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MAXIMAL CONVERGENCES AND MINIMAL PROPER CONVERGENCES IN l-GROUPS

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In this paper the notion of convergence on a lattice ordered group G will be applied in the same sense as in [11]. This notion was investigated (for the abelian case) also in [5], [8], [10], [14]. Particular cases of convergences on lattice ordered groups were dealt with in [6] and [17].

The system of all convergences on G will be denoted by Conv G. This system is partially ordered by inclusion (cf. [9], [11], [14]). Assume that G is a direct product of lattice ordered groups G_i $(i \in I)$ with $G_i \neq \{0\}$ for each $i \in I$. To each system $(\alpha_i: i \in I)$ with $\alpha_i \in \operatorname{Conv} G_i$ for each $i \in I$ there corresponds in a natural way an element $\alpha \in \operatorname{Conv} G$. Let S be the set of all $\alpha \in \operatorname{Conv} G$ which can be constructed in this way. Under the notation as above, α is said to be the product of the system $(\alpha_i: i \in I)$.

If I is finite, then $S = \operatorname{Conv} G$ (cf. [9]). It will be shown below that if I is infinite, then

$$\operatorname{card}(\operatorname{Conv} G \setminus S) \geqq 2^{\aleph_0}.$$

The question arises whether a direct product of maximal elements α_i of Conv G_i must be a maximal element of Conv G. Analogous questions were studied for topological groups and for convergence groups. In both these cases the answers are "No", cf. [4], [5], [7]. (Let us remark that in [4], [7] the term "coarse" is applied instead of "maximal".)

For the case of lattice ordered groups the following positive result will be established:

(A) *Let* $\alpha \in \operatorname{Conv} G$, *where* $G = \prod_{i \in I} G_i$. *Then the following conditions are equivalent*:

(i) α *is a maximal element of* Conv G;

(ii) *there is a system* $(\alpha_i: i \in I)$ *such that for each* $i \in I$, α_i *is a maximal element of* Conv G_i, *and* α *is the product of the system* $(\alpha_i: i \in I)$.

The least element of Conv G will be denoted by $d(G)$. If $\alpha \in \operatorname{Conv} G$ and $\alpha \neq d(G)$ then α will be said to be a proper convergence in G. A minimal proper convergence in G is called an atom of Conv G.

The atoms of Conv G of an abelian lattice ordered group will be dealt with. The following results (B), (C) and (D) will be established:

(B) *Let α be an atom of* Conv G. *Then the interval* $[d(G), \alpha]$ *of* Conv G *is a direct factor of* Conv G.

(C) *Assume that* Conv G *has an atom. Then the following conditions are equivalent*: (i) Conv G *has a greatest element*; (ii) *each atom of* Conv G *has a pseudocomplement*; (iii) *there exists an atom in* Conv G *which has a pseudocomplement.*

(D) *Let S be as above. Then each atom of* Conv G *belongs to S.*

1. PRELIMINARIES

In this section we recall basic notions concerning the convergence lattice ordered groups (cf. [11], [14]).

Let G be a lattice ordered group (shortly: l-group). Let N be the set of all positive integers. An element of the direct product $\prod_{n \in N} G_n$, where $G_n = G$ for each $n \in N$, will be denoted by $(g_n)_{n \in N}$ (or, if no misunderstanding can occur, by (g_n)). If there is $g \in G$ such that $g_n = g$ for each $n \in N$, then we denote $(g_n) = \text{const}(g)$. (g_n) is called a sequence in G. The notion of a subsequence has the usual meaning. A subset A of the positive cone $(G^N)^+$ of the l-group G^N is said to be G-normal, if

$$\text{const}(g) + (g_n) - \text{const}(g) \in A \quad \text{whenever} \quad g \in G \quad \text{and} \quad (g_n) \in A .$$

Let α be a convex G-normal subsemigroup of $(G^N)^+$ such that the following conditions are satisfied:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.

(II) If (g_n) is a sequence in G^+ such that each subsequence of (g_n) has a subsequence belonging to α, then (g_n) belongs to α.

(III) Let $g \in G$. Then const (g) belongs to α if and only if $g = 0$.

Under these assumptions α is said to be a convergence on G. Let Conv G be the system of all convergences on G; this system is partially ordered by inclusion.

For $(g_n) \in G^N$ and $g \in G$ we put $g_n \to_\alpha g$ if and only if $(|g_n - g|) \in \alpha$. Then G with the convergence $\to_\alpha$ is a FLUSH-convergence group in the sense of [15], [16] (cf. [11]).

