Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 39 (1989), No. 2, 323–347

Persistent URL: `http://dml.cz/dmlcz/102305`

Czechoslovak Mathematical Journal, 39 (114) 1989, Praha

ON BOUNDEDNESS OF SUPERPOSITION OPERATORS IN SPACES OF TRIEBEL-LIZORKIN TYPE

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(Received January 29, 1988)

0. INTRODUCTION

During the last twenty years many papers devoted to the problem of boundedness of superposition operators have been published. We give a short but of course not complete history. In 1965 S. Mizohata [11] (cf. also J. Rauch [14]) proved the nowadays well-known fact: $H^s_2(\mathbb{R}_n)$, $s > n/2$ is invariant under nonlinear mappings of the type $T_G: f \to G(f)$, $G \in C^\infty(\mathbb{R}_1)$. Here $H^s_2(\mathbb{R}_n)$ denotes the Bessel-potential space. The same was obtained for Slobodeckij spaces $W^s_p(\mathbb{R}_n)$, $1 \leqq p \leqq \infty$, $s > n/p$ by J. Peetre [12] (actually he proved the invariance of the more general Besov spaces but we do not deal here with this type of spaces in general). Later on D. R. Adams [1] proved the counterpart also for the spaces $H^s_p(\mathbb{R}_n)$, $1 < p < \infty$, $s > n/p$.

In the eighties a new development was started by Y. Meyer [10]. Using the elegant method of paradifferential operators he gave a new proof of Adams' result. Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}_n)$ generalize simultaneously Bessel-potential spaces and Sobolev spaces. Th. Runst [15] has shown that the method of paradifferential operators works also for the spaces $F^s_{p,q}(\mathbb{R}_n)$, $0 < p, q < \infty$, $s > \max(n/p, (n/q) - n)$. Further extensions, obtained by this method, are due to M. Yamazaki [24] in connection with anisotropic Triebel-Lizorkin spaces and to J. Marshall [9] in the case of weigthed Triebel-Lizorkin spaces. In all results mentioned above the inside part of the superposition is supposed to be bounded.

In the case of unbounded functions (in the language of the function spaces considered here this means $s < n/p$) the following is known: G. Stampacchia [19] showed that the superposition operator T_G maps a Sobolev space $W^1_p(\mathbb{R}_n)$, $1 \leqq p < \infty$ into itself if $G' \in L_\infty(\mathbb{R}_1)$. M. Marcus, V. J. Mizel [8] gave characterizations of all those functions $G: \mathbb{R}_1 \to \mathbb{R}_1$ which map via superposition a Sobolev space $W^1_p(\mathbb{R}_n)$, $1 \leqq p < \infty$ into itself. Finally, using characterizations of the underlying function spaces in terms of first order differences the boundedness of T_G, $G' \in L_\infty(\mathbb{R}_1)$ was also obtained for Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}_n)$, $0 < p$, $q < \infty$, $\sigma_{p,q} < s < 1$ (here

$$\sigma_{p,q} = n\left(\frac{1}{\min(1, p, q)} - 1\right),$$

cf. G. Bourdaud [2], H. Triebel [22], D. E. Edmunds, H. Triebel [4], Th. Runst [16]).

