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ON THE BASE AND THE ESSENTIAL BASE IN PARABOLIC POTENTIAL THEORY

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INTRODUCTION

Several decades ago it was recognized that methods of classical potential theory can be applied to investigation of the heat equation. In particular it is possible to define the PWB-solution of the Dirichlet problem and the notion of the regular point, see [10], [2].

While the necessary and sufficient condition of the regularity of a boundary point in the form of the so called series of Wiener's type has been known in the classical case since 1924, see [29], the way to the analogous criterion in the heat case, in spite of considerable similarity with the classical case, took more than 50 years.

First sufficient conditions of the regularity of a region with continuous or differentiable boundary were proved in [23] and [21]. Necessary or sufficient conditions of the regularity were shown in [22], [17]. In 1982, Evans and Gariepy proved the heat analogue of the Wiener test, see [12]. Results from [12] generalized to parabolic equations with variable coefficients can be found in [13]. Regularity of a boundary point is very closely connected with the notion of thinness of a set at a given point, see e.g. [10]. M. Brelot proved the Wiener test of thinness in the classical potential theory in [6]. One of the main results of this paper is the test of Wiener's type of thinness in the heat potential theory, see Theorem 1.11. In our proof, results of [12] are used in an essential way. The probabilistic approach to the criterion of thinness for analytic sets is given in [25]. The paper [24] deals with a probabilistic interpretation of thinness of a set.

In modern potential theory there is a possibility of formulating a number of fundamental results in terms of the base and the essential base of a set (see Definitions 1.12 and 2.7 below). Theorem 1.13 gives necessary and sufficient conditions for a point to belong to the base of a set.

In the second part, a continuous capacity is introduced using continuous potentials, and its fundamental properties are established. Relations between capacity and continuous capacity are cleared up in Theorem 2.9 using results from [16]. Theorem 2.16 is the test of Wiener type for a point to belong to the essential base of a Borel set.

Papers [14], [11], [18], [27] deal with the so called "tusk condition" for regularity of a boundary point. Corollary 2.21 and Corollary 3.5 are "tusk conditions" for a Borel set to be semipolar at a point (see Definition 2.7) and for a point to belong to the Choquet boundary.

The results of this paper were presented with more details in [8] and announced in [9].

0. NOTATION

The set of all positive integers is denoted by N. For $n \in N$, the symbol $R^{n+1} = R^n \times R^1$ will stand for the $(n+1)$-dimensional Euclidean space. We will often write a typical point $z \in R^{n+1}$ as $z = (x, t)$, $x \in R^n$, $t \in R^1$. Let the Euclidean norm be denoted by $|\cdot|$, the set difference by $A \setminus B$, and $R^{n+1} \setminus A$ by A^c. For $A \subset R^{n+1}$, $\bar{A}$ denotes the closure of A and ∂A the boundary of A. For $z = (x, t) \in R^{n+1}$, let us define

$$F(z) = \begin{cases} (4\pi t)^{-n/2} \exp\left[-|x|^2/4t\right] & t > 0 \\ 0 & t \leqq 0 ; \end{cases}$$

this is the fundamental solution of the heat equation. Further, for $z \in R^{n+1}$ and $c > 0$ denote

$$B(z, c) = \{w \in R^{n+1};\ F(z - w) \geqq (4\pi c)^{-n/2}\} \cup \{z\} ,$$

the so called "heat ball", and for $k \in N$ put

$$B_k(z) = B(z, 2^{-k}) , \quad A_k(z) = \mathrm{cl}\,(B_k(z) \setminus B_{k+1}(z)) .$$

1. THERMAL CAPACITY, THINNESS AND BASE

For a set $E \subset R^{n+1}$, let us denote by $\mathscr{M}^+(E)$ the collection of all nonnegative Radon measures on R^{n+1} with compact support in E; the support is denoted by spt. For $\mu \in \mathscr{M}^+(R^{n+1})$ we set

$$P_\mu(z) = \int_{R^{n+1}} F(z - w)\, \mathrm{d}\mu(w) , \quad z \in R^{n+1} ;$$

P_μ is the heat potential of μ. The heat potentials are lower semicontinuous but not continuous in general. The coarsest topology making every heat potential continuous is called the fine topology for the heat equation. Topological concepts related to the fine topology will be used with the attribute fine.

1.1. Definition. Let $K \subset R^{n+1}$ be a compact set. The *capacity* (in detail: the *thermal capacity*) of K is defined as

$$\gamma(K) = \sup\{\mu(R^{n+1});\ \mu \in \mathscr{M}^+(K),\ P_\mu \leqq 1 \text{ in } R^{n+1}\} ;$$

cf. [17].

Fundamental properties of the termal capacity are summarized in the following lemma.

1.2. Lemma. *Let* K, K_j, $j \in N$, *be compact subsets of* R^{n+1}. *For* $s > 0$ *define* $s \odot K = \{(sx, s^2t); (x, t) \in K\}$. *Then*

(1) $\gamma(K) < \infty$;

(2) $\gamma(K_1 \cup K_2) \leqq \gamma(K_1) + \gamma(K_2)$ (*subadditivity of* γ);

(3) $K_1 \subset K_2$ *implies* $\gamma(K_1) \leqq \gamma(K_2)$ (*monotonicity of* γ);

(4) $\gamma(\{z\}) = 0$ *for all* $z \in R^{n+1}$;

(5) $\gamma(s \odot K) = s^n \gamma(K)$;

(6) *if* $\{K_j\}_{j=1}^{\infty}$ *is a decreasing sequence of sets with the intersection* K, *i.e.* $K_j \searrow K$, *then*

$$\lim_{j\to\infty} \gamma(K_j) = \gamma(K) .$$

Proof. See [17], pp. 85, 89.

1.3. Definion. Let $E \subset R^{n+1}$ be an arbitrary set. Then

$$\gamma_*(E) = \sup \{\gamma(K);\ K \subset E, K \text{ compact}\}$$

is called the *inner thermal capacity of* E and

$$\gamma^*(E) = \inf \{\gamma_*(G);\ G \supset E, G \text{ open}\}$$

the *outer thermal capacity of* E.