Let $A \subseteq (G^N)^+$, $A \neq \emptyset$. We denote by δA the system of all subsequences of sequences belonging to A. The convex closure in G^N of the set $A \cup \{\text{const}(0)\}$ will be denoted by $[A]$. Next, let $\langle A \rangle$ be the G-normal subsemigroup of G^N generated by the set A; hence $\langle A \rangle$ is the set of all sequences (x_n) such that there are sequences $(y_n^1), (y_n^2), \ldots, (y_n^k)$ in A and elements $g_1, g_2, \ldots, g_k$ of G with

$$x_n = \sum_{m=1}^{k} (g_m + y_n^m - g_m) \quad \text{for each} \quad n \in N .$$

Finally, the symbol A^* will denote the set of all sequences in G^+ for which each subsequence has a subsequence belonging to A.

For each $\emptyset \neq A \subseteq (G^N)^+$ put $T(A) = [\langle \delta A \rangle]^*$. The set A will be said to be regular if, whenever $0 \neq g \in G$, then const $(g) \notin T(A)$.

1.1. Theorem (cf. [11], Theorem 2.2). *Let $A \subseteq (G^N)^+$, $A \neq \emptyset$. If A is regular, then $T(A)$ is the least element of* Conv G *which contains A as a subset. In the opposite case there exists no $\alpha \in$* Conv G *with $A \subseteq \alpha$.*

2. DIRECT PRODUCTS

Let I be a nonempty set and for each $i \in I$ let $G_i \neq \{0\}$ be an l-group. Put $G = \prod_{i \in I} G_i$. If $i \in I$ and $g \in G$, then the i-th component of g will be denoted by $g(i)$.

Let i be a fixed element of I and let $h \in G_i$. If no misunderstanding can occur, then the element h will be identified with the element h' of G such that $h'(i) = h$ and $h'(j) = 0$ for each $j \in I \setminus \{i\}$. In this sense G_i is considered to be a subset of G.

For $X \subseteq G^N$ and $i \in I$ we denote

$$X(i) = \{(x_n(i))_{n \in N} : (x_n) \in X\} .$$

If $i \in I$ and $A \subseteq (G_i^N)^+$, $A \neq \emptyset$, then $T^{(i)}(A)$ has an analogous meaning as above with the distinction that we take G_i instead of G. Then we obviously have

2.1. Lemma. *Let $\emptyset \neq A \subseteq (G^N)^+$, $i \in I$. Then $T^{(i)}(A(i)) = (T(A))(i)$.*

2.2. Corollary. *Let $\emptyset \neq A \subseteq (G^N)^+$. Assume that A fails to be regular. Then there is $i \in I$ such that $T^{(i)}(A(i))$ is not regular.*

2.3. Definition. *For each $i \in I$, let $\emptyset \neq \alpha_i \subseteq (G_i^N)^+$. Let α be the set of all elements (g_n) of $(G^N)^+$ such that $(g_n(i)) \in \alpha_i$ for each $i \in I$. Then α is called the product of the convergences α_i and we write $\alpha = \prod_{i \in I} \alpha_i$.*

Next, we denote by S the set of all elements $\beta \subseteq (G_n)^+$ having the property that there are $\beta_i \in$ Conv G_i with $\beta = \prod_{i \in I} \beta_i$.

2.4. Lemma. *For each $i \in I$, let $\alpha_i \in$* Conv G_i. *Put $\alpha = \prod_{i \in I} \alpha_i$. Then $\alpha \in$* Conv G.

Proof. It is a routine to verify that α is a convex G-normal subsemigroup of $(G^N)^+$ and satisfies the conditions (I), (II) and (III).

Let $d(G)$ be the set of all $(x_n) \in (G^N)^+$ such that there is $n_0 \in N$ with $x_n = 0$ whenever $n > n_0$. It is clear that $d(G)$ is the least element of Conv G. Let $d(G_i)$ have an analogous meaning (with G replaced by G_i).

Put $d_0 = \prod_{i \in I} d(G_i)$. In view of 2.4, $d_0 \in$ Conv G. Let us remark that if I is infinite, then clearly $d_0 > d(G)$.

Now let us assume that I is infinite. Then there exists a system $\{M_j : j \in J\}$ of pairwise disjoint subsets of I such that

(i) card M_j = card I for each $j \in J$;

(ii) card J = card I.

