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ON THE AFFINE NORMAL

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In what follows, I am going to define (in the analytic as well as the geometric way) the affine normal of a hypersurface in the affine space. The construction may be compared with the papers [1] and [2] and the literature cited therein.

1. In this preliminary section, we put together several lemmas needed in the sequel; the proofs are elementary (using the local coordinates).

Let M^N be an N-dimensional differentiable manifold with local coordinates $(\xi) = (\xi^1, \ldots, \xi^N)$. On M^N, be given an affine connection ∇ by means of the functions Γ^k_{ij}. The torsion and curvature tensors of ∇ are given by

$$(1.1) \qquad T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}, \quad R^j_{ikl} = \partial_l \Gamma^j_{ik} - \partial_k \Gamma^j_{il} + \Gamma^r_{ik}\Gamma^j_{rl} - \Gamma^r_{il}\Gamma^j_{rk}$$

resp. For a function $f\colon M^N \to \mathbb{R}$ and a tangent field X on M^N, we write $\partial_i f := \partial f/\partial \xi^i$ and $D_X f = \nabla_X f := Xf$.

Lemma 1.1. *Be given (M^N, ∇). Let $f\colon M^N \to \mathbb{R}$ be a function, ω a section of $T^*(M^N)$, Ω a symmetric section of $T^*(M^N) \otimes T^*(M^N)$, $S_\xi\colon T_\xi(M^N) \to T_\xi(M^N)$ a field of endomorphisms. Then*

$$(1.2) \qquad [\nabla_X, \nabla_Y] f = D_{T(X,Y)} f,$$

$$(1.3) \qquad [\nabla_X, \nabla_Y]\, \omega(Z) = \nabla_{T(X,Y)}\, \omega(Z) + \omega(R(X, Y)\, Z),$$

$$(1.4) \qquad [\nabla_X, \nabla_Y]\, \Omega(Z, T) = \\ = \nabla_{T(X,Y)} \Omega(Z, T) + \Omega(R(X, Y)\, Z, T) + \Omega(R(X, Y)\, T, Z),$$

$$(1.5) \qquad D_X D_Y f = \nabla_X D_Y f + D_{\nabla_X Y} f,$$

$$(1.6) \qquad \nabla_Y D_{S(X)} f = D_{\nabla_Y S(X)} f + \nabla_{S(X)} D_Y f + D_{T(Y,S(X))} f.$$

Lemma 1.2. *Be given (M^N, ∇) and an affine connection ∇^* on M^N; let $\varrho(X, Y) := (\nabla^* - \nabla)(X, Y)$ be the difference tensor. Let ω be a section of $T^*(M^N)$ and Ω a section of $T^*(M^N) \otimes T^*(M^N)$. Then*

$$(1.7) \qquad \nabla^*_X \omega(Y) = \nabla_X \omega(Y) - \omega(\varrho(Y, X)),$$

$$(1.8) \qquad \nabla^*_X \Omega(Y, Z) = \nabla_X \Omega(Y, Z) - \Omega(\varrho(Y, X), Z) - \Omega(Y, \varrho(Z, X)).$$

Let $h(X, Y)$ be a symmetric bilinear form on M^N; suppose h to be regular. Then there exists the inverse symmetric bilinear form $\tilde{h}$ as a section of $T(M^N) \otimes T(M^N)$ in local coordinates, it is defined by

$$(1.9) \qquad \tilde{h}^{ir} h_{jr} = \delta^i_j .$$

Let G be a $(p + 2)$-linear form on M^N; define the p-linear form

$$(1.10) \qquad \mathrm{Tr}_{\tilde{h}\{X_i, X_j\}}\, G(X_1, \ldots, X_p)$$

by means of the local coordinates as

$$\tilde{h}^{rs} g_{k_1 \ldots k_{i-1} r k_{i+1} \ldots k_{j-1} s k_{j+1} \ldots k_{p+2}} .$$

The following is trivial.

Lemma 1.3. *We have*

$$(1.11) \qquad \mathrm{Tr}_{\tilde{h}\{X,Y\}}\, h(X, Y) = N , \quad \mathrm{Tr}_{\tilde{h}\{Z,T\}}\, h(X, Z)\, h(Y, T) = h(X, Y) .$$

If $h^* = \alpha^{-1} h$, *then* $\tilde{h}^* = \alpha \tilde{h}$.

2. Let A^{N+1} be the affine space, V^{N+1} its vector space, M^N a differentiable manifold and $m\colon M^N \to A^{N+1}$ an immersion. The *normalization* $\mathcal{N}$ of m is the choice of a mapping $n\colon M^N \to V^{N+1}$ such that $n(\xi)$ is transversal to $m(M^N)$ at the point $m(\xi)$ for each $\xi \in M^N$.

