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WEAK BASES IN MODULAR LATTICES

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A subset H of a lattice L is called *weakly independent* iff for all $h, h_1, \ldots, h_n \in H$ which satisfy $h \leqq h_1 \vee \ldots \vee h_n$ there exists an $i(1 \leqq i \leqq n)$ such that $h \leqq h_i$. A maximal weakly independent subset is called a *weak basis* of L.

In a lattice of finite length any chain is a weakly independent subset and any maximal chain is a weak basis. In a finite distributive lattice any set of join-irreducible elements is weakly independent and the set of all join-irreducibles is a weak basis. Thus the following theorem which was proved in [1] generalizes the well-known fact that in a finite distributive lattice the number of elements in a maximal chain equals the number of join-irreducible elements.

Theorem A. *Any two weak bases of a finite distributive lattice have the same number of elements.*

An example given in [1] shows that Theorem A will not be true if we change distributivity for modularity. However, as it was proved in [2] any lattice of finite length with the property that any two bases of it have the same number of elements must be modular. The aim of this paper is to present two classes of modular lattices in which Theorem A is true.

The breadth of a lattice L is the least natural number b such that for any finite $X \subseteq L$ there exists $Y \subseteq X$ with $|Y| \leqq b$ and $\bigvee X = \bigvee Y$. We shall use

Theorem B (see [4]). *Every finitely generated modular lattice of finite length and breadth at most two is finite.*

We also need the notion of c-sublattices. A sublattice L' of a lattice L is said to be a *c-sublattice* if for all $u, v \in L'$ u covers v in L' iff u covers v in L.

Theorem C (see [3]). *A finite modular lattice is distributive if and only if it contains no c-sublattices isomorphic to M_3 (M_3 is the five-element non-distributive modular lattice).*

Theorem 1. *Let L be a modular lattice of finite length and breadth at most two (or equivalently a dismantlable modular lattice of finite length,* cf. [5]). *Then for any two weak bases $H_1, H_2 \subseteq L$ we have $|H_1| = |H_2|$.*

Proof. First observe that in a lattice with no infinite chains any weak basis H

is finite. Indeed, let $a_1, a_2, \dots$ be an enumeration of elements from H such that for $i = 1, 2, \dots$ a_i is minimal in $\{a_i, a_{i+1}, \dots\}$. Then φ defined by $\varphi(a_i) = a_1 \vee \dots \vee a_i$ maps injectively $\{a_1, a_2, \dots\}$ to some chain.

Thus in view of Theorem B, the sublattice L' generated by $H_1 \cup H_2$ is finite. Since H_1 and H_2 are bases in L' too, we may suppose that L itself is finite.

Clearly, it is enough to show that the number of elements in any weak basis H is $l(L) + 1$. For distributive lattices this is Theorem A, thus we can assume that L contains a c-sublattice M isomorphic to M_3.

Let x_1, x_2 and x_3 be the pairwise incomparable elements of M and let $a = x_1 \wedge x_2 \wedge x_3$. For $i = 1, 2, 3$ choose a join-irreducible element $j_i \in L$ with $a \vee j_i = x_i$. It is striaghtforward to check that j_1, j_2 and j_3 are pairwise incomparable. Now there are three pairwise incomparable doubly irreducible elements y_1, y_2 and y_3 in L (see [6]). Since $y_1 \vee y_2 \vee y_3 = y_i \vee y_j$ for some $1 \leqq i, j \leqq 3$, one of y_1, y_2 and y_3, say y_1, is not contained in H. But then H is a weak basis also in the sublattice $L' = L \setminus \{y_1\}$. Moreover, $l(L') = l(L)$ and the assertion follows by induction on $|L|$.

Let L be a finite lattice. For any interval $[a, b]$ of length two in L let $N_{a,b}$ be a (possibly empty) set of new elements such that $N_{a,b} \cap N_{c,d} = \emptyset$ if $a \neq c$ or $b \neq d$. We define a lattice $\tilde{L}$ containing L as a c-sublattice on the set $L \cup \bigcup_{l([a,b])=2} N_{a,b}$ by adding to the Hasse diagram of L the covering relations $a \prec u$ and $u \prec b$ for any $[a, b]$ of length two in L and for any $u \in N_{a,b}$. Then we say that $\tilde{L}$ can be obtained by inserting new elements into L. Let $\mathscr{M}_0$ denote the class of modular lattices which can be obtained by inserting new elements into some finite distributive lattice.

We need the well-known

Lemma D (see [3]). *Let D be a finite distributive lattice. If for the elements $j, x_1, \dots, x_n \in D$ we have $j \in J_0(D)$ (= the set of join-irreducibles of D) and $j \leqq x_1 \vee \dots \vee x_n$ then $j \leqq x_i$ for some i, $1 \leqq i \leqq n$.*

Theorem 2. *If $L \in \mathscr{M}_0$ then for any two weak bases $H_1, H_2 \subseteq L$ we have $|H_1| = |H_2|$.*

Proof. As in the proof of Theorem 1 we have to show that any weak basis H satisfies $|H| = l(L) + 1$.

Let D be a distributive lattice with the property that L can be obtained by inserting new elements into D. We may suppose that if $[a, b]$ is an interval of length two in D and $N_{a,b} \neq \emptyset$ then $|[a, b]| = 4$. Indeed, let $|[a, b]| = 3$ and $N_{a,b} \neq \emptyset$. Now we can add a new element from $N_{a,b}$ to D. By Theorem C, one can easily see that D remains then distributive.