Let $\mathscr{K}$ be the family of all nonempty subsets of the system $\{M_j : j \in J\}$. For $K \in \mathscr{K}$

we denote by $\alpha(K)$ the set of all elements (g_n) of $(G^N)^+$ such that $(g_n) \in d_0$ and there is $m \in N$ such that for each $n > m$ and each $i \in I \setminus K_0$ we have $x_n(i) = 0$, where K_0 is the join of all M_j $(j \in J)$ with $M_j \in K$.

2.5. Lemma. *For each* $K \in \mathscr{K}$, $T(\alpha(K))$ *belongs to* Conv G. *If* $K_1, K_2 \in \mathscr{K}$, $K_1 \not\subseteq K_2$, *then* $T(\alpha(K_1)) \not\subseteq T(\alpha(K_2))$.

Proof. Let $K \in \mathscr{K}$. Then $\alpha(K) \subseteq d_0$, hence $\alpha(K)$ is regular. Thus in view of 1.1 we have $T(\alpha(K)) \in \text{Conv } G$.

Let K_1 and K_2 be elements of $\mathscr{K}$ such that K_1 fails to be a subset of K_2. Hence there is $M_{j(1)} \in K_1 \setminus K_2$. Next, there are distinct elements $s(1), s(2), s(3), \ldots$ in $M_{j(1)}$. For each $n \in N$, let $0 < g_n \in G_{s(n)}$. Then $(g_n) \in \alpha(K_1) \subseteq T(\alpha(K_1))$, but (g_n) does not belong to $T(\alpha(K_2))$. Hence $T(\alpha(K_1)) \not\subseteq T(\alpha(K_2))$.

2.5.1. Corollary. *Let* K_1 *and* K_2 *be distinct elements of* $\mathscr{K}$. *Then* $T(\alpha(K_1)) \neq T(\alpha(K_2))$.

2.6. Lemma. *Let* $K \in \mathscr{K}$, $K \neq \{M_j : j \in J\}$. *Then* $T(\alpha(K))$ *does not belong to* S.

Proof. Put $\beta = T(\alpha(K))$. By way of contradiction, assume that β belongs to S. Hence there are β_i $(i \in I)$ such that

(i) $\beta_i \in \text{Conv } G_i$ for each $i \in I$, and

(ii) $\beta = \prod_{i \in I} \beta_i$. Because $\beta_i \supseteq d(G_i)$ is valid for each $i \in I$, we obtain $\beta \supseteq \prod_{i \in I} d(G_i) = d_0 \supseteq T(\alpha(K_1))$ for each $K_1 \in \mathscr{K}$, contradicting 2.5.

Since card $\mathscr{K} \geqq 2^{\aleph_0}$, from 2.5.1 and 2.6 we obtain:

2.7. Theorem. *Let I be an infinite set and for each $i \in I$ let G_i be a nonzero l-group. Let* $G = \prod_{i \in I} G_i$. *Then* card $(\text{Conv } G \setminus S) \geqq 2^{\aleph_0}$.

3. MAXIMAL ELEMENTS

As above, let G be an l-group. We denote by $M(G)$ the set of all maximal elements of Conv G. We apply Axiom of Choice; then in view of Zorn Lemma, the set $M(G)$ is nonempty.

3.1. Lemma. *Let* $\alpha \in \text{Conv } G$. *Then the following conditions are equivalent:*

(i) $\alpha \in M(G)$;

(ii) *if* (g_n) *is a sequence in* G^+ *with* $(g_n) \notin \alpha$, *then the set* $\alpha \cup \{(g_n)\}$ *fails to be regular*;

(iii) *if* (g_n) *is a sequence in* G^+ *with* $(g_n) \notin \alpha$, *then there are* $(h_n^1), (h_n^2), \ldots, (h_n^k) \in \alpha \cup \delta(g_n)\}$ *and elements* $t_1, t_2, \ldots, t_k \in G$, $g \in G$, $g > 0$ *such that for each* $n \in N$ *the relation* $\sum_{j=1}^{k} (t_j + h_n^j - t_j) \geqq g$ *is valid.*

Proof. This is an immediate consequence of 1.1.