The fundamental equations of a normalized hypersurface are

$$(2.1) \qquad \partial_i m = m_i , \quad \partial_j m_i = \Gamma^k_{ij} m_k + h_{ij} n , \quad \partial_i n = -S^j_i m_j + \tau_i n ;$$

it is easy to see that Γ^k_{ij} induce a linear connection ∇ on M^N. Using this, (2.1) may be rewritten in the form

$$(2.2) \qquad \nabla_Y \mathrm{D}_X m = h(X, Y)\, n , \quad \mathrm{D}_X n = -\mathrm{D}_{S(X)} m + \tau(X)\, n .$$

Lemma 2.1. *The integrability conditions of* (2.2) *are*

$$(2.3) \qquad h(X, Y) = h(Y, X) ,$$

$$(2.4) \qquad \nabla_Z h(X, Y) + h(X, Y)\, \tau(Z) = \nabla_Y h(X, Z) + h(X, Z)\, \tau(Y) ,$$

$$(2.5) \qquad R(X, Y)\, Z = h(Z, X)\, S(Y) - h(Z, Y)\, S(X) ,$$

$$(2.6) \qquad \nabla_X \tau(Y) - h(X, S(Y)) = \nabla_Y \tau(X) - h(Y, S(X)) ,$$

$$(2.7) \qquad \nabla_X S(Y) + \tau(Y)\, S(X) = \nabla_Y S(X) + \tau(X)\, S(Y) .$$

Proof. Using (1.2) for $f = m$, we get

$$(2.8) \qquad 0 = [\nabla_Y, \nabla_X]\, m = \{h(X, Y) - h(Y, X)\}\, n ,$$

i.e., (2.3). From (2.2) and (1.3) for $\omega(X) = \mathrm{D}_X m$, we get

$$(2.9) \qquad \mathrm{D}_{R(X,Y)Z} m = [\nabla_X, \nabla_Y]\, \mathrm{D}_Z m = \\ = \nabla_X h(Z, Y)\, n + h(Z, Y)\{-\mathrm{D}_{S(X)} m + \tau(X)\, n\} - \\ - \nabla_Y h(Z, X)\, n - h(Z, X)\{-\mathrm{D}_{S(Y)} m + \tau(Y)\, n\} ,$$

i.e.,

$$(2.10)\qquad (D_{R(X,Y)Z} + D_{h(Z,Y)S(X)} - D_{h(Z,X)S(Y)})\, m = \\ = \{\nabla_X h(Z, Y) + h(Z, Y)\,\tau(X) - \nabla_Y h(Z, X) - h(Z, X)\,\tau(Y)\}\, n\,,$$

and we have (2.4) and (2.5). From (1.2) and (1.6), we obtain

$$(2.11)\qquad 0 = [\nabla_X, \nabla_Y]\, n = \\ = -\nabla_Y D_{S(X)} m + \nabla_X \tau(Y)\, n + \tau(Y)\{-D_{S(X)} m + \tau(X)\, n\} + \\ + \nabla_X D_{S(Y)} m - \nabla_Y \tau(X)\, n - \tau(X)\{-D_{S(Y)} m + \tau(Y)\, n\} = \\ = - D_{\nabla_X S(Y)} m - \nabla_{S(Y)} D_X m + \nabla_X \tau(Y)\, n - D_{\tau(Y)S(X)} m + \\ + D_{\nabla_Y S(X)} m + \nabla_{S(X)} D_Y m - \nabla_Y \tau(X)\, n + D_{\tau(X)S(Y)} m\,;$$

according to (2.2_1), $\nabla_{S(Y)} D_X m = h(X, S(Y))\, n$. Comparing the corresponding terms as in (2.10), we get (2.6) + (2.7). QED.

Proposition 2.1. *The forms*

$$(2.12)\qquad F_3(X, Y, Z) = \nabla_Z h(X, Y) + h(X, Y)\,\tau(Z)\,,$$

$$(2.13)\qquad F_4(X, Y, Z, T) = \nabla_T F_3(X, Y, Z) + F_3(X, Y, Z)\,\tau(T) + \\ + h(X, Y)\, h(Z, S(T)) + h(Y, Z)\, h(X, S(T)) + h(Z, X)\, h(Y, S(T))\,,$$

$$(2.14)\qquad F_5(X, Y, Z, T, U) = \nabla_U F_4(X, Y, Z, T) + F_4(X, Y, Z, T)\,\tau(U) + \\ + h(X, Y)\, F_3(Z, T, S(U)) + h(X, Z)\, F_3(Y, T, S(U)) + \\ + h(X, T)\, F_3(Y, Z, S(U)) + h(Y, Z)\, F_3(X, T, S(U)) + \\ + h(Y, T)\, F_3(X, Z, S(U)) + h(Z, T)\, F_3(X, Y, S(U)) + \\ + h(X, S(U))\, F_3(Y, Z, T) + h(Y, S(U))\, F_3(X, Z, T) + \\ + h(Z, S(U))\, F_3(X, Y, T) + h(T, S(U))\, F_3(X, Y, Z)$$

are symmetric.

