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GRAPHS WITH A GIVEN EDGE NEIGHBOURHOOD

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0. INTRODUCTION

All graphs considered in this paper are connected finite graphs without loops and multiple edges.

If the edge $e \in G$ joins the vertices x, y then denote by $N_G(e)$ or $N_G(x, y)$ the subgraph of the graph G induced by the set of all vertices adjacent to at least one of the vertices x, y (except the vertices x, y). Analogously, denote by $N_G(x)$ the subgraph of G induced by the set of all vertices adjacent to x.

A given graph H is called *edge-realizable* or shortly *e-realizable* (*vertex-realizable* or *v-realizable*) if there exists a graph G in which the neighbourhood $N_G(e)$ of every edge e ($N_G(x)$ of every vertex x) is isomorphic to H; in such a case G is called an *e-realization* (*a v-realization*) of H. The set of all e-realizations (v-realizations) of H is denoted by $\mathscr{R}_e(H)$ ($\mathscr{R}_v(H)$).

The notion of v-realizable graphs was introduced by A. A. Zykov [4] and many authors have studied the properties of some families of these graphs. B. Zelinka [3] introduced the notion of e-realizable graphs and showed some families of them.

In this article some generalizations of the results of [3] are given.

1. e-REALIZATIONS OF THE COMPLETE MULTIPARTITE GRAPHS

Theorem A (Zelinka [3]). *The complete bipartite graph $K_{n,m}$ is e-realizable.*

A similar proposition for v-realizable graphs was suggested by B. Alspach and observed also by J. Doyen, X. Hubaud and M. Reynaert (see [2]).

Theorem B ([2]). *The complete multipartite graph $K_{n_1,n_2,\dots,n_k}$ is not v-realizable unless $n_1 = n_2 = \dots = n_k$.*

The next generalization of Theorem A is an analogue of Theorem B for e-realizable graphs.

Theorem 1. *The complete multipartite graph $K_{n_1,n_2,\dots,n_k}$ is e-realizable if and only if*

$$n_1 + 1 = n_2 + 1 = n_3 = \dots = n_k .$$

To prove this theorem, we will use the following

Lemma 1.1. *Let G be isomorphic to* $K_{n_1,n_2,\dots,n_k}$ $(k \geqq 3)$. *Then* $N_G(e) \simeq N_G(f)$ *for each pair of edges e, f of G if and only if* $n_1 = n_2 = \dots = n_k$.

Proof of this lemma is simple and can be left to the reader.

Proof of Theorem 1. ($\Leftarrow$) If $G \simeq K_{n,n,\dots,n}$ then $N_G(e) \simeq K_{n-1,n-1,n,\dots,n}$.

($\Rightarrow$) Let G be an e-realization of $K_{n_1,n_2,\dots,n_k}$. Then $N_G(y_1, y_2) \simeq K_{n_1,n_2,\dots,n_k}$ for each pair of the adjacent vertices y_1, y_2. Denote the parts of $N_G(y_1, y_2)$ by $P_1, P_2, \dots, P_k$ and the vertices of P_i by $x_1^i, x_2^i, \dots, x_{n_i}^i$ for each $i = 1, 2, \dots, k$. Without losing generality we can suppose that

$$(1) \qquad n_1 \leqq n_2 \leqq \dots \leqq n_k .$$

Now explore the neighbourhood of the edge x_i^1, x_j^2. As $G \in \mathscr{R}_e(K_{n_1,n_2,\dots,n_k})$ hence $N_G(x_i^1, x_j^2) \simeq K_{n_1,n_2,\dots,n_k}$. Denote by $P_1', P_2', \dots, P_k'$ the parts of $N_G(x_i^1, x_j^2)$. We can see that $N_G(x_i^1, x_j^2)$ contains the vertices y_1, y_2 and the graph $F \simeq K_{n_1-1,n_2-1,n_3,\dots,n_k}$ with the parts $P_1 - x_i^1, P_2 - x_j^2, P_3, \dots, P_k$. Since the vertices y_1, y_2 are adjacent, each part of $N_G(x_i^1, x_j^2)$ can contain at most one of these vertices. It follows from (1) that $P_3' = P_3, P_4' = P_4. \dots, P_k' = P_k$ and each of the parts P_1', P_2' contains exactly one of the vertices y_1, y_2. Without loss of generality we can suppose that $P_1' = P_1 - x_i^1 + y_1$ and $P_2' = P_2 - x_j^2 + y_2$. Therefore the vertex y_1 is adjacent to x_j^2 and to all vertices of the parts $P_2', P_3', \dots, P_k'$. Analogously, y_2 is adjacent to x_i^1 and to all vertices of the parts $P_1', P_3', P_4', \dots, P_k'$. Thus G contains a subgraph isomorphic to $K_{n_1+1,n_2+1,n_3,\dots,n_k} = K$.