In particular, for the abelian case we obtain:

3.2. Corollary. *Let G be an abelian l-group and let $\alpha \in \operatorname{Conv} G$. Then the following conditions are equivalent:*

(i) $\alpha \in M(G)$;

(ii) *if (g_n) is a sequence in G^+ with $(g_n) \notin \alpha$, then there are $(g_n^1), (g_n^2), \ldots, (g_n^k) \in \delta\{(g_n)\}$, $(h_n) \in \alpha$ and $0 < g \in G$ such that for each $n \in N$ the relation $g_n^1 + g_n^2 + \ldots + g_n^k + h_n \geqq g$ is valid.*

3.3. Theorem. *Let $G = \prod_{i\in I} G_i$, where $G_i \neq \{0\}$ for each $i \in I$. Let $\alpha_i \in M(G_i)$ for each $i \in I$, $\alpha = \prod_{i\in I} \alpha_i$. Then $\alpha \in M(G)$.*

Proof. By way of contradiction, assume that there is $\beta \in \operatorname{Conv} G$ with $\alpha < \beta$. Thus there exists $(g_n) \in \beta \setminus \alpha$. Therefore there exists $j \in I$ such that

$$(g_n(j)) \notin \alpha_j . \tag{1}$$

We have $0 \leqq g_n(i) \leqq g_n$ for each $n \in N$ and $i \in I$, hence $(g_n(i)) \in \beta$. Also, if $i \in I$, then $\alpha_i \leqq \alpha < \beta$, whence $\alpha_i \cup \{(g_n)\} \subseteq \beta$. Thus $\alpha_i \cup \{(g_n)\}$ is a regular subset of $(G^N)^+$. Therefore in view of 2.1, $\alpha_i \cup \{(g_n(i))\}$ is a regular subset of $(G_i^N)^+$. Hence there is $\gamma \in \operatorname{Conv} G_j$ with $\alpha_j \cup \{(g_n(j))\} \subseteq \gamma$. Then according to (1) we have $\alpha_j \subset \gamma$. Since $\alpha_j \in M(G_j)$, we have arrived at a contradiction.

3.4. Theorem. *Let G be as in 3.3. Let $\alpha \in M(G)$. Then $\alpha(i) \in M(G_i)$ for each $i \in I$ and $\alpha = \prod_{i\in I} \alpha(i)$.*

Proof. Let $i \in I$. We obviously have $\alpha(i) \in \operatorname{Conv} G_i$. By way of contradiction, assume that $\alpha(i)$ does not belong to $M(G_i)$. Hence there is $\alpha' \in M(G_i)$ with $\alpha(i) < \alpha'$. Put $\beta_i = \alpha'$ and $\beta_j = \alpha(j)$ for each $j \in I \setminus \{i\}$. Let $\beta = \prod_{i\in I} \beta_i$. According to 2.4 we have $\beta \in \operatorname{Conv} G$ and clearly $\alpha < \beta$, which is a contradiction. Hence $\alpha(i) \in M(G_i)$ for each $i \in I$.

Next, we have $\alpha \leqq \prod_{i\in I} \alpha(i)$. Since α is maximal, the relation $\alpha = \prod_{i\in I} \alpha(i)$ must be valid.

In view of 3.3 and 3.4 we infer that (A) holds.

4. ATOMS IN Conv G

In this section the abelian l-groups will be investigated. The assertions (B), (C) and (D) formulated above will be proved.

Let G be an l-group. If X is a sequence in G, then its n-th member will be denoted by $X(n)$. For a subset C of G we denote $C^\perp = \{g \in G: |g| \wedge |c| = 0 \text{ for all } c \in C\}$.

4.1. Lemma. *Let G be an l-group, C its convex non-trivial linearly ordered l-subgroup, and let H be an l-subgroup generated by $C \vee C^\perp$. If $\gamma \in \operatorname{Conv} G$ and $X \in \gamma$, then there exists a positive integer m such that $X(n) \in H$ for each $n \geqq m$.*

Proof. Since $C \neq \{0\}$, there exists $c_1 \in C$, $c_1 > 0$. The set $\{n \in N: X(n) \geqq c_1\}$ is finite: if not, there is a subsequence Y of X such that $Y(n) \geqq c_1$ for each $n \in N$.

Because $Y \in \gamma$, we have $\text{const}(c_1) \in \gamma$. Therefore $c_1 = 0$, which contradicts our assumption.