Proof. The form (2.12) is symmetric because of (2.3) and (2.4). The form F_4 is symmetric in X, Y, Z by definition. Using (1.4), (2.5) and (2.6), we get

$$(2.15)\qquad F_4(X, Y, Z, T) - F_4(X, Y, T, Z) = \\ = [\nabla_T, \nabla_Z]\, h(X, Y) + \nabla_T h(X, Y)\,\tau(Z) - \nabla_Z h(X, Y)\,\tau(T) + \\ + h(X, Y)\{\nabla_T \tau(Z) - \nabla_Z \tau(T)\} + \\ + \nabla_Z h(X, Y)\,\tau(T) - \nabla_T h(X, Y)\,\tau(Z) + \\ + h(X, Y)\{\tau(Z)\,\tau(T) - \tau(T)\,\tau(Z)\} + \\ + h(X, Y)\{h(Z, S(T)) - h(T, S(Z))\} + h(Y, Z)\, h(X, S(T)) - \\ - h(Y, T)\, h(X, S(Z)) + h(Z, X)\, h(Y, S(T)) - h(T, X)\, h(Y, S(Z)) = \\ = h(R(T, Z)\, X, Y) + h(X, R(T, Z)\, Y) + h(Y, Z)\, h(X, S(T)) -$$

$$- h(Y, T)\, h(X, S(Z)) + h(Z, X)\, h(Y, S(T)) - h(T, X)\, h(Y, S(Z)) =$$
$$= h(h(X, T)\, S(Z) - h(X, Z)\, S(T), Y) + h(X, h(Y, T)\, S(Z) -$$
$$- h(Y, Z)\, S(T)) + h(Y, Z)\, h(X, S(T)) - h(Y, T)\, h(X, S(Z)) +$$
$$+ h(Z, X)\, h(Y, S(T)) - h(T, X)\, h(Y, S(Z)) = 0\,.$$

The proof for F_5 goes along the same lines, it is just a little bit longer. QED.

Let A be a tangent vector field on M^N and $\alpha: M^N \to \mathbb{R}$, $\alpha(\xi) \neq 0$ for each $\xi \in M^N$, a function. Consider a new normalization $\mathcal{N}^*$ given by

$$n^* = \mathrm{D}_A m + \alpha n\,. \tag{2.16}$$

It induces a new linear connection ∇^* on M^N, and we get

Lemma 2.2. *Let*

$$\nabla^*_Y \mathrm{D}_X m = h^*(X, Y)\, n^*\,, \quad \mathrm{D}_X n^* = -\mathrm{D}_{S^*(X)} m + \tau^*(X)\, n^* \tag{2.17}$$

be equations analogous to (2.2). *Then*

$$(\nabla^* - \nabla)(X, Y) = -\alpha^{-1} h(X, Y)\, A\,, \tag{2.18}$$

$$h^*(X, Y) = \alpha^{-1} h(X, Y)\,, \tag{2.19}$$

$$\tau^*(X) = \tau(X) + \alpha^{-1}\{h(A, X) + \mathrm{D}_X \alpha\}\,, \tag{2.20}$$

$$S^*(X) = \alpha\, S(X) - \nabla_X A + \{\tau(X) + \alpha^{-1} h(A, X) + \alpha^{-1} \mathrm{D}_X \alpha\}\, A\,. \tag{2.21}$$

Proof. From $(2.17_1) + (2.2_1)$,

$$(\nabla^*_Y - \nabla_Y)\, \mathrm{D}_X m = \{\alpha h^*(X, Y) - h(X, Y)\}\, n + h^*(X, Y)\, \mathrm{D}_A m\,. \tag{2.22}$$

From this, we get (2.19). Further, consider (1.7) with $\omega(X) = \mathrm{D}_X m$. We immediately see that $\varrho(X, Y) = -\alpha^{-1} h(X, Y)\, A$, i.e., we have (2.18). Further, using (1.5), we get

$$\begin{aligned} \mathrm{D}_X n^* &= \mathrm{D}_X \mathrm{D}_A m + \mathrm{D}_X \alpha n + \alpha \mathrm{D}_X n = \\ &= \mathrm{D}_{\nabla_X A} m + \nabla_X \mathrm{D}_A m + \mathrm{D}_X \alpha n + \alpha\{-\mathrm{D}_{S(X)} m + \tau(X)\, n\}\,. \end{aligned} \tag{2.23}$$

We have $\nabla_X \mathrm{D}_A m = h(X, Y)\, n$; inserting into (2.17_2),

$$\begin{aligned} &\mathrm{D}_{\nabla_X A} m + h(X, Y)\, n + \mathrm{D}_X \alpha n - \mathrm{D}_{\alpha S(X)} m + \alpha\, \tau(X)\, n = \\ &= -\mathrm{D}_{S^*(X)} m + \tau^*(X)\, (\mathrm{D}_A m + \alpha n)\,, \end{aligned} \tag{2.24}$$

and (2.20) + (2.21) follow. QED.