1.4. Lemma. *Let* E, E_j, $j \in N$, *be arbitrary subsets of* R^{n+1}. *Then*

(1) $0 \leqq \gamma_*(E) \leqq \gamma^*(E)$;

(2) $\gamma^*(E_1 \cup E_2) \leqq \gamma^*(E_1) + \gamma^*(E_2)$;

(3) $E_1 \subset E_2$ *implies* $\gamma_*(E_1) \leqq \gamma_*(E_2)$, $\gamma^*(E_1) \leqq \gamma^*(E_2)$;

(4) *if* $s > 0$, *then* $\gamma^*(s \odot E) = s^n \gamma^*(E)$;

(5) *if* $\{E_j\}_{j=1}^{\infty}$ *is an increasing sequence of sets with the union* E, *i.e.* $E_j \nearrow E$, *then*

$$\lim_{j\to\infty} \gamma^*(E_j) = \gamma^*(E) .$$

Proof. See [26], p. 352.

1.5. Lemma. *Let* $t \in R^1$, *let* F *be a Borel subset of* R^n, $K = \{(x, t) \in R^{n+1}; x \in F\}$, *and let* λ_n *stand for the Lebesgue measure in* R^n. *Then*

$$\gamma(K) = \lambda_n(F) .$$

Proof. See [26], p. 355.

1.6. Remark. It follows from the definition of the inner and outer capacities and from Lemma 1.2 (6) that $\gamma(K) = \gamma_*(K) = \gamma^*(K)$ whenever K is a compact subset of R^{n+1}. A subset E of R^{n+1} is said to be γ-capacitable, if $\gamma_*(E) = \gamma^*(E)$. It can be shown, see [26], p. 352, that all Borel subsets of R^{n+1} are γ-capacitable. We can extend the set function γ which is defined for compact sets only to γ-capacitable sets $E \subset R^{n+1}$ by defining $\gamma(E) = \gamma_*(E)$. In particular, we will write $\gamma(E)$ instead of $\gamma_*(E)$ and $\gamma^*(E)$ whenever $E \subset R^{n+1}$ is capacitable.

1.7. Definition. The balayage of the function identically equal to 1 on a subset E of R^{n+1} will be denoted by $\hat{R}_1^E$. For $z = (x, t) \in R^{n+1}$ and $r > 0$, let

$$C(z, r) = \{(\chi, \tau) \in R^{n+1};\ |\chi - x| \leqq r,\ -r^2 \leqq \tau - t \leqq 0\}.$$

We say that a set E is *thin* at a point $z \in R^{n+1}$ if there exists $r > 0$ such that

$$\hat{R}_1^{E \cap C(z,r)}(z) < 1.$$

It can be shown, see [10], p. 158 and p. 141, that this notion of thinness coincides with the notion usually adopted in potential theory (see e.g. [10], p. 149).

1.7. Remark. If $E \subset R^{n+1}$ is thin at a point $z \in R^{n+1}$ and $E' \subset E$, then E' is also thin at z. If the set $E \subset R^{n+1} \setminus \{z\}$ is thin at a point z, then there exists an open set $G \subset R^{n+1}$ such that $E \subset G$ and G is thin at z.

1.9. Lemma. *Let $K \subset R^{n+1}$ be a compact set. Then*

(1) $\hat{R}_1^K = 1$ *on* int K, *the interior of* K;

(2) *there exists a unique Radon measure* $\tilde{\mu} \in \mathcal{M}^+(K)$ *such that*

$$\hat{R}_1^K = P_{\tilde{\mu}} \quad \textit{and} \quad \tilde{\mu}(R^{n+1}) = \gamma(K)$$

($\tilde{\mu}$ is called the equilibrium measure for K);

(3) *if $\tilde{\mu}$ is the equilibrium measure for K, then $P_\mu \leqq P_{\tilde{\mu}}$ for all $\mu \in \mathcal{M}^+(K)$ such that $P_\mu \leqq 1$ in R^{n+1}.*

Proof. See [17], pp. 86–88.

In [12], Evans and Gariepy proved the criterion of regulatity for the heat equation. If we use the well known relation between regularity and thinness of a set at a given point, see e.g. [17], p. 94, we obtain the following Theorem 1.10 a generalization of which to arbitrary sets is contained in Theorem 1.11.

1.10. Theorem. *Let $F \subset R^{n+1}$ be a closed set and $z \in R^{n+1}$. Then F is thin at z if and only if the series*

$$\sum_{k=1}^{\infty} 2^{nk/2} \gamma(F \cap A_k(z))$$

is convergent.

Proof. See [12], p. 295 and p. 298.

1.11. Theorem. *Let $E \subset R^{n+1}$ be an arbitrary set and $z \in R^{n+1}$. Then E is thin at z if and only if the series*

$$\sum_{k=1}^{\infty} 2^{nk/2} \gamma^*(E \cap A_k(z)) \tag{1.1}$$

is convergent.

Proof. Assume that the series in (1.1) is convergent. Choose $\varepsilon_k > 0$, $k \in N$, such that the series

$$(1.2) \qquad \sum_{k=1}^{\infty} 2^{nk/2} \varepsilon_k$$

is convergent. Let $G_k \subset R^{n+1}$, $k \in N$, be open sets such that $E \cap A_k(z) \subset G_k$ and $\gamma(G_k) \leqq \gamma(E \cap A_k(z)) + \varepsilon_k$. It follows from (1.1) and (1.2) that the series

$$(1.3) \qquad \sum_{k=1}^{\infty} 2^{nk/2} \gamma(G_k)$$

is convergent. Let $G = (\bigcup_{k=1}^{\infty} (G_k \cap A_k(z))) \cap B_1^c(z)$. Clearly G is a Borel set and $E \subset G$. We are going to show that the set G is thin at z, which in turn implies that E is thin at z. To this end, we shall show that there exists $r_0 > 0$ such that

$$\hat{R}^{G \cap C_{r_0}}(z) < 1 ;$$

here C_r is used instead of $C(z, r)$. Let us choose $0 < \eta < 1$ and show that there exists $r_0 > 0$ such that $\hat{R}_1^{K \cap C_{r_0}}(z) < \eta$ whenever K is a compact subset of G. Then $\hat{R}_1^{G \cap C_{r_0}}(z) < 1$ according to [10], p. 132, because $\hat{R}_1^{G \cap C_{r_0}} = \sup \{\hat{R}_1^{K \cap C_{r_0}}; K \subset G, K$ compact$\}$. So let $0 < \eta < 1$, let K be an arbitrary compact subset of G and $r > 0$ arbitrary. Putting $D_k(r) = A_k(z) \cap K \cap C_r$, $k \in N$, $D_0 = \mathrm{cl}\,(K \cap C_r) \setminus \bigcup_{k=1}^{\infty} D_k(r)$), we get

$$(1.4) \qquad K \cap C_r \subset \bigcup_{k=0}^{\infty} D_k(r) \subset C_r \,.$$