It is obvious there is a gap between 1 and n/p. Concerning this gap B. E. J. Dahlberg [3] has proved the following: A C^∞-function $G: \mathbb{R}_1 \to \mathbb{R}_1$ which maps a Sobolev space $W_p^m(\mathbb{R}_n)$, $1 \leqq p \leqq \infty$, $1 + 1/p < m < n/p$, into itself is necessarily a linear function. Using the same construction for the counterexample as B. E. J. Dahlberg, G. Bourdaud [2] and Th. Runst [16] have extended this to more general classes of functions. In the case of Triebel-Lizorkin spaces we have (cf. Th. Runst [16]): *Let* $0 < p < \infty$ *and* $0 < q \leqq \infty$. *Let* $n \,.\, \max(0, (1/p) - 1) + 1 + 1/p < s < n/p$. *Then every function* $G: \mathbb{R}_1 \to \mathbb{R}_1$, $G \in C^2(\mathbb{R}_1)$, *which maps* $F^s_{p,q}(\mathbb{R}_n)$ *into itself is necessarily of the type* $G(t) = c \,.\, t$, $c \in \mathbb{R}_1$. In view of this result a natural question is to find the minimal defect of smoothness in the case of a superposition $G(f)$, where the inside part f belongs to a space $F^s_{p,q}(\mathbb{R}_n)$, $1 < s < n/p$ and the outside part G is a C^∞-function. It is the aim of the paper to answer this question correctly. This means we have both affirmative and negative results to show the correctness of the number

$$\frac{(s-1)((n/p)-s)}{(n/p) - s + 1}$$

which represents this defect.

This paper is organized as follows. In Section 1 we collect some information about Triebel-Lizorkin spaces. Section 2 contains our main results concerning the superposition operators T_G, $G \in C^\infty(\mathbb{R}_1)$, without proofs. Finally, in Section 3 the proofs are given. As an important substep of the proof we study there the mapping properties of operators of the type T_G with $G(t) = t^\mu$, $G(t) = |t|^\mu$, $\mu > 1$. The results are partially announced in [18].

1. SOME INFORMATION ABOUT TRIEBEL-LIZORKIN SPACES

If not otherwise stated all functions are defined on the Euclidean n-space $\mathbb{R}_n$ and so we omit $\mathbb{R}_n$ in our notation.

Let S be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}_n$. By S' we denote its topological dual. If $\varphi \in S$ then

$$\mathscr{F}\,\varphi(x) = (2\pi)^{-n/2} \int_{R_n} e^{-ix\xi}\, \varphi(\xi)\, d\xi\,, \quad x \in \mathbb{R}_n$$

denotes the Fourier transform $\mathscr{F}\varphi$ of φ. As usual, $\mathscr{F}^{-1}\varphi$ means the inverse Fourier transform of φ. Let $\psi \in S$ be a function with the following properties

$$\text{(1.1)} \qquad \begin{cases} \psi(x) = 1 & \text{if } \ |x| \leqq 1\,, \\ \psi(x) = 0 & \text{if } \ |x| \geqq 3/2\,. \end{cases}$$

By

$$\text{(1.2)} \qquad \begin{cases} \varphi_0(x) := \psi(x)\,, \quad \varphi_1(x) = \psi(x/2) - \psi(x)\,, \\ \varphi_j(x) := \varphi_1(2^{-j+1}x)\,, \quad x \in \mathbb{R}_n\,, \quad j = 1, 2, \ldots \end{cases}$$

we define a smooth partition, of unity, i.e.

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for all} \quad x \in \mathbb{R}_n .$$

Further, observe that

$$\text{supp}\, \varphi_j \subset \{x \mid x \in \mathbb{R}_n,\ 2^{j-1} \leqq |x| \leqq 3 \,.\, 2^{j-1}\}\,, \quad j = 1, 2, \ldots . \tag{1.3}$$

Let $0 < p \leqq \infty$ and $0 < q \leqq \infty$. If $\{f_j\}_{j=0}^{\infty}$ is a sequence of complex-valued Lebesgue-measurable functions on $\mathbb{R}_n$ then we put

$$\|f_j \mid L_p(l_q)\| = \|\{f_j\}_{j=0}^{\infty} \mid L_p(l_q)\| = \left(\int_{\mathbb{R}_n} \left(\sum_{j=0}^{\infty} |f_j(x)|^q\right)^{p/q} \mathrm{d}x\right)^{1/p}$$

(with the usual modifications if $\max(p, q) = \infty$).