If L is distributive then the assertion follows from Theorem A. Suppose that L is not distributive. Now there is an interval $[a, b]$ of length two in L such that $|[a, b] \cap D| = 4$ and $N_{a,b} \neq \emptyset$. Let $[a, b] \cap D = \{a, b, x, y\}$ and let $u \in N_{a,b}$. If $u \notin H$ then H is also a basis in $L' = L \setminus \{u\}$ and the assertion follows by induction.

Let $u \in H$. For any $p \in D$ set

$$J(p) = \{j \mid j \leqq p \text{ and } j \in J_0(D)\}.$$

Then we have $J(x) = J(a) \cup \{j\}$ and $J(y) = J(a) \cup \{k\}$ for some $j, k \in J_0(D)$, moreover $j \vee k = b$. Indeed, if $j, j' \in J(x) \setminus J(a)$ then $a \vee j = a \vee j' = x$ since x covers a. By Lemma D we have $j \geqq j'$ and $j' \geqq j$, i.e. $j = j'$. To see $j \vee k = b$ recall that in a modular lattice the mapping $z \to z \vee q$ is an isomorphism between the intervals $[p \wedge q, p]$ and $[q, p \vee q]$ for any p and q. Choose $p = j \vee k$ and $q = a$. Since $p \vee q = j \vee k \vee a = x \vee y = b$, there is an element v in L such that $p \wedge q < v < p$ and $v \vee a = u$. As u is join-irreducible in L and $a < u$ we must have $v = u$. Then $j \vee k = p > v = u$, i.e. $j \vee k = b$.

We define a mapping $x \to \bar{x}$ of L to D by

$$\bar{x} = \begin{cases} \text{the unic upper cover of } x \text{ if } x \in L \setminus D\,; \\ x \text{ if } x \in D\,. \end{cases}$$

For any join $x_1 \vee \ldots \vee x_n$ in L we have either

$$x_1 \vee \ldots \vee x_n = \bar{x}_1 \vee \ldots \vee \bar{x}_n$$

or

$$x_1 \vee \ldots \vee x_n = x_i$$

for some i, $1 \leqq i \leqq n$.

Define $x \in X \subseteq L$ and $y \in Y \subseteq L$ by

$$X = \{p \in L \mid \bar{p} \geqq j \text{ and } p \ngeqq b \text{ and } p \notin N_{a,b}\}\,,$$
$$Y = \{q \in L \mid \bar{q} \geqq k \text{ and } q \ngeqq b \text{ and } q \notin N_{a,b}\}\,.$$

Note that for any $p \in X$ and for any $q \in Y$ we have $p \vee q \geqq j \vee k = b \geqq u$ and $p \ngeqq u$, $q \ngeqq u$. This implies that either $H \cap X = \emptyset$ or $H \cap Y = \emptyset$ and without loss of generality we may assume that $H \cap X = \emptyset$. First observe that $L' = L \setminus X$ is a sublattice of L.

Indeed, if $p \in N_{a,b}$ or $q \in N_{a,b}$ then either $p \vee q \in N_{a,b}$ or $p \vee q \geqq b$ and either $p \wedge q \in N_{a,b}$ or $p \wedge q \leqq a$, i.e. $p \vee q$, $p \wedge q \in L'$. If $p \notin N_{a,b}$ and $q \notin N_{a,b}$ then we have four possibilities:

1. $\bar{p} \ngeqq j$ and $\bar{q} \ngeqq k$. Then $p \vee q = p$ or $p \vee q = q$ or $p \vee q = \bar{p} \vee \bar{q}$ and in the latter case by Lemma D $p \vee q \ngeqq j$, i.e. $p \vee q \in L'$. Since $\overline{p \wedge q} \leqq \bar{p}$, $p \wedge q \in L'$ is trivial.

2. $\bar{p} \ngeqq j$ and $q \geqq b$. Then by $p \vee q \geqq b$ and by $\overline{p \wedge q} \ngeqq j$ we have $p \vee q$, $p \wedge q \in L'$.

3. $p \geqq b$ and $\bar{q} \ngeqq k$. This case is similar to case 2.

4. $p \geqq b$ and $q \geqq b$. Then $p \vee q$, $p \wedge q \geqq b$.

On the other hand $l(L') = l(L)$ as $A \cup \{u\} \cup B$ is a subset of L' where $A = \{a' \in L \mid a' \leqq a\}$ and $B = \{b' \in L' \mid b' \geqq b\}$. This implies that if for some $v \in N_{c,d}$ we have $v \in L'$ then $c, d \in L'$ holds too. Then L' can be obtained by inserting new elements into $D' = D \cap L'$. H is obviously a basis in L' and by induction we have $|H| = l(L') + 1 = l(L) + 1$.

From [5] we know that any planar lattice is dismantlable. On the other hand it can be easily shown that $\mathcal{M}_0$ too contains the class of planar modular lattices. Thus either Theorem 1 or Theorem 2 implies

Corollary. *In a planar modular lattice any two weak bases have the same number of elements.*

Remarks. 1. The example exhibited in [1] shows that Theorem 1 will not be true in general for modular lattices of breadth three. The same example shows that Theorem 2 does not remain true if we consider modular lattices which can be obtained in two steps by adding new elements to some finite distributive lattice.

2. It can be easily seen that not one of the classes of modular lattices considered in Theorem 1 and Theorem 2 contains the other one.

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