Let $\mu \in \mathcal{M}^+(K \cap C_r)$ be the equilibrium measure for $K \cap C_r$ (see Lemma 1.9 (2)), ν_k the restriction of the measure μ to the set $D_k(r)$, $k \in N \cup \{0\}$, and $\nu'_k \in \mathcal{M}(D_k(r))$ the equilibrium measure for $D_k(r)$, $k \in N \cup \{0\}$. According to Lemma 1.9 (3), (2) $P_{\nu_k} \leqq$ $\leqq P_{\nu'_k}$, $k \in N \cup \{0\}$, and

$$\hat{R}_1^{K \cap C_r}(w) = P_{\tilde{\mu}}(w) = \int_{K \cap C_r} F(w - v) \, \mathrm{d}\tilde{\mu}(v) \,.$$

Now (1.4) yields

$$(1.5) \qquad \hat{R}_1^{K \cap C_r}(w) \leqq \sum_{k=0}^{\infty} \int_{D_k(r)} F(w - v) \, \mathrm{d}\tilde{\mu}(v) \leqq \sum_{k=0}^{\infty} P_{\nu'_k}(w) \,.$$

Since $D_k(r) \subset A_k(z)$ for $k \in N$, we have

$$F(z - v) \leqq \left(\frac{2^k}{2\pi}\right)^{n/2}$$

for $v \in D_k(r)$. The same inequality holds for $k = 0$ because $D_0(r) \subset (\mathrm{int}\, B_1(z))^c$. Consequently

$$P_{\nu'_k}(z) \leqq \left(\frac{2^k}{2\pi}\right)^{n/2} \nu'_k(D_k(r)) \,, \quad k \in N \cup \{0\} \,,$$

and we obtain from (1.5)

$$\hat{R}_1^{K \cap C_r}(z) \leqq \left(\frac{1}{2\pi}\right)^{n/2} \sum_{k=0}^{\infty} 2^{nk/2} \, v_k'(D_k(r)) \, .$$

According to Lemma 1.9 (2) $v_k'(D_k(r)) = \gamma(D_k(r))$, and so

$$\hat{R}_1^{K \cap C_r}(z) \leqq \left(\frac{1}{2\pi}\right)^{n/2} \left(\gamma(C_r) + \sum_{k=1}^{\infty} 2^{nk/2} \, \gamma(K \cap A_k(z) \cap C_r)\right) .$$

Since $K \subset G$ and the inclusion

$$G \cap A_k(z) \subset G_{k-1} \cup G_k \cup G_{k+1} \cup \{z\}$$

holds for $k \in N$ (with $G_0 = \emptyset$), it follows from monotonicity and subaditivity of γ that

$$\gamma(K \cap A_k(z) \cap C_r) \leqq \gamma(G_{k-1}) + \gamma(G_k) + \gamma(G_{k+1}) \, , \quad k \in N \, .$$

Since $\gamma(C_r) \to 0$ for $r \to 0^+$ and the series (1.3) converges, we easily establish the existence of $r_0 > 0$ such that

$$\hat{R}_1^{K \cap C_{r_0}}(z) < \eta < 1$$

whenever K is a compact subset of G. The first part of the proof is complete.

Suppose now that the set E is thin at z. We can assume that $z \notin E$. Let $G \supset E$ be an open set thin at z. Let ε_k be strictly positive numbers satisfying (1.2). Since $G \cap A_k(z)$ is a K_σ – set there exist compact sets $K_k \subset G \cap A_k(z)$ such that

$$(1.6) \qquad \gamma(G \cap A_k(z)) \leqq \gamma(K_k) + \varepsilon_k \, .$$

Obviously, the set $K = \bigcup_{k=1}^{\infty} K_k \cup \{z\}$ is compact and $K \subset G \cup \{z\}$. Consequently, the set K is thin at z. According to Theorem 1.10, the series

$$(1.7) \qquad \sum_{k=1}^{\infty} 2^{nk/2} \gamma(K \cap A_k(z))$$

is convergent. From $E \subset G$, from the inequality (1.6) and from the monotonicity of γ it follows that

$$\gamma^*(E \cap A_k(z)) \leqq \gamma(K_k) + \varepsilon_k \leqq \gamma(K \cap A_k(z)) + \varepsilon_k \, , \quad k \in N \, .$$

The last inequality, the convergence of the series in (1.7) and (1.2) imply that the series in (1.1) is convergent.

The proof of Theorem 1.10 is complete.

1.12. Definition. Let E be an arbitrary subset of R^{n+1}. The set $b(E)$ of all points $z \in R^{n+1}$ such that E is not thin at z will be called the *base of E*.

1.13. Theorem. *For an arbitrary set $E \subset R^{n+1}$, the following conditions are equivalent:*

(1) $\qquad z \in b(E)$;

(2) $$\int_0^1 \gamma^*(E \cap B(z, c))/c^{n/2+1}\,\mathrm{d}c = \infty\,;$$

(3) $$\sum_{k=1}^{\infty} 2^{nk/2}\,\gamma^*(E \cap B_k(z)) = \infty\,;$$

(4) $$\sum_{k=1}^{\infty} 2^{nk/2}\,\gamma^*(E \cap A_k(z)) = \infty\,.$$

Proof. (1) eq. (4). This follows from Theorem 1.11 and Definition 1.12.

(4) implies (3). Since $E \cap A_k(z) \subset E \cap B_k(z)$, according to Lemma 1.4 (3) we have $\gamma^*(E \cap A_k(z)) \leqq \gamma^*(E \cap B_k(z))$; this implies (3).

