g_i\colon T_i \to X_i$ *be* $\mathscr{P}_i$*-measurable,* $i = 1, \ldots, d$. *Then* $(g_i) \in \mathscr{L}_1(\Gamma)$ *if and only if the following condition holds:*

(C_{L_1}): *If* $(h_{i,n}) \in \mathsf{X}S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$, $|h_{i,n}| \leqq |g_i|$ *for each* $n = 1, 2, \ldots$ *and each* $i = 1, \ldots, d$, $h_{i,n} \to h_i\colon T_i \to X_i$ *for each* $i = 1, \ldots, d$, *and if at least one* h_i, $i \in \{1, \ldots, d\}$ *is identically the zero function, then* $\lim_{n\to\infty} \int_{(T_i)} (h_{i,n})\, d\Gamma = 0$.

Proof. The necessity of (C_{L_1}) is a consequence of the LDCT in $\mathscr{L}_1(\Gamma)$, i.e., of Theorem XI.10.

Suppose (C_{L_1}) holds. Let $f_i\colon T_i \to X_i$ be $\mathscr{P}_i$-measurable and let $|f_i| \leqq |g_i|$ for each $i = 1, \ldots, d$. We have to show that $(f_i) \in \mathscr{I}(\Gamma)$. For each $i = 1, \ldots, d$ take a sequence $f_{i,n} \in S(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ such that $f_{i,n} \to f_i$ and $|f_{i,n}| \nearrow |f_i|$. We assert that $\lim_{n\to\infty} \int_{(A_i)} (f_{i,n})\, d\Gamma \in Y$ exists for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, which implies $(f_i) \in \mathscr{I}_1(\Gamma)$. Suppose the contrary. Then there are: $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, $\varepsilon > 0$, and a subsequence $\{n_k\} \subset \{n\}$ such that

$$\left|\int_{(A_i)} (f_{i,n_{k+1}})\, d\Gamma - \int_{(A_i)} (f_{i,n_k})\, d\Gamma\right| > \varepsilon$$

for each $k = 1, 2, \ldots$. Since

$$\int_{(A_i)} (f_{i,n_{k+1}})\, \mathrm{d}\Gamma - \int_{(A_i)} (f_{i,n_k})\, \mathrm{d}\Gamma = \int_{(A_i)} (f_{1,n_{k+1}} - f_{1,n_k}, f_{2,n_{k+1}}, \ldots, f_{d,n_{k+1}})\, \mathrm{d}\Gamma + \ldots + \int_{(A_i)} (f_{1,n_k}, \ldots, f_{d-1,n_k}, f_{d,n_{k+1}} - f_{d,n_k})\, \mathrm{d}\Gamma ,$$

there is an $i_0 \in \{1, \ldots, d\}$ and a subsequence $\{k_j\} \subset \{k\}$ such that the inequality

$$\left|\int_{(A_i)} (\ldots, f_{i_0-1,n_{kj}}, f_{i_0,n_{kj+1}} - f_{i_0,n_{kj}}, f_{i_0+1,n_{kj+1}}, \ldots)\, \mathrm{d}\Gamma\right| > \varepsilon d^{-1}$$

for each $j = 1, 2, \ldots$. Put $h_{i,j} = f_{i,n_{kj}}\chi_{A_i}$ for $i < i_0$, $h_{i_0,j} = (f_{i_0,n_{kj+1}} - f_{i_0,n_{kj}})\, \chi_{A_{i0}}$, and $h_{i,j} = f_{i,n_{j+1}}\chi_{A_i}$ for $i > i_0$, $j = 1, 2, \ldots$. Then we have a contradiction with the condition (C_{L_1}).

Analogously the following characterization can be proved:

Theorem 16. *The polymeasure Γ has Beppo Levi property if and only if the following condition holds:*

(C_{BL}): *If: $(h_{i,n}) \in XS(\mathscr{P}_i, X_i) \cap \mathscr{L}_1(\Gamma)$, $n = 1, 2, \ldots$, $\hat{\Gamma}[(\sup_n |h_{i,n}|), (T_i)] < +\infty$, $h_{i,n} \to h_i$: $T_i \to X_i$ for each $i = 1, \ldots, d$, and at least one h_i, $i \in \{1, \ldots, d\}$ is identically equal to the zero function, then $\lim_{n\to\infty} \int_{(T_i)} (h_{i,n})\, \mathrm{d}\Gamma = 0$.*

Another usefull characterization of elements of $\mathscr{L}_1(\Gamma)$ is given by the following

Theorem 17. *Let g_i: $T_i \to X_i$ be $\mathscr{P}_i$-measurable, $i = 1, \ldots, d$. Then $(g_i) \in \mathscr{L}_1(\Gamma)$ if and only if $\hat{\Gamma}[(g_i), (T_i)] < +\infty$ and $(h_i) \in \mathscr{I}(\Gamma)$ whenever $(h_i) \in XE(\mathscr{P}_i, X_i)$ and $|h_i| \leqq |g_i|$ for each $i = 1, \ldots, d$.*

Proof. The necessity follows from Theorem XIII.6 and the definition of $\mathscr{L}_1(\Gamma)$.