Definition 1. Let $-\infty < s < \infty$, $0 < p < \infty$ and $0 < q \leqq \infty$. Then we put

$$F^s_{p,q} = \{f \mid f \in S',\ \|f \mid F^s_{p,q}\|^\psi = \|2^{sj} \mathscr{F}^{-1}[\varphi_j(\xi)\, \mathscr{F} f(\xi)](\cdot) \mid L_p(l_q)\| < \infty\}\,.$$

Remark 1. The spaces $F^s_{p,q}$ are quasi-Banach spaces (Banach spaces if $\min(p, q) \geqq \geqq 1$), independent of the special choice of ψ (equivalent quasi-norms). As mentioned in Introduction these spaces generalize the Bessel-potential and Sobolev spaces. More precisely, we have

(i) $F^0_{p,2} = L_p$, $1 < p < \infty$,
(ii) $F^m_{p,2} = W^m_p$, $1 < p < \infty$, $m = 1, 2, \ldots$, where W^m_p denote the Sobolev spaces,
(iii) $F^s_{p,2} = H^s_p$, $1 < p < \infty$, $-\infty < s < \infty$, where H^s_p denote the Bessel-potential spaces,
(iv) $F^s_{p,p} = W^s_p$, $1 \leqq p < \infty$, $s > 0$, $s \neq$ integer, where W^s_p denote the Slobodeckij spaces.

In each case equality means the existence of an equivalent quasinorm. Moreover, as it is well-known for the Slobodeckij spaces, any space $F^s_{p,q}$ can be characterized via differences and derivatives, at least if s is large enough. We do not go into detail, referring the reader to H. Triebel [21, 2.5].

Remark 2. The following continuous embeddings are of some interest for us:

(i)
$$F^s_{p,q} \hookrightarrow \bigcup_{1 \leqq r \leqq \infty} L_r \quad \text{if} \quad s > n \max(0, (1/p) - 1)\,.$$

(ii)
$$F^{s+\varepsilon}_{p,\infty} \hookrightarrow \bigcap_{0<q\leqq\infty} F^s_{p,q} \hookrightarrow W^s_p \hookrightarrow \bigcup_{0<q\leqq\infty} F^s_{p,q} = F^s_{p,\infty}\,, \quad 1 < p < \infty$$

for any $\varepsilon > 0$.

Since the last number will often appear in what follows we introduce the notation

$$\sigma_p = n \max(0, (1/p) - 1)\,, \quad 0 < p \leqq \infty\,. \tag{1.4}$$

Remark 3. More information and proofs of the facts listed in Remarks 1 and 2 may be found in H. Triebel [21]. For Bessel-potential and Slobodeckij spaces we refer also to E. M. Stein [20].

Since we will consider in general real-valued functions we have to introduce the real-valued part of $F^s_{p,q}$.

Definition 2. Let $0 < p < \infty$, $0 < q \leqq \infty$, and $s > \sigma_p$. Then $\tilde{F}^s_{p,q}$ is the subspace of $F^s_{p,q}$ consisting of all real-valued functions and equipped with the same quasi-norm as $F^s_{p,q}$.

2. BOUNDEDNESS OF SUPERPOSITION OPERATORS IN SPACES OF TRIEBEL-LIZORKIN TYPE. THE CRITICAL CASE $1 < s < n/p$

As usual, $C^m(\mathbb{R}_1)$, $m = 1, 2, \ldots$ is the set of all functions f satisfying

(i) $f, f^{(1)}, \ldots, f^{(m)}$ are uniformly continuous functions on $\mathbb{R}_1$,
(ii) $\|f \mid C^m(\mathbb{R}_1)\| = \max\limits_{0 \leqq l \leqq m} \sup\limits_{t \in R_1} |f^{(l)}(t)| < \infty$.

Moreover, we put

$$C^\infty(\mathbb{R}_1) = \bigcap_{m=1}^{\infty} C^m(\mathbb{R}_1) .$$

For brevity we use the notation

$$\varrho = \varrho(s, n/p) = \frac{n/p}{(n/p) - s + 1} . \tag{2.1}$$

Now we are in position to formulate the first result.