(3) implies (4). Since $B_k(z) \subset A_k(z) \cup B_{k+1}(z)$, according to Lemma 1.4 (2), (3) we have

$$\gamma^*(E \cap B_k(z)) \leqq \gamma^*(E \cap A_k(z)) + \gamma^*(E \cap B_{k+1}(z))\,.$$

Multiplying this inequality by $2^{nk/2}$ and summing from $k = 1$ to M we easily obtain

$$(1 - 2^{-n/2})\sum_{k=2}^{M} 2^{nk/2}\,\gamma^*(E \cap B_k(z)) + 2^{n/2}\,\gamma^*(E \cap B_1(z)) \leqq$$
$$\leqq \sum_{k=1}^{M} 2^{nk/2}\,\gamma^*(E \cap A_k(z)) + 2^{Mn/2}\,\gamma^*(E \cap B_{M+1}(z))\,.$$

Further, $\gamma^*(E \cap B_1(z)) < \infty$. There is c such that

$$2^{Mn/2}\,\gamma^*(E \cap B_{M+1}(z)) \leqq c\,.$$

This and the previous relations imply (4).

(4) eq. (2). For $k \in N \cup \{0\}$,

$$2^{(k+1)n/2}(1 - 2^{-n/2})\,\gamma^*(E \cap B_{k+1}(z)) \leqq$$
$$\leqq \int_{2^{kn/2}}^{2^{(k+1)n/2}} \gamma^*(E \cap B(z, t^{-2/n}))\,\mathrm{d}t \leqq 2^{nk/2}(2^{n/2} - 1)\,\gamma^*(E \cap B_k(z))\,.$$

Summing from $k = 1$ to M we conclude that the series in (3) is divergent if and only if

(1.8) $$\int_1^{\infty} \gamma^*(E \cap B(z, t^{-2/n}))\,\mathrm{d}t = \infty\,.$$

The change of variables $c = t^{-2/n}$ transforms the integral in (1.8) to the integral in (2). Hence the theorem is proved.

1.14. Corollary. *Let $E \subset R^{n+1}$ be an arbitrary set and $z \in R^{n+1}$. If E is thin at z, then*

$$\lim_{t \to 0+} \frac{\gamma^*(E \cap B(z, t))}{\gamma^*(B(z, t))} = 0\,.$$

Proof. Since E is thin at z we obtain from the proof of Theorem 1.13 that the integral

$$\int_1^{\infty} \gamma^*(E \cap B(z, t^{-2/n}))\,\mathrm{d}t$$

is convergent. Hence

$$\lim_{t \to 0+} t\,.\,\gamma^*(E \cap B(z, t^{-2/n})) = 0\,,$$

i.e.

$$\lim_{c \to 0+} \frac{\gamma^*(E \cap B(z, c))}{c^{n/2}} = 0 .$$

The assertion is then obtained by virtue of Lemma 1.4 (4) and from $\gamma(B(z, 1)) > 0$.

1.15. Remark. The so called "tusk condition" from [11], [14] and [18] can be then deduced using Corollary 1.14.

2. CONTINUOUS THERMAL CAPACITY AND ESSENTIAL BASE

2.1. Definition. Let K be a compact set. The *continuous capacity* (in detail: the *continuous thermal capacity*) of K is defined as

$$\alpha(K) = \sup \{\mu(R^{n+1});\ \mu \in \mathcal{M}^+(K),\ P_\mu \leqq 1 \text{ and continuous in } R^{n+1}\} ;$$

cf. [1]. Obviously, $\alpha(K) \leqq \gamma(K)$ for every compact subset K of R^{n+1}.

For an arbitrary subset E of R^{n+1} the inner continuous capacity $\alpha_*(E)$ and the outer continuous capacity $\alpha^*(E)$, respectively, are defined in a similar way as $\gamma_*(E)$ and $\gamma^*(E)$.

Let $K = [0, 1]^n \times \{0\}$ and $K_j \subset R^{n+1}$, $j \in N$, be compact sets such that $K_{j+1} \subset \subset \operatorname{int} K_j$, $j \in N$, and $K = \bigcap_{j=1}^{\infty} K_j$. Then $\alpha(K) = 0$ and $\alpha(K_j) \geqq 1$; see Theorem 2.9.

Consequently, the continuous thermal capacity is not the Choquet capacity in the sense of [7]. Further, $\alpha_*(K) < \alpha^*(K)$ for $K = [0, 1]^n \times \{0\}$. (This follows immediately from Definition 2.7 and Theorem 2.9.) Hence we do not define the α-capacitable sets. Of course, $\alpha_*(K) = \alpha(K)$ whenever $K \subset R^{n+1}$ is a compact set.

Proofs of the following lemmas are left to the reader.

2.2. Lemma. *Let K, K_1, K_2 be compact subsets of R^{n+1}. Then*

(1) $\alpha(K) < \infty$;

(2) $\alpha(K_1 \cup K_2) \leqq \alpha(K_1) + \alpha(K_2)$ (*subadditivity of* α);

(3) $K_1 \subset K_2$ *implies* $\alpha(K_1) \leqq \alpha(K_2)$ (*monotonicity of* α),

(4) $\alpha(\{z\}) = 0$ *for all* $z \in R^{n+1}$;

(5) *if* $s > 0$, *then* $\alpha(s \odot K) = s^n \alpha(K)$.

2.3. Lemma. *Let E, E_1, E_2 be arbitrary subsets of R^{n+1}. Then*

(1) $0 \leqq \alpha_*(E) \leqq \alpha^*(E)$;

(2) $\alpha^*(E_1 \cup E_2) \leqq \alpha^*(E_1) + \alpha^*(E_2)$;

(3) $E_1 \subset E_2$ *implies* $\alpha_*(E_1) \leqq \alpha_*(E_2)$, $\alpha^*(E_1) \leqq \alpha^*(E_2)$;

(4) *if* $s > 0$, *then* $\alpha_*(s \odot E) = s^n \alpha_*(E)$, $\alpha^*(s \odot E) = s^n \alpha^*(E)$.

2.4. Definition. A subset T of R^{n+1} is called *totally thin* if it is thin at every point $z \in R^{n+1}$, i.e. if $b(T) = 0$. A subset S of R^{n+1} is called *semipolar* if it is a countable union of totally thin sets.