Thus there exists $m \in N$ such that $X(n) \ngeq c_1$ for each $n \geqq m$. Take $n \in N$, $n \geqq m$. Then $X(n) \ngeq c_1$ and we shall show that $X(n) \in H$. Clearly, $X(n) = X(n) - (X(n) \wedge \wedge c_1) + (X(n) \wedge c_1)$ and $X(n) \wedge c_1 \in C$. Denote $b = X(n) - (X(n) \wedge c_1)$,

$$c_2 = c_1 - (X(n) \wedge c_1).$$

It is easy to see that $b \geqq 0$, $c_2 > 0$, $b \wedge c_2 = 0$ and $c_2 \in C$. For completing the proof it suffices to verify that $b \in C^\perp$. Let c be an element of C^+. Then $b \wedge c$ is an element of C and thus it is comparable with c_2. However, if $b \wedge c \geqq c_2$, then $b \geqq b \wedge c \geqq \geqq c_2$ and then $0 = b \wedge c_2 = c_2 > 0$, a contradiction. So we have $b \wedge c < c_2$ and then $b \wedge c \leqq b \wedge c_2 = 0$. Hence for each $c \in C^+$ the relation $b \wedge c = 0$ is valid, i.e., $b \in C^\perp$.

4.2. Remark. Let H be a convex l-subgroup of an abelian l-group G. For each $\alpha \in \operatorname{Conv} G$ we denote by $\varphi_H(\alpha)$ the set $\alpha \cap (H^N)^+$ and for each $\beta \in \operatorname{Conv} H$ we denote by $\psi_G(\beta)$ the set of all $X \in (G^N)^+$ such that there are $T \in \beta$ and $m \in N$ with $T(m + n)_{n \in N} \in \beta$. In [13], it was shown (cf. Lemma 5.1) that
(i) if $\alpha \in \operatorname{Conv} G$, then $\varphi_H(\alpha) \in \operatorname{Conv} H$ and $\psi_G(\varphi_H(\alpha)) \subseteq \alpha$,
and
(ii) if $\beta \in \operatorname{Conv} H$, then $\psi_G(\beta) \in \operatorname{Conv} G$ and $\varphi_H(\psi_G(\beta)) = \beta$.

Let H be a convex l-subgroup of G generated by $C \vee C^\perp$, where C is a nontrivial convex linearly ordered subgroup of G. Let φ_H, ψ_G be as above. In this case, the assertion (i) can be improved in the following way:
(i') if $\alpha \in \operatorname{Conv} G$, then $\varphi_H(\alpha) \in \operatorname{Conv} H$ and $\psi_G(\varphi_H(\alpha)) = \alpha$. In fact, if $\alpha \in \operatorname{Conv} G$ and $X \in \alpha$, by Lemma 4.1 there exists $m \in N$ such that $X(n) \in H$ for each $n \in N$, $n \geqq m$; i.e., $X(m + n)_{n \in N} \in \varphi_H(\alpha)$. Therefore $X \in \psi_G(\varphi_H(\alpha))$.

4.3. Lemma. *Let H be a convex l-subgroup of G generated by $C \vee C^\perp$, where C is a nontrivial convex linearly ordered subgroup of G. Let φ_H, ψ_G be as above. If $\gamma \in \operatorname{Conv} G$, $\beta \in \operatorname{Conv} H$ such that $\gamma \subseteq \psi_G(\beta)$, then $\psi_G(\varphi_H(\gamma)) = \gamma$.*

Proof. Straightforward.

4.4. Corollary. *Let G be an abelian l-group containing a non-trivial convex linearly ordered l-subgroup C. Let H be a convex l-subgroup of G generated by $C \vee C^\perp$. Then φ_H and ψ_G are mutually inverse isomorphisms of partially ordered sets* Conv G *and* Conv H.

Let ξ be an isomorphism of a partially ordered set P onto a direct product $A' \times B'$. Assume that $0_P, 0_{A'}, 0_{B'}$ are the least elements of P, A', B'. Denote $A = \xi^{-1}\{(a, 0_{B'})$: $a \in A'\}$ and let B be defined analogously. Then A and B are convex subsets of P and each element p of P can be uniquely represented in the form $p = p_A \vee p_B$ where $p_A \in A$, $p_B \in B$. Conversely, if this condition is fulfilled then the mapping $\eta(p) = (a, b)$ is obviously an isomorphism of P onto $A \times B$. Motivated by the above observation we introduce the following definition.