Theorem 1. *Let* $0 < p < \infty$, $0 < q \leqq \infty$ *and*

$$\sigma_p + 1 < s < n/p . \tag{2.2}$$

Furthermore, let

$$\varrho > \sigma_p . \tag{2.3}$$

Let $G\colon \mathbb{R}_1 \to \mathbb{R}_1$ *be a function with the properties*

$$G(0) = 0 , \tag{2.4}$$

$$G \in C^\infty(\mathbb{R}_1) . \tag{2.5}$$

Then $T_G\colon f \to G(f)$ *is a bounded mapping of*

$$\bigcup_{0<q\leqq\infty} \tilde{F}^s_{p,q} \quad \text{into} \quad \bigcap_{0<r\leqq\infty} F^\varrho_{p,r} .$$

Moreover, for any $r > 0$ there exists a constant c such that

$$\|G(f) \mid F^\varrho_{p,r}\| \leqq c(\|f \mid F^s_{p,\infty}\| + \|f \mid F^s_{p,\infty}\|^\varrho) \tag{2.6}$$

for all $f \in \tilde{F}^s_{p,\infty}$.

We shall discuss the conditions of Theorem 1.

Remark 4. The assumption (2.2) implies

$$1 < \varrho < s . \tag{2.7}$$

(to see this use the equality

$$\frac{n/p}{(n/p)-s+1} = s - (\varrho - 1)\left(\frac{n}{p} - s\right)\Big).$$

So, in view of (2.3), Remark 2 and the properties of G, (2.4) and (2.5), we have $G(f) \in \bigcup_{1 \leqq p \leqq \infty} L_p$ for any $f \in \bigcup_{0<q\leqq\infty} \tilde{F}^s_{p,q}$. Hence $G(f)$ can be again interpreted as a regular and tempered distribution. Consequently, we can deal with T_G as with a mapping from a certain subspace of S' into S' and this makes the theorem meaningful.

Remark 5. Why we restrict ourselves to the values $1 < s < n/p$ is pointed out in Introduction. Some further remarks to the range $1 < s \leqq \sigma_p + 1$ are given in Section 3, Lemma 3. The restriction of ϱ to values larger than $n \max(0, (1/p) - 1)$ has, may be, merely technical reasons. Nevertheless, taking into consideration formulae (1.4) and (2.7) we can see that (2.3) is a restriction only in the case $p < n/(n + 1) < 1$. So, it does not occur in the case of the classical spaces of Sobolev and Slobodeckij types.

Remark 6. Let us discuss the conditions on G. $G(0) = 0$ is necessary, since we measure $G(f)$ in spaces which depend on integrability properties. $G \in C^\infty(\mathbb{R}_1)$ is not necessary, one needs a finite number of derivatives only. More exactly, if $\varrho = N_\varrho + \tau_\varrho$ where N_ϱ is a positive natural number and $0 < \tau_\varrho \leqq 1$ then $G \in C^{N_\varrho+1}(\mathbb{R}_1)$ is always sufficient. The dependence of the constant c which appears in (2.6) on G can be also calculated more precisely. We can choose

$$c = c' \|G \mid C^{N_\varrho+1}(\mathbb{R}_1)\| , \tag{2.8}$$

c' being independent of G.

Remark 7. Of course, $\bigcup_{0<q\leqq\infty} F^s_{p,q} = \tilde{F}^s_{p,\infty}$. We have prefered the above formulation to make the independence on q more obvious. Such a behaviour (independence on the parameter q) is also known for superposition operators of the type $T_m: f \to f^m$, $m = 2, 3, \dots$ (cf. [17] and Section 3).