Lemma 2.3. *Let the form $F_3^*(X, Y, Z)$ be associated to the normalization $\mathcal{N}^*$. Then*

$$\begin{aligned} F_3^*(X, Y, Z) &= \alpha^{-1} F_3(X, Y, Z) + \\ &+ \alpha^{-2}\{h(X, Y)\, h(Z, A) + h(Y, Z)\, h(X, A) + h(Z, X)\, h(Y, A)\}\,. \end{aligned} \tag{2.25}$$

Proof. Using (1.8) with $\varrho(X, Y) = -\alpha^{-1} h(X, Y)\, A$, we get

$$F_3^*(X, Y, Z) = \nabla^*_Z h^*(X, Y) + h^*(X, Y)\, \tau^*(Z) = \tag{2.26}$$

$$= \nabla^*_Z(\alpha^{-1}h(X, Y)) + h^*(X, Y)\,\tau^*(Z) =$$
$$= -\alpha^{-2}D_Z\alpha h(X, Y) + \alpha^{-1}\{\nabla_Z h(X, Y) + h(\alpha^{-1}h(X, Z)\,A, Y) +$$
$$+ h(X, \alpha^{-1}h(Y, Z)\,A)\} + \alpha^{-2}h(X, Y)\,h(Z, A) +$$
$$+ \alpha^{-2}D_Z\alpha h(X, Y) + \alpha^{-1}h(X, Y)\,\tau(Z),$$

and (2.25) follows. QED.

Supposition 2.1. Let us restrict ourselves to hypersurfaces with h regular.

Using the notation (1.10) and Lemma 1.3, we get

$$\text{(2.27)} \qquad \mathrm{Tr}_{\bar{h}^*\{X,Y\}}F_3^*(X, Y, Z) = \mathrm{Tr}_{\bar{h}\{X,Y\}}F_3(X, Y, Z) + (N + 2)\,\alpha^{-1}h(A, Z)$$

from (2.26).

Definition 2.1. The normalization $\mathcal{N}$ is called *good* if

$$\text{(2.28)} \qquad \mathrm{Tr}_{\bar{h}\{X,Y\}}F_3(X, Y, Z) = 0 \quad \text{for each } Z\,.$$

Proposition 2.2. *There exist good normalizations. If $\mathcal{N}$ and $\mathcal{N}^*$ are two good normalizations, we have $A = 0$, i.e.,*

$$\text{(2.29)} \qquad n^* = \alpha n\,.$$

Proof follows immediately from (1.27). QED.

Lemma 2.4. *Let $\mathcal{N}$ and $\mathcal{N}^*$ be good normalizations. Then*

$$\text{(2.30)} \qquad \nabla^* = \nabla\,, \quad h^*(X, Y) = \alpha^{-1}h(X, Y)\,,$$
$$\tau^*(X) = \tau(X) + \alpha^{-1}D_X\alpha\,, \quad S^*(X) = \alpha\,S(X)\,,$$

and we have

$$\text{(2.31)} \qquad F_3^*(X, Y, Z) = \alpha^{-1}F_3(X, Y, Z)\,, \quad F_4^*(X, Y, Z, T) = \alpha^{-1}F_4(X, Y, Z, T)\,.$$

Proof. See (2.18)–(2.21) with $A = 0$. QED.

3. Because the formulas are going to be too complicated, I will express them in the usual tensor slang. Let us restrict ourselves to a domain of M^N with the local coordinates $(\xi) = (\xi^1, \ldots, \xi^N)$. For the tangent vector fields $X = x^i\,\partial/\partial\xi^i, \ldots, U = u^i\,\partial/\partial\xi^i$, write

$$\text{(3.1)} \qquad h(X, Y) = h_{ij}x^iy^j\,, \quad F_3(X, Y, Z) = a_{ijk}x^iy^jz^k\,,$$
$$F_4(X, Y, Z, T) = a_{ijkl}x^iy^jz^kt^l\,, \quad F_5(X, Y, Z, T, U) = a_{ijklp}x^iy^jz^kt^lu^p\,;$$

we have

$$\text{(3.2)} \qquad a_{ijk} = \nabla_k h_{ij} + h_{ij}\tau_k\,,$$
$$a_{ijkl} = \nabla_l a_{ijk} + a_{ijk}\tau_l + (h_{ij}h_{kr} + h_{jk}h_{ir} + h_{ki}h_{jr})\,S^r_l\,,$$
$$a_{ijklp} = \nabla_p a_{ijkl} + a_{ijkl}\tau_p + (h_{ij}a_{klr} + h_{ik}a_{jlr} + h_{il}a_{jkr} + h_{jk}a_{ilr} +$$
$$+ h_{jl}a_{ikr} + h_{kl}a_{ijr} + h_{ir}a_{jkl} + h_{jr}a_{ikl} + h_{kr}a_{ijl} + h_{lr}a_{ijk})\,S^r_p\,.$$