4.5. Definition. *Let $(P, \leqq)$ be a partially ordered set containing the least element. Let A and B be convex subsets of P. Then P will be called the direct product of A and B if for each $p \in P$ there exists exactly one pair (p_A, p_B) of elements of P such that $p_A \in A$, $p_B \in B$ and $p = p_A \vee p_B$. The sets A and B will be called direct factors of P.*

4.6. Lemma. *Let H be an abelian l-group containing a convex linearly ordered l-subgroup C such that H is generated by $C \vee C^{\perp}$. Let β be an atom of* Conv H *such that $\beta \cap C_N \neq d(C)$. Then* Conv H *is a direct product of the prime interval $[d(H), \beta]$ and of the set $\{\varrho \in \text{Conv } H: \beta \cap \varrho = d(H)\}$.*

Proof. Denote

$$A = [d(H), \beta] \quad \text{and}$$
$$B = \{\varrho \in \text{Conv } H: \beta \cap \varrho = d(H)\} .$$

Since β is an atom of Conv H, then A is a convex subset of Conv H. It is easy to verify that B is convex as well. Take $\gamma \in \text{Conv } H$ and denote $\gamma_A = \gamma \cap \beta$ and $\gamma_B = \bigvee\{\varrho \in \text{Conv } H: \varrho \subseteq \gamma \text{ and } \beta \cap \varrho = d(H)\}$. We will verify that

(1) $\gamma_A \in A$,

(2) $\gamma_B \in B$,

(3) $\gamma = \gamma_A \vee \gamma_B$, and

(4) if $\gamma = \varrho_A \vee \varrho_B$ for some $\varrho_A \in A$, $\varrho_B \in B$, then $\varrho_A = \gamma_A$ and $\varrho_B = \gamma_B$.

Since $\beta \cap \gamma \in \text{Conv } H$ (cf. [11], Lemma 2.1) and β is an atom of Conv H, (1) is true. By [11] (Lemma 2.3), $\gamma_B \in \text{Conv } H$. According to Thm. 2.5 (c) of [11], the closed interval $[d(H), \gamma]$ is a complete Brouwerian lattice, threefore (cf. [1]) the infinite meet-distributive law holds there. Hence $\beta \cap \gamma_B = d(H)$, and the assertion (2) holds.

Assume $X \in \gamma$. The relations $\gamma \subseteq (H^+)^N$ and $H^+ = C^+ \times (C^{\perp})^+$ (see [2], Prop. 3.5.8) imply that there exist $X_A \in C^N$ and $X_B \in (C^{\perp})^N$ such that $X = X_A + X_B$. First, $\text{const}(0) \leqq X_A \leqq X \in \gamma$, thus $X_A \in \gamma$. In view of [10], Thm. 3.9, we have $\text{Conv } C = [d(C), \ \beta \cap C^N]$. Since $\gamma \cap C^N \in \text{Conv } C$, we obtain $X_A \in \gamma \cap C^N \subseteq \beta \cap C^N \subseteq \beta$; finally, $x_A \in \gamma_A$. Secondly, $\text{const}(0) \leqq X_B \leqq X$, thus $X_B \in \gamma$; hence the set $\{X_B\}$ is regular, $T(\{X_B\}) \in \text{Conv } H$ and $T(\{X_B\}) \subseteq \gamma$. Let $R \in \beta \cap T(\{X_B\})$.

Then there exists $m \in N$ such that $R(n) \in C$ for each $n \geqq m$. On the other hand, we have $R(n) \leqq X_B^1(n) + X_B^2(n) + \ldots + X_B^k(n)$ where $k \in N$ and $X_B^1, X_B^2, \ldots, X_B^k$ are subsequences of X_B. Because $X_B \in (C^{\perp})^N$ and $C^{\perp}$ is a convex subgroup of H, we have obtained that $R(n) \in C^{\perp}$ for each $n \in N$. Hence for $n \in N$, $n \geqq m$ we get $R(n) \in C \cap C^{\perp} = \{0\}$. We have shown that $R \in d(H)$, thus $\beta \cap T(\{X_B\}) = d(H)$ and $X_B \in T(\{X_B\}) \subseteq \gamma_B$. In this way the inclusion $\gamma \subseteq \gamma_A \vee \gamma_B$ holds; the converse inclusion is trivial. Suppose $\gamma = \gamma_A \vee \gamma_B = \varrho_A \vee \varrho_B$ for some $\gamma_A, \varrho_A \in A$, $\gamma_B, \varrho_B \in B$. In the same way as when proving (2), we get $\varrho_A \cap \gamma_B = \gamma_A \cap \varrho_B = d(H)$. Since $\beta \in A$, we have