Obviously, every subset of a semipolar set is semipolar and every countable union of semipolar sets is semipolar.

2.5. Lemma. *Let S be a Borel subset of R^{n+1}. Then S is semipolar if and only if $\alpha_*(S) = 0$.*

Proof. Let S be a semipolar set and $K \subset S$ an arbitrary compact set. According to [19], p. 121, $\alpha(K) = 0$, consequently $\alpha_*(S) = 0$. If S is not semipolar, then S contains a nonsemipolar compact set K; see [16], p. 498. According to [19], p. 121, $\alpha(K) > 0$ and so $\alpha_*(S) > 0$.

2.6. Remark. It follows from Definition 2.1 and the proof of Lemma 2.5 that $\alpha_*(S) = 0$ for every semipolar set. The converse implication is not true as shown by the following example. We say that a subset A of R^1 is of Bernstein's type if the intersection of both A and A^c with every closed uncountable set is nonempty. Let T be a set of Bernstein's type (see [20] for the existence) and $S_1 = R^n \times T$, $S_2 = R^n \times T^c$. Let K be an arbitrary compact subset of S_1. Then $K \subset R^n \times L$ for a suitable countable set $L \subset T$. But the set $R^n \times L$ is semipolar and by Lemma 2.5 and the monotonicity of α we have $\alpha(K) = 0$. Consequently, $\alpha_*(S_1) = 0$. In a similar way, $\alpha_*(S_2) = 0$. Since $S_1 \cup S_2 = R^{n+1}$, at least one of the sets S_i, $i = 1, 2$, is not semipolar. Consequently, there exists a nonsemipolar subset A of R^{n+1} such that $\alpha_*(A) = 0$. According to Lemma 2.5, A cannot be a Borel set.

2.7. Definition. Let E be an arbitrary subset of R^{n+1} and $z \in R^{n+1}$. Then E is said to be *semipolar* at z if there exists a fine neighborhood V of z such that the set $E \cap V$ is semipolar. Let $\beta(E)$ be the set of all points $z \in R^{n+1}$ such that E is not semipolar at z. The set $\beta(E)$ is called the *essential base* of E.

(In [16], the essential base is denoted by ϱ and called the quasibase. It can be shown, see [3], p. 184, that $\beta(E)$ is a G_δ-set.)

2.8. Theorem. *Let B be a Borel subset of R^{n+1}. Then*

$$\hat{R}_1^{\beta(B)} = \sup \{P_\mu; \mu \in \mathcal{M}^+(R^{n+1}),\ P_\mu \leqq 1 \text{ and continuous in } R^{n+1},\ \operatorname{spt} \mu \subset B\}.$$

Proof. See [16], p. 502.

2.9. Theorem. *Let B be a Borel subset of R^{n+1}. Then*

$$\alpha_*(B) = \gamma(\beta(B)).$$

Proof. We will use the following notation. If $E \subset R^{n+1}$, then $^*E = \{(x, t) \in R^{n+1}; (x, -t) \in E\}$. If $E \subset R^{n+1}$ is a Borel set and $\nu \in \mathcal{M}^+(R^{n+1})$, then we define $^*\nu(E) = \nu(^*E)$. Clearly, $^*(^*\nu) = \nu$, $^*\nu \in \mathcal{M}^+(R^{n+1})$ and

$$\int_{R^{n+1}} F(z - w)\, d\nu(w) = \int_{R^{n+1}} F(w - z)\, d(^*\nu)(w). \tag{2.1}$$

We first assume that B is a bounded Borel set. From [10], p. 133 and p. 127, and from [28], p. 279 it follows that there exists $\mu \in \mathcal{M}^+(\operatorname{cl} \beta(B))$ such that $\hat{R}_1^{\beta(B)} = P_\mu$.

Let $L \subset R^{n+1}$ be a compact set such that

(2.2) $\quad \operatorname{cl} \beta(B) \subset \operatorname{int} L .$

According to Lemma 1.9 (1), (2) there exists $\nu \in \mathcal{M}^+(L)$ such that $\hat{R}_1^L = P_\nu$ and $P_\nu = 1$ on int L. Consequently,

$$\mu(R^{n+1}) = \int_{R^{n+1}}(\int_{R^{n+1}} F(z - w) \, d\nu(w)) \, d\mu(z) .$$

Applying Fubini's theorem and the relation (2.1) we obtain

(2.3) $\quad \mu(R^{n+1}) = \int_{R^{n+1}}(\int_{R^{n+1}} F(w - z) \, d\mu(z)) \, d(*\nu)(w) .$

Since $P_\mu = \hat{R}_1^{\beta(B)}$ and $\{P_{\mu'}; \mu' \in \mathcal{M}^+(R^{n+1}), P_{\mu'} \leqq 1$ and continuous in R^{n+1}, spt $\mu' \subset B\}$ is an upper directed family of continuous functions, see e.g. [10], p. 40, then according to Deny-Cartan's lemma, see e.g. [7], and to Theorem 2.8 we have

$$\mu(R^{n+1}) = \sup \{\int_{R^{n+1}} P_{\mu'} \, d(*\nu); \mu' \in \mathcal{M}^+(R^{n+1}), P_{\mu'} \leqq 1 \text{ and continuous in } R^{n+1}, \operatorname{spt} \mu' \subset B\} .$$

This together with (2.1) and (2.2) and an application of Fubini's theorem yields

$$\mu(R^{n+1}) = \sup \{\mu'(R^{n+1}); \mu' \in \mathcal{M}^+(R^{n+1}), P_{\mu'} \leqq 1 \text{ and continuous in } R^{n+1}, \operatorname{spt} \mu' \subset B\} .$$

Consequently,

(2.4) $\quad \mu(R^{n+1}) = \alpha_*(B) .$

According to [10], p. 132, $\hat{R}_1^{\beta(B)} = \sup \{\hat{R}_1^K; K \subset \beta(B), K \text{ compact}\}$. Using (2.3) and $\hat{R}_1^{\beta(B)} = P_\mu$ we get

$$\mu(R^{n+1}) = \int_{R^{n+1}} \sup \{\hat{R}_1^K; K \subset \beta(B), K \text{ compact}\} \, d(*\nu) .$$