Simple calculations yield

Lemma 3.1. *We have*

$$\nabla_k \tilde{h}^{ij} = \tilde{h}^{ij}\tau_k - \tilde{h}^{ir}\tilde{h}^{js}a_{rsk}, \tag{3.3}$$

$$\nabla_j(\tilde{h}^{rs}a_{rsi}) = \tilde{h}^{rs}a_{rsij} - \tilde{h}^{rr'}\tilde{h}^{ss'}a_{rsi}a_{r's'j} - (N+2)\,h_{ir}S^r_j. \tag{3.4}$$

Proposition 3.1. *If $\mathcal{N}$ is a good normalization, we have*

$$S^j_i = (N+2)^{-1}\,\tilde{h}^{tj}(\tilde{h}^{rs}a_{rsti} - \tilde{h}^{rr'}\tilde{h}^{ss'}a_{rsi}a_{r's't}), \tag{3.5}$$

$$\tau_i = -N^{-1}\tilde{h}^{rs}\nabla_i h_{rs} \tag{3.6}$$

and

$$\mathrm{d}\tau = 0; \tag{3.7}$$

here, d *is the exterior differential.*

Proof. If $\mathcal{N}$ is good, we have $\tilde{h}^{rs}a_{rsi} = 0$, and (3.4) implies (3.5). Looking at the right-hand side of (3.4), we see that

$$h_{ir}S^r_j = h_{jr}S^r_i. \tag{3.8}$$

The integrability condition (2.6) being

$$\nabla_i\tau_j - h_{ir}S^r_j = \nabla_j\tau_i - h_{jr}S^r_i, \tag{3.9}$$

(3.8) simplifies it to

$$\nabla_i\tau_j = \nabla_j\tau_i. \tag{3.10}$$

From $\mathrm{d}(\tau_i\,\mathrm{d}\xi^i) = \partial_j\tau_i\,\mathrm{d}\xi^j \wedge \mathrm{d}\xi^i$, we see that $\mathrm{d}\tau = 0$ if and only if $\partial_j\tau_i = \partial_i\tau_j$, this being equivalent to (3.10). Finally, (3.6) follows from (3.2_1). QED.

Lemma 3.2. *If $\mathcal{N}$ is a good normalization, we have*

$$\begin{aligned}\nabla_i(\tilde{h}^{rr'}\tilde{h}^{ss'}\tilde{h}^{pp'}a_{rsp}a_{r's'p'}) &= \tilde{h}^{rr'}\tilde{h}^{ss'}\tilde{h}^{pp'}a_{rsp}a_{r's'p'}\tau_i + \\ &+ \tilde{h}^{ss'}\tilde{h}^{pp'}(2\tilde{h}^{rr'}a_{rsp}a_{r's'p'i} - 3\tilde{h}^{rq}\tilde{h}^{r'q'}a_{rsp}a_{r's'p'}a_{qq'i}), \\ \nabla_i(\tilde{h}^{rr'}\tilde{h}^{ss'}a_{rr'ss'}) &= \tilde{h}^{rr'}\tilde{h}^{ss'}a_{rr'ss'}\tau_i - \\ &- 2\tilde{h}^{rr'}\tilde{h}^{ss'}\tilde{h}^{pp'}a_{rsi}a_{r's'pp'} + \tilde{h}^{rr'}\tilde{h}^{ss'}a_{rr'ss'i}.\end{aligned} \tag{3.11}$$

Writing

$$\operatorname{Tr} S = S^r_r, \tag{3.12}$$

we have

$$\begin{aligned}(N+2)\,\nabla_i \operatorname{Tr} S &= (N+2)\operatorname{Tr} S\tau_i + \tilde{h}^{rr'}\tilde{h}^{ss'}a_{rr'ss'i} - \\ &- 2\tilde{h}^{rr'}\tilde{h}^{ss'}\tilde{h}^{pp'}(a_{rsi}a_{r's'pp'} + a_{rsp}a_{r's'p'i}) + \\ &+ 3\tilde{h}^{rq}\tilde{h}^{r'q'}\tilde{h}^{ss'}\tilde{h}^{pp'}a_{rsp}a_{r's'p'}a_{qq'i}.\end{aligned} \tag{3.13}$$

Proof. We obtain (3.11) by a direct calculation as well as (3.13) using (3.5). QED.

Lemma 3.3. *If $\mathcal{N}$ and $\mathcal{N}^*$ are good normalizations, we have*

$$\operatorname{Tr} S^* = \alpha \operatorname{Tr} S \,. \tag{3.14}$$

Proof follows from (2.30_4). QED.

Supposition 3.1. We restrict ourselves to hypersurfaces with

$$\operatorname{Tr} S \neq 0 \tag{3.15}$$

using a good normalization.

Definition 3.1. The good normalization $\mathcal{N}$ is called *very good* if

$$\operatorname{Tr} S = N \,. \tag{3.16}$$

Proposition 3.2. *There is exactly one very good normalization $\mathcal{N}$ of a given hypersurface.*

Proof is trivial.