$$\gamma_A = \gamma_A \wedge \beta = (\gamma_A \wedge \beta) \vee (\gamma_B \wedge \beta) = (\gamma_A \vee \gamma_B) \wedge \beta =$$
$$= (\varrho_A \vee \varrho_B) \wedge \beta = (\varrho_A \wedge \beta) \vee (\varrho_B \wedge \beta) = \varrho_A \wedge \beta = \varrho_A$$

and thus

$$\gamma_B = \gamma_B \wedge \gamma = \gamma_B \wedge (\varrho_A \vee \varrho_B) = (\gamma_B \wedge \varrho_A) \vee (\gamma_B \wedge \varrho_B) =$$
$$= \varrho_B \wedge \gamma_B = (\varrho_B \wedge \gamma_B) \vee (\varrho_B \wedge \gamma_A) = \varrho_B \wedge (\gamma_B \vee \gamma_A) = \varrho_B \wedge \gamma = \varrho_B .$$

Now Conv H is a direct product of the sets A and B.

4.7. Theorem. *Let G be an abelian l-group and let α be an atom of* Conv G. *Then* Conv G *is a direct product of the prime interval* $[d(G), \alpha]$ *and of the set* $\{\gamma \in \text{Conv } G: \alpha \cap \gamma = d(G)\}$.

Proof. By [11] (Thm. 3.6), there exists a convex linearly ordered l-subgroup C of G containing a decreasing sequence belonging to α. Let H denote a convex l-subgroup of G generated by $C \vee C^{\perp}$. By Lemma 4.6, $[d(H), \varphi_H(\alpha)]$ is a direct factor of Conv H. Applying the isomorphisms φ_H and ψ_G, namely their properties (i′) and (ii) of Remark 4.2, we obtain that Conv G is a direct product of $[d(G), \alpha]$ and $\{\gamma \in \text{Conv } G: \alpha \cap \gamma = d(G)\}$.

4.8. Definition. *Let $(P, \leqq)$ be a partially ordered set containing the least element p_0 and let $p_1 \in P$. Then an element* $\text{pc}(p_1) \in P$ *will be called a pseudocomplement of p_1 if* $\text{pc}(p_1)$ *is the greatest element of the set* $\{p \in P: p \wedge p_1 = p_0\}$.

4.9. Theorem. *Let G be an abelian l-group and let* Conv G *have an atom. Then the following conditions are equivalent*:

(i) Conv G *has a greatest element*;

(ii) *each atom of* Conv G *has a pseudocomplement*;

(iii) *there exists an atom in* Conv G *which has a pseudocomplement.*

Proof. (i) implies (ii): If we denote by γ the greatest element of Conv G and by α an atom of Conv G, then by Theorem 4.7 (under the notation as in the proof of 4.6) there exist $\gamma_A, \gamma_B \in \text{Conv } G$ such that $\gamma = \gamma_A \vee \gamma_B$ and $\alpha \cap \gamma_B = d(G)$. Since γ is the greatest element of Conv G, we conclude that γ_B is a pseudocomplement of α.

In view of the assumption of the theorem, (ii) implies (iii). (iii) implies (i): Let α be an atom of Conv G, $\text{pc}(\alpha)$ its pseudocomplement and β an arbitrary element of Conv G. Let us again apply the notation introduced in the proof of 4.6. According to Theorem 4.7, there exist $\beta_A \in [d(G), \alpha]$ and $\beta_B \in \text{Conv } G$ such that $\beta = \beta_A \vee \beta_B$ and $\alpha \cap \beta_B = d(G)$. Since $\beta_A \subseteq \alpha$ and $\beta_B \subseteq \text{pc}(\alpha)$, we have $\beta \subseteq \alpha \vee \text{pc}(\alpha)$. Thus $\alpha \vee \text{pc}(\alpha)$ is the greatest element of Conv G.

From the results of [10] (Sections 4, 5) we get the following lemma (for the definition of the lex-sum of linearly ordered groups see [3]).

4.10. Lemma. *Let G_i be an abelian linearly ordered group for each $i \in \in \{1, 2, \ldots, m\}$. If G is a lexico-sum of $G_1, G_2, \ldots, G_m$, then the partially ordered set* Conv G *is isomorphic to the direct product of the partially ordered sets* Conv G_1, Conv $G_2, \ldots,$ Conv G_m.