Sufficiency. Let f_i: $T_i \to X_i$ be $\mathscr{P}_i$-measurable and let $|f_i| \leqq |g_i|$ for each $i = 1, \ldots, d$. By assertion 3) of Theorem 3 for each $i = 1, \ldots, d$ there is a sequence $h_{i,n} \in E(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ such that $|h_{i,n}| \nearrow |f_i|$ and $|f_i - h_{i,n}| \leqq (1/n)\, |f_i|$ for each $n = 1, 2, \ldots$. Hence

$$|h_{i,n}| \leqq \left(1 + \frac{1}{n}\right) |f_i| \leqq \left(1 + \frac{1}{n}\right) |g_i|$$

for each $n = 1, 2, \ldots$ and each $i = 1, \ldots, d$. Thus

$$\left(\left(1 + \frac{1}{n}\right)^{-1} h_{i,n}\right) \in \mathscr{I}(\Gamma),$$

hence $(h_{i,n}) \in \mathscr{I}(\Gamma)$ for each $n = 1, 2, \ldots$ by assumption. Since $h_{i,n}(t_i) \to f_i(t_i)$ for each $t_i \in T_i$ and each $i = 1, \ldots, d$, and since

$$\left|\int_{(A_i)} (h_{i,n})\, \mathrm{d}\Gamma - \int_{(A_i)} (h_{i,k})\, \mathrm{d}\Gamma\right| \leqq$$
$$\leqq \left|\int_{(A_i)} (h_{1,n} - h_{1,k}, h_{2,n}, \ldots, h_{d,n})\, \mathrm{d}\Gamma\right| + \ldots$$
$$\ldots + \left|\int_{(A_i)} (h_{1,k}, \ldots, h_{d-1,k}, h_{d,n} - h_{d,k})\, \mathrm{d}\Gamma\right| \leqq$$

$$\leqq \frac{2d}{n_0} \hat{\Gamma}[(f_i), (T_i)] \leqq \frac{2d}{n_0} \hat{\Gamma}[(g_i), (T_i)]$$

for each $(A_i) \in X\sigma(\mathscr{P}_i)$ and each k, $n \geqq n_0$, $(f_i) \in \mathscr{I}(\Gamma)$ by Theorem IX.4 – 1). Since (f_i) with required properties was arbitrary, $(g_i) \in \mathscr{L}_1(\Gamma)$.

Corollary. $\mathscr{L}_1\mathscr{M}(\Gamma) = \mathscr{L}_1(\Gamma)$ *if and only if* $\mathscr{L}_1\mathscr{M}(\Gamma) \cap XE(\mathscr{P}_i, X_i) \subset \mathscr{L}_1\mathscr{I}(\Gamma)$.

Theorem 18. *Let* $g_i\colon T_i \to X_i$ *be* $\mathscr{P}_i$*-measurable,* $i = 1, \ldots, d$. *Then* $(g_i) \in \mathscr{L}_1(\Gamma)$ *if and only if the following condition holds:*

$(C^*_{L_1})$: $\hat{\Gamma}[(g_i), (T_i)] < +\infty$, *and if* $(h_{i,n}) \in XS(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$, $|h_{i,n}| \leqq |g_i|$ *for each* $n = 1, 2, \ldots$ *and* $h_{i,n} \cdot h_{i,k} = 0$ *for* $n \neq k$, $i = 1, \ldots, d$, *imply*

$$\lim_{N\to\infty} \left|\int_{(T_i)} \left(\sum_{n=1}^{N+1} h_{i,n}\right) d\Gamma - \int_{(T_i)} \left(\sum_{n=1}^{N} h_{i,n}\right) d\Gamma\right| = 0\,.$$