Since $\{\hat{R}_1^K; K \subset \beta(B), K \text{ compact}\}$ is obviously an upper directed family of lower semicontinuous functions we ave according to Deny-Cartan's lemma, see [7],

$$\mu(R^{n+1}) = \sup \{\int_{R^{n+1}} \hat{R}_1^K \, d(*\nu); K \subset \beta(B), K \text{ compact}\} .$$

Lemma 1.9 (2) implies the existence of a Radon measure $\mu_K \in \mathcal{M}^+(K)$ such that $\hat{R}_1^K = P_{\mu_K}$ and

(2.5) $\quad \gamma(K) = \mu_K(R^{n+1}) .$

Fubini's theorem and (2.1) give

$$\mu(R^{n+1}) = \sup \{\int_{R^{n+1}} P_\nu \, d\mu_K; K \subset \beta(B), K \text{ compact}\} .$$

Hence using the inclusion (2.2), the equality $P_\nu = 1$ on int L, the relation (2.5) and the definition of γ_*, we obtain $\mu(R^{n+1}) = \gamma(\beta(B))$. This together with (2.4) implies the desired equality.

Now let B be an arbitrary Borel set. Let $U_k \subset R^{n+1}$, $k \in N$, be bounded open sets such that $U_k \nearrow R^{n+1}$. Since $\beta(B) \cap U_k \subset \beta(B \cap U_k)$, $k \in N$, the above proved equality and the monotonicity of α_* and γ imply

$$\alpha_*(B) \geqq \gamma(\beta(B) \cap U_k), \quad k \in N.$$

Since $\beta(B) \cap U_k \nearrow \beta(B)$, Lemma 1.4 (5) yields

$$\lim_{k \to \infty} \gamma(\beta(B) \cap U_k) = \gamma(\beta(B)),$$

and so

$$(2.6) \qquad \alpha_*(B) \geqq \gamma(\beta(B)).$$

Let $K \subset B$ be an arbitrary compact set. Since $\alpha_*(K) = \gamma(\beta(K))$ and $\beta(K) \subset \beta(B)$, the monotonicity of γ gives $\alpha(K) \leqq \gamma(\beta(B))$. Consequently,

$$(2.7) \qquad \alpha_*(B) \leqq \gamma(\beta(B)).$$

The inequalities (2.6) and (2.7) complete the proof.

2.10. Remark. Let S_1 and S_2 be as in Remark 2.6. Then $\alpha_*(S_1) = \alpha_*(S_2) = 0$ and $R^{n+1} = \beta(R^{n+1}) = \beta(S_1) \cup \beta(S_2)$. Consequently,

$$\infty = \gamma(R^{n+1}) \leqq \gamma(\beta(S_1)) + \gamma(\beta(S_2)).$$

This implies that there exists a set $B \subset R^{n+1}$ such that $\alpha_*(B) \neq \gamma(\beta(B))$. Theorem 2.9 fails for arbitrary sets.

2.11. Lemma. *Let B be a Borel subset of R^{n+1}. Then there exists a Borel semipolar set $S \subset B$ such that*

$$\alpha_*(B) = \gamma(B \setminus S).$$

Proof. Obviously, $S = B \setminus \beta(B)$ is a Borel semipolar set. Further, $B \setminus S \subset \beta(B)$. From the monotonicity of γ and from Theorem 2.9 we obtain $\gamma(B \setminus S) \leqq \alpha_*(B)$. According to [5], p. 296, $\beta(B \setminus S) = \beta(B)$. This, Theorem 2.9 and the relation betweeen α_* and γ_* imply $\alpha_*(B) \leqq \gamma(B \setminus S)$.

2.12. Remark. Given a compact set $K \subset R^{n+1}$, let us denote $\alpha_1(K) = \gamma(\beta(K))$ and

$$(2.8) \qquad \alpha_2(K) = \inf\{\gamma_*(K \setminus S);\ S \subset R^{n+1},\ S \text{ semipolar}\}.$$

Then $\alpha_1(K) = \alpha_2(K) = \alpha(K)$ whenever K is a compact subset of R^{n+1}. The equalities can be easily deduced from Theorem 2.9, from Lemma 2.11 and from the fact that every semipolar set is contained in a Borel semipolar set, see e.g. [5], p. 282. Lemma 2.11 says that the right hand side of (2.8) attains the minimum.

2.13. Lemma. *Let U be an arbitrary subset of R^{n+1} and let $z \in R^{n+1}$. Then U is a fine neighborhood of z if and only if $z \in U$ and U^c is thin at z.*

Proof. See [10], p. 152.

2.14. Theorem. *Let B be a Borel set and let $z \in R^{n+1}$. The set B is semipolar at z*

if and only if the series

$$(2.9) \qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \alpha_*(B \cap A_k(z))$$

is convergent. If B is an arbitrary set semipolar at z, then the series in (2.9) is convergent.

Proof. Assume that B is a Borel set and the series in (2.9) is convergent. For every $k \in N$ let S_k be a Borel semipolar set such that $S_k \subset B \cap A_k(z)$ and

$$(2.10) \qquad \gamma((B \cap A_k(z)) \setminus S_k) = \alpha_*(B \cap A_k(z))\,;$$

see Lemma 2.11. Consequently, the set $S = \{z\} \cup \bigcup_{k=1}^{\infty} S_k$ is semipolar. For $k \in N$ we have $(B \setminus S) \cap A_k(z) \subset (B \cap A_k(z)) \setminus S_k$. Thus (2.10) and the monotonicity of γ yield

$$\gamma((B \setminus S) \cap A_k(z)) \leqq \alpha_*(B \cap A_k(z))\,.$$

Since the series in (2.9) is convergent, we obtain from the above relation that the series

$$\sum_{k=1}^{\infty} 2^{nk/2}\, \gamma((B \setminus S) \cap A_k(z))$$

is convergent. According to Theorem 1.11, the set $B \setminus S$ is thin at z, hence $V = (B \setminus S)^c$ is a fine neighborhood of the point z (see Lemma 2.13). Since the set $V \cap B$, being a subset of S, is semipolar, the set B is semipolar at z by Definition 2.7.