Let us remark the following two facts. If $\mathcal{N}$ is a very good normalization, we may calculate τ from (3.13), the left-hand side being equal to zero. The form $h(X, Y)$ induces then a pseudoriemannian metric on M^N; it is simple to calculate its associated connection as

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = \Gamma^i_{jk} + \tfrac{1}{2}(\tilde{h}^{ir} a_{jks} - \delta^i_j \tau_k - \delta^i_k \tau_j + \tilde{h}^{ir} h_{jk} \tau_r) \,, \tag{3.17}$$

δ^i_j being the Kronecker deltas.

4. We are going to present a geometric description of the very good normalization. Let us start with a normalized hypersurface $m\colon M^N \to A^{N+1}$ given by (2.1); let $\xi_0 \in M^N$ be a fixed point in a coordinate neighborhood $(\xi) = (\xi^1, \ldots, \xi^N)$ of M^N. Be given a curve $\gamma\colon (-\varepsilon, \varepsilon) \to M^N$ such that $\gamma(0) = \xi_0$ by $\xi^i = f^i(t)$, and consider the curve $m \circ \gamma\colon (-\varepsilon, \varepsilon) \to A^{N+1}$ given by $m(t) = m(f^i(t))$; we have

$$m(t) = m(\xi_0) + t\,\frac{\mathrm{d}m(0)}{\mathrm{d}t} + \tfrac{1}{2}t^2\,\frac{\mathrm{d}^2 m(0)}{\mathrm{d}t^2} + \tfrac{1}{6}t^3\,\frac{\mathrm{d}^3 m(0)}{\mathrm{d}t^3} + \ldots \,. \tag{4.1}$$

Write

$$F^i := \frac{\mathrm{d}f^i(0)}{\mathrm{d}t}\,, \quad G^i := \frac{\mathrm{d}^2 f^i(0)}{\mathrm{d}t^2}\,, \tag{4.2}$$

and introduce the coordinates of $y \in A^{N+1}$ at $m(\xi_0)$ by

$$y = m(\xi_0) + y^i\, m_i(\xi_0) + y^{N+1}\, n(\xi_0) \,. \tag{4.3}$$

The curve $m(t)$ (4.1) is then given by

$$\begin{aligned} y^i &= y^i(t) = tF^i + \tfrac{1}{2}t^2(G^i + \Gamma^i_{rs}F^r F^s) + O(t^3)\,, \\ y^{N+1} &= y^{N+1}(t) = \tfrac{1}{2}t^2 h_{rs} F^r F^s + \\ &\quad + \tfrac{1}{6}t^3\{(\partial_p h_{rs} + h_{rs}\tau_p + h_{qs}\Gamma^q_{pr})\, F^p F^r F^s + 3h_{rs} F^r G^s\} + O(t^4)\,. \end{aligned} \tag{4.4}$$

A general hyperquadric $Q^N \subset A^{N+1}$ is given by

$$(4.5) \qquad Q(y^i, y^{N+1}) \equiv A_{ij}y^iy^j + B_iy^iy^{N+1} + C(y^{N+1})^2 + + D_iy^i + Ey^{N+1} + F = 0 .$$

We say that the curve $m(t)$ (4.4) has a contact of order k (at least) with the hyperquadric Q (4.5) at $m(\xi_0)$ if

$$(4.6) \qquad Q(y^i(t), y^{N+1}(t)) = O(t^{k+1}) .$$

Inserting (4.4) into (4.5) and looking at the terms at t^0, t^1, t^2, we easily prove

Lemma 4.1. *Each of the hyperquadrics*

$$(4.7) \qquad -\tfrac{1}{2}Eh_{ij}y^iy^j + B_iy^iy^{N+1} + C(y^{N+1})^2 + Ey^{N+1} = 0$$

has a contact of order 2 *(at least) with any curve* $m \circ \gamma$ *at the point* $m(\xi_0)$.

The hyperquadrics (4.7) are the so-called *osculating hyperquadrics* of the hypersurface $m: M^N \to A^{N+1}$ at the point $m(\xi_0)$.

Comparing further the terms at t^3, we get

Lemma 4.2. *Let* $F \in T_{\xi_0}(M^N)$, $F = F^i\, \partial/\partial\xi^i$. *Each curve* $m \circ \gamma$ *with* $\mathrm{d}\gamma(\mathrm{d}/\mathrm{d}t|_{t=0}) = F$ *has a contact of order* 3 *(at least) with the osculating quadric* (4.7) *if and only if*

$$(4.8) \qquad f_{ijk}F^iF^jF^k = 0 \quad \text{with} \quad f_{ijk} = Ea_{ijk} + B_ih_{jk} + B_jh_{ki} + B_kh_{ij} .$$

Definition 4.1. The osculating hyperquadric is said to be a *Darboux hyperquadric* if the cone (4.8) is apolar to the so-called asymptotic hypercone $h_{ij}F^iF^j = 0$.