Proof. Let $(g_i) \in \mathscr{L}_1(\Gamma)$. Then $\hat{\Gamma}[(g_i), (T_i)] < +\infty$ by Theorem XIII.12. Let $(h_{i,n}) \in XS(\mathscr{P}_i, X_i)$, $n = 1, 2, \ldots$ satisfy the assumptions of $(C^*_{L_1})$. Put $h_i = \sum_{n=1}^{\infty} h_{i,n}$, $i = 1, \ldots, d$. Then $(h_i) \in XE(\mathscr{P}_i, X_i)$ and $|h_i| \leqq |g_i|$ for each $i = 1, \ldots, d$. Hence $(h_i) \in \mathscr{I}(\Gamma)$ by Theorem 17. Put $\gamma(A_i) = \int_{(A_i)} (h_i)\, d\Gamma$, $(A_i) \in X\sigma(\mathscr{P}_i)$. Since $H_{i,N} = \{t_i \in T_i, \sum_{n=1}^{N} h_{i,n}(t_i) \neq 0\} \nearrow H_i = \{t_i \in T_i, h_i(t_i) \neq 0\}$ for each $i = 1, \ldots, d$,

$$\lim_{N\to\infty} \gamma(H_{i,N}) = \lim_{N\to\infty} \int_{(T_i)} \left(\sum_{n=1}^{N} h_{i,n}\right) d\Gamma \in Y$$

exists by Theorem VIII.1.

Suppose $(C^*_{L_1})$ holds. Let $(h_i) \in XE(\mathscr{P}_i, X_i)$ and let $|h_i| \leqq |g_i|$ for each $i = 1, \ldots, d$. According to Theorem 17 it is enough to show that $(h_i) \in \mathscr{I}(\Gamma)$. Each h_i, $i = 1, \ldots, d$ is of the form $h_i = \sum_{j=1}^{\infty} x_{i,j}\chi_{A_{i,j}}$, where $x_{i,j} \in X_i$ and $A_{i,j} \in \mathscr{P}_i$, $j = 1, 2, \ldots$ are pairwise disjoint. For $i = 1, \ldots, d$ and $k = 1, 2, \ldots$ put $u_{i,k} = \sum_{j=1}^{k} x_{i,j}\chi_{A_{i,j}} \in S(\mathscr{P}_i, X_i)$. Clearly $u_{i,k} \to h_i$ for each $i = 1, \ldots, d$. We assert that $\lim_{k\to\infty} \int_{(A_i)} (u_{i,k})\, d\Gamma \in Y$ exists for each $(A_i) \in X\sigma(\mathscr{P}_i)$, and this by Theorem IX.1 will imply the integrability of (h_i). Suppose the contrary. Then there is an $(A_i) \in X\sigma(\mathscr{P}_i)$, and $\varepsilon > 0$, and a subsequence $\{k_n\} \subset \{k\}$ such that

$$\left|\int_{(A_i)} (u_{i,k_{n+1}})\, d\Gamma - \int_{(A_i)} (u_{i,k_n})\, d\Gamma\right| > \varepsilon$$

for each $n = 1, 2, \ldots$. For $n = 1, 2, \ldots$ and $i = 1, \ldots, d$ put $h_{i,n} = (u_{i,k_n} - u_{i,k_{n-1}}) \cdot \chi_{A_i}$, where $u_{i,k_0} = 0$. Then we have a contradiction with the assertion in $(C^*_{L_1})$. Hence $(h_i) \in \mathscr{I}(\Gamma)$, what we wanted to show.

Analogously our last characterization can be proved.

Theorem 19. *The polymeasure* Γ *has Beppo Levi property if and only if the following condition holds:*

(C^*_{BL}): *If* $(h_{i,n}) \in XS(\mathscr{P}_i, X_i) \cap \mathscr{L}_1(\Gamma)$, $n = 1, 2, \ldots$, $h_{i,n} \,.\, h_{i,k} = 0$ *for* $n \neq k$, $n, k = 1, 2, \ldots$, $i = 1, \ldots, d$, *and* $\lim_{N \to \infty} \hat{\Gamma}[(\sum_{n=1}^{N} h_{i,n}), (T_i)] = \hat{\Gamma}[(\sum_{n=1}^{\infty} h_{i,n}), (T_i)] < +\infty$, then $\lim_{N \to \infty} \left| \int_{(T_i)} (\sum_{n=1}^{N} h_{i,n}) \, d\Gamma - \int_{(T_i)} (\sum_{n=1}^{\infty} h_{i,n}) \, d\Gamma \right| = 0$.

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Author's address: 842 15 Bratislava, Obrancov mieru 49, Czechoslovakia (MÚ SAV).