Let B be a Borel set semipolar at z. Then there exists a fine neighborhood V of the point z such that the set $V \cap B$ is semipolar. Since z has a fundamental system of fine neighborhoods which are compact in the Euclidean topology we can assume that V is compact. Since $B \cap A_k(z)$ and $(B \setminus V) \cap A_k(z)$ differ for a semipolar set, according to Theorem 2.9 we have

$$(2.11) \qquad \alpha_*(B \cap A_k(z)) = \alpha_*((B \setminus V) \cap A_k(z))\,.$$

As V is a fine neighborhood of the point z, the series

$$(2.12) \qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \gamma(V^c \cap A_k(z))$$

is convergent by Theorem 1.11 and Lemma 2.13. Using the equality (2.11), the relation between α_* and γ_*, the inclusion $(B \setminus V) \cap A_k(z) \subset V^c \cap A_k(z)$ and the monotonicity of γ_* we obtain

$$\alpha_*(B \cap A_k(z)) \leqq \gamma(V^c \cap A_k(z))\,.$$

This and the convergence of the series in (2.12) imply that the series (2.9) is convergent.

Now let B be an arbitrary set semipolar at z. From Definition 2.7 and from [5], p. 285, it follows that there exists a Borel set B' such that $B \subset B'$ and B' is semipolar at z. Applying the assertion proved above for the Borel set B' and using the monotonicity of α_* we obtain the convergence of the series in (2.9).

The proof of Theorem 2.14 is complete.

2.15. Remark. Let $A \subset R^{n+1}$ be a nonsemipolar set such that $\alpha_*(A) = 0$, see Remark 2.6. Since A is nonsemipolar, $\beta(A) \neq \emptyset$ according to [5], p. 296. Consequently, there exists a point $z \in R^{n+1}$ such that A is not semipolar at the point z. As $\alpha_*(A) = 0$, the series in (2.9) (for A instead of B) is convergent. Consequently, the assumption in Theorem 2.14 that B is a Borel set is essential.

2.16. Theorem. *For an arbitrary Borel set B, the following conditions are equivalent:*

(1) $\qquad z \in \beta(B)$;

(2) $\qquad \int_0^1 \alpha_*(B \cap B(z, c))/c^{n/2+1}\,\mathrm{d}c = \infty$;

(3) $\qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \alpha_*(B \cap B_k(z)) = \infty$;

(4) $\qquad \sum_{k=1}^{\infty} 2^{nk/2}\, \alpha_*(B \cap A_k(z)) = \infty$.

Proof. The assertions are proved using Theorem 2.14 in an analogous way as in the proof of Theorem 1.13.

2.17. Corollary. *Let $B \subset R^{n+1}$ be a set and $z \in R^{n+1}$. If B is semipolar at z, then*

$$\lim_{t \to 0+} \frac{\alpha_*(B \cap B(z, t))}{\alpha_*(B(z, t))} = 0 .$$

Proof. In a similar way as in the proof of Corollary 1.14 we obtain from Theorem 2.16 and from Theorem 2.14

$$\lim_{t \to 0+} \frac{\alpha_*(B \cap B(z, t))}{t^{n/2}} = 0 .$$

Since the set $B(z, 1)$ is not semipolar, $\alpha_*(B(z, 1)) > 0$ according to Lemma 2.5. The assertion follows from Lemma 2.3 (4).

2.18. Definition. Let E be an arbitrary subset of R^{n+1} and let $z = (x, t) \in R^{n+1}$. We say that E *lies parabolically below* z provided there is $b > 0$ such that

$$\tau - t < -b|\chi - x|^2$$

for any $(\chi, \tau) \in E$. For $c > 0$ and $z = (x, t) \in R^{n+1}$ put

$$D(z, c) = \{(\chi, \tau) \in R^{n+1};\ t - c \leqq \tau \leqq t\} .$$

2.19. Corollary. *Let B be a subset of R^{n+1} and let $z \in R^{n+1}$. If B lies parabolically* *below z and B is semipolar at z, then*

$$\lim_{c\to 0^+} \frac{\alpha_*(B \cap D(z, c))}{c^{n/2}} = 0 .$$

Proof. Since B lies parabolically below $z = (x, t)$, there exists $b > 0$ such that $(\tau - t) < -b|\chi - x|^2$ for any $(\chi, \tau) \in B$. We put $k = \exp(1/nb)$. An easy calculation shows that $B \cap D(z, c) \subset B \cap B(z, k \cdot c)$. This and Corollary 2.17 give the assertion.

2.20. Definition. Let T be a subset of R^1 and let $t \in R^1$. The point t is said to be a *condensation point of* T if the set $]t - \varepsilon, t + \varepsilon[\cap T$ is uncountable whenever $\varepsilon > 0$.

2.21. Corollary. *Let* B_0 *be an arbitrary subset of* R^1, $T \subset]0, \infty[$, *let* 0 *be a condensation point of* T *and*

$$B = \{(tx, -t^2) \in R^{n+1};\ x \in B_0,\ t \in T\} .$$

If B *is semipolar at* $(0, 0)$, *then* $\lambda_n(B_0) = 0$.

Proof. We first assume that B_0 is a bounded set and B is semipolar at $z = (0, 0)$. Since 0 is a condensation point of T, there exists a decreasing sequence of positive numbers $\{c_j\}_{j=1}^{\infty}$ such that

$$(2.14) \qquad \lim_{j\to\infty} c_j = 0$$

and for every $j \in N$ the set $T \cap]c_j/2, c_j[$ is uncountable. It follows from Definition 2.7 and from [5], p. 285 that there exists a Borel set $\tilde{B}$ such that $B \subset \tilde{B}$ and $\tilde{B}$ is semipolar at z. Since B_0 is bounded, we can assume that $\tilde{B}$ lies parabolically below z. For every $j \in N$ let $S^j \subset R^{n+1}$ be a semipolar Borel set such that

$$(2.15) \qquad \alpha_*(\tilde{B} \cap D(z, c_j^2)) = \gamma(\tilde{B} \cap D(z, c_j^2) \setminus S^j) ;$$

see Lemma 2.9. For $M \subset R^{n+1}$ and $t \in R^1$ we define

$$(M)_t = \{(x, -t) \in R^{n+1};\ (x, -t) \in M\} .$$

Putting $S = \bigcup_{j=1}^{\infty} S^j$, we get a semipolar Borel set and $S = \bigcup \{(S)_r;\ r \in R^1\}$. According to [15] and Lemma 1.5, the set $P = \{p \in R^1;\ \lambda_n((S)_{p^2}) > 0\}$ is countable. We put $T' = T \setminus P$. For every $j \in N$ there exists $d_j \in]c_j/2, c_j[\cap T'$ such that

$$(2.16) \qquad \lambda_n((\tilde{B} \setminus S^j)_{d^2_j}) = \lambda_n((\tilde{B})_{d^2_j}) .$$