Elementary analytic geometry implies the validity of the following

Proposition 4.1. *The Darboux hyperquadrics form a pencil*

$$(4.9) \qquad (N + 2)\, h_{ij}y^iy^j + 2\tilde{h}^{rs}a_{rsi}y^iy^{N+1} - 2(N + 2)\, y^{N+1} + \lambda(y^{N+1})^2 = 0 , \quad \lambda \in \mathbb{R} ,$$

and the (proper) centers of them are situated on a straight line l. *We have* $l = \{x = m + sn, s \in \mathbb{R}\}$ *if and only if* $\mathcal{N}$ *is a good normalization.*

This is the geometric description of the good affine normal straight line.

Be given a hypersurface m with a good normalization $\mathcal{N}$. On each normal straight line be chosen a point

$$(4.10) \qquad F = m + fn .$$

$f: M^N \to \mathbb{R}$ a function. The set of points $F(\xi)$, $\xi \in M^N$, is called a *focal set* if there is a non-vanishing tangent vector field X on M^N such that $\tan(\mathrm{D}_XF) = 0$. For $X = x^i\, \partial/\partial x^i$,

$$(4.11) \qquad \mathrm{D}_XF = \mathrm{D}_{X - fS(X)}m + \{\mathrm{D}_Xf + f\,\tau(X)\}\, n ,$$

i.e., (4.10) is a focal set if and only if there is a $X \neq 0$ such that

$$(4.12) \qquad X - f\, S(X) = 0 .$$

This means that we must have

(4.13) $$\det \|J_N - fS\| = \det \|\delta_i^j - fS_i^j\| = 0\,.$$

On the normal line $m + tn$, $t \in \mathbb{R}$, let us pass to the homogeneous coordinates $t = t_1/t_0$; the point F (4.10) is then a focus if and only if

(4.14) $$\varphi(f_0, f_1) \equiv \det \|f_0\delta_i^j - f_1 S_i^j\| = 0\,.$$

On each normal line there are, in general, N foci (some of them may coincide or be the improper point of the normal line).

Definition 4.2. The *central point* of the normal line is the point apolar to m with respect to the N-tuple of foci.

Let (s_0, s_1) be the homogeneous coordinates of the central point; the homogeneous coordinates of the point m are, of course, $\mu_0 = 1$, $\mu_1 = 0$. In general, the point apolar to the point (μ_0, μ_1) satisfies

(4.15) $$s_0 \frac{\partial\varphi(\mu_0, \mu_1)}{\partial f_0} + s_1 \frac{\partial\varphi(\mu_0, \mu_1)}{\partial f_1} = 0\,.$$

In our case, (4.15) becomes

(4.16) $$Ns_0 - \operatorname{Tr} S s_1 = 0\,.$$

They are two possibilities. In the case $\operatorname{Tr} S = 0$, the central point is the improper point of the normal line. In the case $\operatorname{Tr} S \neq 0$, we may pass to the very good normalization (see Definition 3.1), and then the central point is exactly the point $m + n$. This gives the geometrical description of the Supposition 3.1 and of the very good normalization.

We may define the *Lie hyperquadric* of our hypersurface as the Darboux hyperquadric with its center in the central point. This is in accord with the case of a hyperbolic surface in A^3; see [1], p. 223.

5. Let us, very briefly, describe the situation of a hypersurface of a space with the affine connection $(\tilde{M}^{N+1}, \tilde{\nabla})$. The proofs are similar to that of paragraph 2, and I am not going to repeat them.

Let $m\colon M^N \to \tilde{M}^{N+1}$ be an immersion; our considerations being local, let us identify M^N with $m(M^N)$. Let us choose a normalization $\mathcal{N}$ as a map $n\colon M^N \to T(\tilde{M}^{N+1})$ such that $n(\xi) \in T_\xi(M^N)$ is transversal to M^N at the point $\xi \in M^N$.

Let $\tilde{T}$ and $\tilde{R}$ be the torsion and curvature of $(\tilde{M}^{N+1}, \tilde{\nabla})$ resp. The forms

(5.1) $$\tilde{T}'_{\mathcal{N}}\colon T(M^N) \times T(M^N) \to \mathbb{R}\,, \quad \tilde{R}'_{\mathcal{N}}\colon \mathsf{X}^3 T(M^N) \to \mathbb{R}$$

be defined by

(5.2) $$\tilde{T}'_{\mathcal{N}}(X, Y)\, n = \operatorname{nor} \tilde{T}(X, Y)\,, \quad \tilde{R}'_{\mathcal{N}}(X, Y)\, Zn = \operatorname{nor} \tilde{R}(X, Y)\, Z\,;$$

here, nor V and tan V are the normal or tangential parts of $V \in T_\xi(\tilde{M}^{N+1})$ at the point $\xi \in M^N$ resp.