4.11. Theorem. *Let G be an abelian l-group which contains m strictly positive* *pairwise disjoint elements* $g_1, g_2, \ldots, g_m$ *but not* $m + 1$ *such elements. Let* k *be the number of such* g_i, $i \in \{1, 2, \ldots, m\}$, *for which the l-subgroup of* G *generated by* g_i *contains a strictly decreasing sequence* X *with* $\inf\{X(n): n \in N\} = 0$. *Then* Conv G *is a Boolean algebra isomorphic to* 2^k.

Proof. Let G_i be an l-subgroup of G generated by g_i for each $i \in \{1, 2, \ldots, m\}$. Since every G_i is linearly ordered, according to [10] (Theorem 3.9) we have card (Conv G_i) $\in \{1, 2\}$. By [3] (Theorem, p. 2.47), [10] (Corollary 3.10) and the assumption, there exist exactly k groups $G_{i(1)}, G_{i(2)}, \ldots, G_{i(k)}$ such that Conv $G_j \simeq 2$ for each $j \in \{i(1), i(2), \ldots, i(k)\}$. The other G_i's possess only the discrete convergence. By Lemma 4.10, Conv G is isomorphic to the direct product of Conv G_i, $i \in \in \{1, 2, \ldots, m\}$. Thus Conv $G \simeq 2^k$.

In 4.12 and 4.13 we assume that I is a non-empty set of indices and $G(i)$ is an abelian l-group for each $i \in I$. Denote by G the direct product of $G(i)$. Since no misunderstanding can occur, we again identify the elements of G which have only one non-zero component with their projections on the corresponding factor (like in Section 2), φ and ψ are as in 4.2.

4.12. Theorem. α *is an atom of* Conv G *if and only if there exist* $i \in I$ *and an atom* β *of* Conv $G(i)$ *such that* $\alpha = \psi_G(\beta)$.

Proof. If α is an atom of Conv G, then by [11] (Thm. 3.6) there exist $C \subseteq G$ and $X \in C^N$ such that C is a convex linearly ordered l-subgroup of G and X is a strictly decreasing sequence. Since C is a convex linearly ordered subset of G, there exists $i \in I$ such that $C \subseteq G(i)$. Denote $\beta = \varphi_{G(i)}(\alpha)$. By [13], $\beta \in$ Conv $G(i)$, $\psi_G(\beta) \in$ Conv G and $\psi_G(\beta) = \psi_G(\varphi_{G(i)}(\alpha)) \subseteq \alpha$. Because α is an atom of Conv G and $X \in \psi_G(\beta) \setminus d(G)$, we have obtained that $\psi_G(\beta) = \alpha$. To show that β is an atom in Conv $G(i)$, let γ be an element of Conv $G(i)$ such that $d(G(i)) \subseteq \gamma \subseteq \beta$. Then by [10] (Thm. 3.9), $d(G) = \psi_G(d(G(i))) \subseteq \psi_G(\gamma) \subseteq \psi_G(\beta) = \alpha$. According to the assumption, α is an atom and therefore either $\psi_G(\gamma) = d(G)$ or $\psi_G(\gamma) = \alpha$.

Thus $\gamma = \varphi_{G(i)}(\psi_G(\gamma)) = d(G(i))$ or $\gamma = \varphi_{G(i)}(\psi_G(\gamma)) = \varphi_{G(i)}(\alpha)$.

Conversely, let β be an atom of Conv $G(i)$ and let $\alpha = \psi_G(\beta)$. In view of [13] (Lemma 5.1), $\alpha \in$ Conv G. If $d(G) \subseteq \gamma \subseteq \alpha$ for some $\gamma \in$ Conv G, then $d(G(i)) = = \varphi_{G(i)}(d(G)) \subseteq \varphi_{G(i)}(\gamma) \subseteq \varphi_{G(i)}(\alpha) = \varphi_{G(i)}(\psi_G(\beta)) = \beta$. Thus $\varphi_{G(i)}(\beta) = d(G(i))$ or $\varphi_{G(i)}(\gamma) = \beta$. By Lemma 4.3, $\gamma = d(G)$ or $\gamma = \alpha$.

4.13. Corollary. *If* α *is an atom of* Conv G, *then* $\alpha \in S$.

Proof. By Theorem 4.12, $\alpha = \psi_G(\beta)$ for some $\beta \in$ Conv $G(i)$. In order to get α as a product convergence, it suffices to take β and all $d(G(j))$ for $j \in I$, $j \neq i$.

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