To see the reasons which create such a difficult behaviour of the defect of smoothness we add an example. To avoid technical difficulties we restrict ourselves to $p \geqq 1$. We put

$$f_\alpha(x) = |x|^{-\alpha} \psi(x) \tag{2.9}$$

where ψ has the meaning from (1.1). Assume $0 < \alpha < n/p$. If $0 < s < n/p$ then we have

$$f_\alpha \in F^s_{p,p} \Leftrightarrow s < \frac{n}{p} - \alpha \tag{2.10}$$

(cf. E. M. Stein [20, Chapt. V., 6.9] or [17]). Let $G \in C^\infty(\mathbb{R}_1)$. Let s be fixed, $0 < s < n/p$. Let m be a natural number with $s - m \geqq 0$. This ensures the equality

(in S')

$$\partial^{\gamma}(G(f_\alpha))(x) = \sum_{l=1}^{|\gamma|} \sum_{\beta^1+\ldots+\beta^l=\gamma} G^{(l)}(f_\alpha(x))\, \partial^{\beta^1} f_\alpha(x) \cdot \ldots \cdot \partial^{\beta^l} f_\alpha(x) \tag{2.11}$$

for $|\gamma| \leqq m$ ($\beta^1, \ldots, \beta^l$ denote also multi-indices of length n, and $|\gamma| = \gamma_1 + \ldots + \gamma_n$). Let us further assume that we can avoid the influence of the factors $G^{(l)}(f_\alpha(x))$ in (2.11). Choose $\gamma = (\underbrace{1, \ldots, 1}_{m\text{-times}}, 0, \ldots, 0)$. In this case the asymptotic behaviour is given by

$$\partial^{\gamma}(G(f_\alpha))(x) \sim |x|^{-(\alpha+1)m} \quad \text{near zero} . \tag{2.12}$$

The assumption $G(f_\alpha) \in F^m_{p,2}$ implies $\partial^\gamma(G(f_\alpha)) \in L_p$ for any γ, $|\gamma| \leqq m$ and hence, from the asymptotic behaviour in the case $\gamma = (\underbrace{1, \ldots, 1}_{m\text{-times}}, 0, \ldots, 0)$, we conclude

$$m(\alpha + 1) < \frac{n}{p} .$$

If α tends to $((n/p) - s)$, this is equivalent to

$$m \leqq \frac{n/p}{(n/p) - s + 1} .$$

By virtue of (2.10) this corresponds to the results stated in our Theorem 1. A comparison of the asymptotic behaviour

$$\partial^\gamma f_\alpha(x) \sim |x|^{-\alpha-m}, \quad |\gamma| = m, \quad |x| \to 0$$

with (2.12) shows that, roughly speaking, differentiation of the composite function $G(f)$ creates stronger singularities than differentiation of the original function.

Since the above calculations give only a feeling of the correctness of ϱ we add the following theorem.

Theorem 2. *Let $T > 0$. Let $0 < p < \infty$ and $1 < s < n/p$. If there exists a natural number m such that*

$$1 \leqq m \leqq \varrho = \frac{n/p}{(n/p) - s + 1} < m + 1 \tag{2.13}$$

and

$$\sigma_p \leqq \varrho - m < 1 \tag{2.14}$$

then for any non-vanishing $(m + 1)$-times continuously differentiable and T-periodic function $G\colon \mathbb{R}_1 \to \mathbb{R}_1$ and for any $\varepsilon > 0$ there exists a function $f \in \tilde{F}^s_{p,\infty}$ with compact support and $G(f) \notin F^{\varrho+\varepsilon}_{p,\infty}$.

Remark 8. The main consequence of Theorem 2 is the fact that Theorem 1 cannot be improved in the framework of Triebel-Lizorkin spaces if p is fixed, that is if we measure f and $G(f)$ in the scale $\{F^s_{p,q};\ 1 < s < n/p,\ 0 < q \leqq \infty\}$. Consequently, it is also impossible to improve the number ϱ in the case of Sobolev spaces. For instance, by the embeddings (ii) from Remark 2 we obtain under the same assumptions as in Theorem 1 that T_G is a bounded mapping from W_p^m into $W_p^{m_0}$ and m_0 is the integer part of

$$\frac{n/p}{(n/p)-s+1}.$$

Moreover, Theorem 2 again with the embeddings from Remark 2 (ii) tells us that in general m_0 is the best possible choice.