The fundamental equations of our hypersurface are

$$(5.3)\qquad \partial_i m = m_i\,,\quad \tilde{\nabla}_j m_i = \Gamma^k_{ij} m_k + h_{ij} n\,,\quad \tilde{\nabla}_i n = -S^j_i m_j + \tau_i n\,;$$

the functions Γ^k_{ij} induce an affine connection ∇ on M^N.

Lemma 5.1. *The integrability conditions of* (5.3) *are*

$$(5.4)\qquad T(X, Y) = \tan \tilde{T}(X, Y)\,,$$

$$(5.5)\qquad h(X, Y) - h(Y, X) = \tilde{T}'_{\mathcal{N}}(X, Y)\,,$$

$$(5.6)\qquad h(X, Z)\, S(Y) - h(X, Y)\, S(Z) + R(Y, Z)\, X = \tan \tilde{R}(Y, Z)\, X\,,$$

$$\nabla_Y h(Z, X) + h(Z, X)\,\tau(Y) - \nabla_X h(Z, Y) - h(Z, Y)\,\tau(X) + $$
$$+ h(Z, T(X, Y)) = \tilde{R}'_{\mathcal{N}}(X, Y)\, Z\,,$$

$$\nabla_X S(Y) - \tau(X)\, S(Y) - \nabla_Y S(X) + \tau(Y)\, S(X) + S(T(Y, X)) =$$
$$= \tan \tilde{R}(X, Y)\, n\,,$$

$$\nabla_X \tau(Y) + h(S(X), Y) - \nabla_Y \tau(X) - h(S(Y), X) + \tau(T(Y, X)) =$$
$$= \tilde{R}'_{\mathcal{N}}(Y, X)\, n\,.$$

Define

$$(5.7)\qquad F_3(X, Y, Z) = \nabla_Z h(X, Y) + h(X, Y)\,\tau(Z)\,.$$

Lemma 5.2. *We have*

$$(5.8)\qquad F_3(X, Y, Z) - F_3(X, Z, Y) = \tilde{R}'_{\mathcal{N}}(Y, Z)\, X + h(X, T(Z, Y))\,,$$

$$F_3(X, Y, Z) - F_3(Y, X, Z) = \tilde{T}'_{\mathcal{N}}(X, Y)\,\tau(Z) + \nabla_Z \tilde{T}'_{\mathcal{N}}(X, Y)\,.$$

Lemma 5.3. *Let A be a tangent vector field on M^N, $\alpha\colon M^N \to \mathbb{R}$ a nowhere vanishing function, and*

$$(5.9)\qquad n^* = A + \alpha n$$

a new normalization $\mathcal{N}^*$. Then we have (5.3*) *with*

$$(5.10)\qquad h^*(X, Y) = \alpha^{-1} h(X, Y)\,,$$

$$(5.11)\qquad \nabla^*_Y X = \nabla_Y X - \alpha^{-1} h(Y, X)\, A\,,$$

$$\tau^*(X) = \tau(X) + \alpha^{-1}\{h(A, X) + D_X \alpha\}\,,$$

$$S^*(X) = \alpha\, S(X) - \nabla_X A + \{\tau(X) + \alpha^{-1} h(A, X) + \alpha^{-1} D_X \alpha\}\, A\,;$$

$$(5.12)\qquad F^*_3(X, Y, Z) = \alpha^{-1} F_3(X, Y, Z) +$$
$$+ \alpha^{-2}\{h(X, Y)\, h(A, Z) + h(X, Z)\, h(A, Y) + h(Y, Z)\, h(X, A)\}\,.$$

Definition 5.1. The *symmetrizations* $\hat{h}$ and $\hat{F}_3$ be defined by

$$(5.13)\qquad \hat{h}(X, Y) = \tfrac{1}{2}\{h(X, Y) + h(Y, X)\}\,,$$

$$\hat{F}_3(X, Y, Z) = \tfrac{1}{6}\{F_3(X, Y, Z) + \ldots + F_3(X, Z, Y)\}\,.$$

Supposition 5.1. We suppose $\hat{h}$ to be regular.

Definition 5.2. The normalization $\mathcal{N}$ is called *good* if $\hat{h}$ and $\hat{F}_3$ are polar.

Proposition 5.1. *There are good normalizations. If* $\mathcal{N}$ *and* $\mathcal{N}^*$ *are good, there is a function* α: $M^N \to \mathbb{R}$ *such that* $n^* = \alpha n$.

To get a *very good* normalization, we may proceed as above. The form (2.13) is not symmetric (in general), but we may pass to its symmetrization $\hat{F}_4$.

References

[1] Affine Differentialgeometrie. Proceedings Math. Forschungsinst. Oberwolfach, 2.—8. Nov. 1986, Technische Univ. Berlin.

[2] *Nomizu K., Pinkall U.:* Cubic form theorem for affine immersions. Results in Math., Vol *13* (1988), 338—362.

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