As $N_G(y_1, y_2) \simeq K_{n_1,n_2,\dots,n_k}$, the number of its vertices is $|N_G(y_1, y_2)| = n_1 + n_2 + \dots + n_k = n_0$. Since $G \in \mathscr{R}_e(K_{n_1,n_2,\dots,n_k})$, the equality $|N_G(f)| = n_0$ holds for every edge f of G. On the other hand, $|N_K(f)| = n_0$ and hence $G = K$.

Under Lemma 1.1 $G \simeq K_{n,n,\dots,n}$ and this yields $n_1 + 1 = n_2 + 1 = n_3 = \dots = n_k$.

2. e-REALIZATIONS OF THE CYCLES

M. Brown and R. Connelly proved the following

Theorem C ([1]). *All cycles are vertex-realizable.*

In his article [3] Zelinka has shown that the cycles C_3, C_4, C_6, C_8 are e-realizable and C_5 is not e-realizable.

The next theorem is a generalization of this result.

Theorem 2. *The cycles* C_{2n+1} *are not e-realizable, with the single exception of* C_3.

To prove this theorem we need the following

Lemma 2.1. *Let the graph* $H = K_4 - e$ *be a subgraph of G. Then G is not an e-realization of* C_{2n+1} *for* $n > 1$.

Proof. Let K_4 be the complete graph with vertices y_1, y_2, y_3, y_4 and let $e = y_3, y_4$. Suppose that $H = K_4 - e$ is a subgraph of G. Let $N_G(y_1, y_2)$ be isomorphic to C_{2n+1}

with the vertices $x_0, x_1, x_2, \ldots, x_{2n}$. Without loss of generality we can identify y_3 with x_0 and y_4 with x_j. It is evident that x_0 is not adjacent to x_j – in the opposite case $N_G(y_1, x_i)$ (or $N_G(y_2, x_i)$) for any $i \neq 0, j$ contains the cycle C_3 induced by the vertices y_2, x_0, x_j (y_1, x_0, x_j). Thus $2 \leqq j \leqq 2n - 1$.

Now suppose that there exists an edge x_i, x_{i+1} $(i \neq 0, j - 1, j, 2n)$ such that x_i is adjacent to y_1 and x_{i+1} is adjacent to y_2. Then $N_G(x_{i+1}, y_2)$ contains the subgraph $K_{1,3}$ with the vertices y_1, x_0, x_j, x_i, which is a contradiction. Analogously, if x_i is adjacent to y_2 and x_{i+1} is adjacent to y_1 then $N_G(x_{i+1}, y_1)$ contains $K_{1,3}$ with the vertices y_2, x_0, x_j, x_i. Thus all the vertices $x_1, x_2, \ldots, x_{j-1}$ have to be adjacent to exactly one vertex of the pair y_1, y_2. Without losing generality we can suppose that it is the vertex y_1.

Analogously, all the vertices $x_{j+1}, x_{j+2}, \ldots, x_{2n}$ are adjacent to exactly one vertex y of the pair y_1, y_2.

Now just one of the following cases occurs:

(i) $y = y_1$. Then $N_G(y_1, x_0)$ contains a subgraph $K_{1,3}$ with the vertices x_{j-1}, x_j, y_2 and hence $G \notin \mathscr{R}_e(C_{2n+1})$.