Remark 9. The proof of Theorem 2 is based on Dahlberg's counterexample and on the characterization of the underlying Triebel-Lizorkin spaces in terms of first order differences. It is possible that the assumption $\varrho - m \geqq \sigma_p$ (instead of the more natural condition $\varrho > \sigma_p$) is only technical. The method of Th. Runst [16] of proving the negative result mentioned in Introduction is not applicable. He reduces the problem via appropriate embeddings to the above stiuation, but the use of embeddings leads to changes of our number ϱ.

Remark 10. The fact that we can choose the function f with a compact support indicates there is no further progress if one replaces the function spaces defined on $\mathbb{R}_n$ by the corresponding spaces on domains.

Remark 11. The assumption on G to be periodic is not necessary. It can be weakened. One needs the existence of a sequence of disjoint intervals $\{I_j\}_{j=0}^{\infty}$ with

$$\inf \lambda(I_j) \geqq B > 0 \tag{2.15}$$

and

$$I_j \subset \{t \mid t \in \mathbb{R}_1,\ |G^{(m+1)}(t)| \geqq A\} \tag{2.16}$$

for some positive numbers A and B (λ denotes the Lebesgue measure on $\mathbb{R}_1$). Note that the disjointness of the intervls I_j implies

$$\lambda(\{t \mid t \in \mathbb{R}_1,\ |G^{(m+1)}(t)| \geqq A\}) = \infty .$$

Remark 12. Also the following consequence of Theorem 1 is of some interest. Let $0 < p < \infty$, $0 < q \leqq \infty$ and $1 < s \leqq 1 + 1/p$. There exist C^∞-functions G which do not map via superposition $F^s_{p,q}$ into itself. Thus Theorem 2 is also a supplement to the negative results of B. E. J. Dahlberg [3], G. Bourdaud [2] and Th. Runst [16].

Remark 13. We add a trivial observation about the defect of smoothness. Let n and p be fixed and such that $n/p > 1$. Put $d(s) = s - \varrho(s, n/p)$. One easily checks that $\lim\limits_{s\downarrow 1} d(s) = \lim\limits_{s\uparrow n/p} d(s) = 0$. Since $\varrho < s$ (cf. Remark 4), the function $d(s)$ has a maximum in $(1, n/p)$. It will be assumed at the point $s = (n/p) - \sqrt{(n/p)} + 1$. In this sense we have a little bit surprising effect that there exist the worst spaces with respect to the superposition mappings, namely $F^s_{p,q}$, $s = (n/p) - \sqrt{(n/p)} + 1$ with the absolutely largest defect on smoothness $(\sqrt{(n/p)} - 1)^2$. Since a function with compact support or rapidly decreasing smooth function like e^{-t} cannot satisfy the assumptions (2.15), (2.16), the next theorem is also of some interest.

Theorem 3. *Let* $0 < p < \infty$ *and* $1 + 1/p < s < n/p$. *If there exists a natural number m such that*

$$(2.17) \qquad m \leqq \varrho^* := \frac{n/p + ((n/p) - s)/p}{(n/p) - s + 1} < m + 1$$

and

$$(2.18) \qquad \sigma_p \leqq \varrho^* - m < 1$$

then for every $G\colon \mathbb{R}_1 \to \mathbb{R}_1$, $(m+1)$*-times continuous differentiable,* $G^{(m+1)}(t)$ *non-vanishing, and for any* $\varepsilon > 0$ *there exists a function* $f \in \tilde{F}^s_{p,\infty}$ *with compact support and* $G(f) \notin F^{\varrho^*+\varepsilon}_{p,\infty}$.