The monotonicity of γ, the relations (2.15) and (2.16) and Lemma 1.5 imply

$$\alpha_*(\tilde{B} \cap D(z, c_j^2)) \geqq \lambda_n((\tilde{B})_{d^2_j}) .$$

Denoting by λ_n^* the outer Lebesgue measure, we have from the inclusions $B \subset \tilde{B}$ that

$$\lambda_n((\tilde{B})_{d^2_j}) \geqq d_j^n\, \lambda_n^*(B_0) .$$

Consequently, $\alpha_*(\tilde{B} \cap D(z, d_j^2)) \geqq d_j^n\, \lambda_n^*(B_0)$. Since $d_j \in]c_j/2, c_j[$, we obtain

$$(2.17) \qquad \frac{\alpha_*(\tilde{B} \cap D(z, d_j^2))}{d_j^n} \geqq \left(\frac{1}{2}\right)^n \lambda_n^*(B_0) .$$

As the set $\tilde{B}$ lies parabolically below z and is semipolar at z, Corollary 2.19 and the relations (2.14) and (2.17) imply that $\lambda_n^*(B_0) = 0$ and hence $\lambda_n(B_0) = 0$.

Now let B_0 be an arbitrary set. Then $B_0 = \bigcup_{k=1}^{\infty} B_{0k}$, where B_{0k}, $k \in N$, are bounded sets. Then by the first part of the proof $\lambda_n(B_{0k}) = 0$ for any $k \in N$. Consequently, $\lambda_n(B_0) = 0$.

2.22. Remark. Suppose that $B_0 = R^n$, $T \subset \,]0, \infty[$ and 0 is not a condensation point of T, i.e. there exists $\varepsilon > 0$ such that $T_\varepsilon = T \cap \,]0, \varepsilon[$ is at most countable. The set $B_\varepsilon = \{(tx, -t^2) \in R^{n+1};\ x \in B_0,\ t \in T_\varepsilon\}$ is semipolar. According to [5], p. 296, $\beta(B_\varepsilon) = \emptyset$. Consequently, B is semipolar at the point $(0, 0)$, while $\lambda_n(B_0) = \infty$. The assumption that 0 is a condensation point of T cannot be omitted in Corollary 2.21.

3. THE CHOQUET BOUNDARY IN PARABOLIC POTENTIAL THEORY

3.1. Definition. Let X be a metrizable compact topological space and let $\mathscr{C}(X)$ be the Banach space of continuous functions on X. Suppose that P is a closed linear subspace of $\mathscr{C}(X)$ which separates points of X and contains the constant functions. For every $x \in X$ the symbol $\mathscr{M}_x$ stands for the set of all positive Radon measures μ on X such that $f(x) = \mu(f)$ whenever $f \in P$. Obviously, the Dirac measure ε_x concentrated at x belongs to $\mathscr{M}_x$. The set

$$\mathrm{Ch}_P X = \{x \in X;\ \mathscr{M}_x = \{\varepsilon_x\}\}$$

is called the *Choquet boundary of X* (with respect to P).

3.2. Notation and Definition. Let U be a bounded open subset of R^{n+1}. A function u is said to be *caloric* on U, if u has continuous second partial derivatives on U and

$$\frac{\partial u}{\partial t}(x, t) - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x, t) = 0 \quad \text{for all} \quad (x, t) \in U\,.$$

The set of all continuous functions on $\overline{U}$ whose restriction to U is caloric will be denoted by $K(U)$. We will apply Definition 3.1 to the following situation: $X = \overline{U}$ and $P = K(U)$.

3.3. Theorem. *Let U be a bounded open subset of R^{n+1}. Then*

$$\mathrm{Ch}_{K(U)} \overline{U} = \beta(U^c) \cap \overline{U}\,.$$

Proof. See [4], pp. 101, 103 and [16], p. 516.

3.4. Theorem. *Let U be a bounded open subset of R^{n+1}. Then the following conditions are equivalent:*

(1) $\quad z \in \mathrm{Ch}_{K(U)} \overline{U}$;

(2) $\quad \int_0^1 \alpha(U^c \cap B(z, c))/c^{n/2+1}\, \mathrm{d}c = \infty$;

(3) $$\sum_{k=1}^{\infty} 2^{nk/2}\, \alpha(U^c \cap B_k(z)) = \infty\ ;$$

(4) $$\sum_{k=1}^{\infty} 2^{nk/2}\, \alpha(U^c \cap A_k(z)) = \infty\ .$$

Proof. The assertions follow from Theorems 3.3 and 2.16.

3.5. Corollary. *Let U be a bounded open subset of R^{n+1} and let $z = (x, t) \in \partial U$. Let B_0 be an arbitrary subset of R^n and T a subset of $]0, \infty[$ such that 0 is a condensation point of T. If*

$$\{(x + \tau\chi,\ t - \tau^2) \in R^{n+1};\ \chi \in B_0,\ \tau \in T\} \subset U^c$$

and $\lambda_n^*(B_0) > 0$, then $z \in \mathrm{Ch}_{K(U)}\bar{U}$.

Proof. We may assume that $z = (0, 0)$. We shall prove that $z \in \beta(U^c)$. Since

$$B = \{(\tau\chi, \tau^2) \in R^{n+1};\ \chi \in B_0,\ \tau \in T\} \subset U^c$$

and $\lambda_n^*(B_0) > 0$, the set B is not semipolar at z by Corollary 2.21, hence $z \in \beta(B)$. Since $B \subset U^c$, we have $\beta(B) \subset \beta(U^c)$ too. Consequently, $z \in \beta(U^c)$.

The proof of Corollary 3.5 is complete.

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