(ii) $y = y_2$. Then $N_G(y_2, x_j)$ contains the path $x_{j+1}, x_{j+2}, \ldots, x_{2n}, x_0, y_1, x_{j-1}$. If $G \in \mathscr{R}_e(C_{2n+1})$ then $N_G(y_2, x_j) \simeq C_{2n+1}$ with the vertices $x_0, y_1, x_{j-1}, z_3, z_4, \ldots$ $\ldots, z_j, x_{j+1}, x_{j+2}, \ldots, x_{2n}$. Suppose that $z_i = x_r$ for any $i \in \{3, 4, \ldots, j\}$, $r \in \{1, 2, \ldots, j - 2\}$. As $G \in \mathscr{R}_e(C_{2n+1})$ hence either x_r is adjacent to x_j (and $N_G(y_1, y_2) \not\simeq C_{2n+1}$), or x_r is adjacent to y_2 (and $N_G(x_r, y_2)$ contains $K_{1,3}$ – see above). Thus $z_i \neq x_r$.

Hence $N_G(y_1, x_j)$ contains the cycle C_{2j} with $2j$ vertices $x_0, x_1, \ldots, x_{j-1}$, $z_3, z_4, \ldots, z_j, x_{j+1}, y_2$, which is a contradiction. Thus $G \notin \mathscr{R}_e(C_{2n+1})$.

Now we are able to prove Theorem 2.

Proof of Theorem 2. Let $e = y_1, y_2$ be any edge of G and let $\{x_0, x_1, \ldots, x_{2n}\}$ be the vertex set of $N_G(e) \simeq C_{2n+1}$. If there exists a vertex x_i which is adjacent to both vertices y_1 and y_2 then the graph G contains the graph H from Lemma 2.1 with the vertex set $\{x_i, x_{i+1}, y_1, y_2\}$, and hence $G \notin \mathscr{R}_e(C_{2n+1})$.

Thus each vertex x_i is adjacent to exactly one vertex of the pair y_1, y_2. Since C_{2n+1} contains an odd number of vertices, there exists a triangle induced by the vertices y_1, x_i, x_{i+1} (or y_2, x_i, x_{i+1}). Without losing generality we can suppose that it is the triangle y_1, x_0, x_1. Then x_2 is adjacent to y_2 (in the opposite case the vertices x_0, x_1, x_2, y_1 induce the graph H) and x_3 has to be also adjacent to y_2 (in the opposite case $N_G(x_1, x_2)$ contains the subgraph $K_{1,3}$ with the vertices y_1, y_2, x_0, x_3, which is a contradiction). Hence the vertices x_{4k}, x_{4k+1} are adjacent to y_1 and the vertices x_{4k+2}, x_{4k+3} are adjacent to y_2. If n is an even number then x_{2n} is adjacent to y_1 and G contains the subgraph H with the vertices x_{2n}, x_0, y_1, y_2, which is a contradiction. If n is an odd number then x_{2n} is adjacent to y_2. In this case x_{2n-1} is adjacent to y_1 and thus $N_G(x_{2n}, x_0)$ contains the subgraph $K_{1,3}$ with the vertices y_1, y_2, x_{2n-1}, x_1, which is also a contradiction. Therefore $G \notin \mathscr{R}_e(C_{2n+1})$ and hence C_{2n+1} is not edge-realizable.

On the other hand, e-realizability of the even cycles was proved by R. Nedela [5].

Theorem D (Nedela). *The cycles C_{2n} are e-realizable for each $n \geqq 2$.*

From this Theorem and our Theorem 2 we obtain the following

Corollary. *A cycle C_n is e-realizable if and only if n is an even number or $n = 3$.*

References

[1] *M. Brown, R. Connelly:* On graphs with a constant link. In: Proof techniques in graph theory (F. Harary ed.), Academic press 1969.

[2] *J. Doyen, X. Hubaud, M. Reynaert:* Finite graphs with isomorphic neighbourhood. In: Colloque CNRS. Problèmes combinatoires et théorie des graphes, Orsay 1976.

[3] *B. Zelinka:* Edge neighbourhood graphs. Czech. Math. Journ. *36* (*111*), 1986, 44—47.

[4] *A. A. Zykov:* Problem 30. In: Theory of graphs and its applications. Proc. Symp. Smolenice 1963 (ed. M. Fiedler), Prague 1964, 164—165.

[5] *R. Nedela:* Graphs which are edge-locally C_n. Submitted to Czech. Math. Journ.

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