Remark 14. Since $0 < s < n/p$ we have $\varrho^* < n/p$. Moreover

$$\varrho^* < s \Leftrightarrow 1 + 1/p < s\,.$$

This is the reason for restricting s to the values larger than $1 + 1/p$.

Remark 15. Affirmative results related to Theorem 3 are not known to the author. Nevertheless, a result of G. Bourdaud [2] is of some interest in this direction:

A twice differentiable function G with $G \in C^1(\mathbb{R}_1)$ and $G'' \in L_1(\mathbb{R}_1)$ maps via superposition the spaces W^2_1 and W^s_1, $0 < s < 2$, $s \neq 1$ into itself.

So, there is some hope that one can improve the number ϱ in the case $s < 1 + 1/p$ with the help of additional integrability properties.

Remark 16. Everybody who is familiar with the modern theory of function spaces (cf. E. M. Stein [20], H. Triebel [21]) would ask also for similar results in the case of Besov spaces $B^s_{p,q}$. All methods applied here work also in this case. Under the same conditions as in Theorem 1 one obtains boundedness of T_G as a mapping from $\tilde{B}^s_{p,p\varrho}$ into $B^\varrho_{p,\infty}$ (again $\sim$ is used as a symbol of restriction to the real-valued part). At the moment it is not clear whether it is natural to restrict the second lower index q to the values less than or equal to $p\varrho$ in order to ensure $G(f) \in B^\varrho_{p,\infty}$. Also the question whether "∞" can be replaced by a smaller value, may be under additional assumptions on the second upper index q of the original, is still open. Only partial answers can be obtained via embeddings in connection with Theorem 1.

3. BOUNDEDNESS OF SUPERPOSITION OPERATORS OF TYPE $T_\mu\colon f \to f^\mu$, $\mu > 1$

The results of this section are of independent interest but they are also strongly related to Section 2. As we shall see Theorem 1 appears mainly as a consequence of some boundedness results for the above type of superposition operators.

3.1. Mappings of type $T_m\colon f \to f^m$, $m = 2, 3, \ldots$. We introduce the following notation: Let $\mu > 0$. Then we put

$$(3.1) \qquad s_\mu := s - (\mu - 1)((n/p) - s)\,.$$

Theorem 4. *Let* $m = 2, 3, \ldots$. *Let* $0 < p < \infty$, $0 < q \leqq \infty$ *and* $0 < s < n/p$. *Moreover, let*

$$s > n \max(0, (1/p) - (1/m)). \tag{3.2}$$

Then for any r, $0 < r \leqq \infty$, *there exists a constant* c *such that*

$$\|f^m \mid F^{s_m}_{p,r}\| \leqq c\|f \mid F^s_{p,q}\|^m \tag{3.3}$$

for all $f \in F^s_{p,q}$.

Remark 17. Theorem 4 in the case $m = 2$ is contained in M. Yamazaki [23] and also in [17]. An extension to the values $m > 2$ may be found in [17, Theorem 12] but with a more restrictive condition on s. To obtain (3.3) in the remaining cases one can apply the method of proof presented in [17] in the case $m = 2$, including the following modification. Instead of an iteration of Theorem 4 in the case $m = 2$ one has to use an iteration of the paramultiplication principle (cf. [17, formula (3.3)–(3.6)]). That means, one has to decompose the product f^m in a finite number of infinite sums of the type $\sum_{k\in\Omega} (\prod_{i=1}^{m} \mathscr{F}^{-1}[\varphi_{k_i}\mathscr{F}f])$, where Ω is an appropriate subset of $\{k \mid k \in \mathbb{R}_m$, k_i non-negative integer, $i = 1, \ldots, m\}$ (recall that φ_i has the meaning from (1.2)). As in the case $m = 2$ this decomposition has to be done in such a way that the support of the Fourier image of each sum is controllable.

Because no other new idea is needed we omit the proof of Theorem 4 and refer to [17]. Observe that a restriction to the spaces $\tilde{F}^s_{p,q}$ is not necessary here. The meaning of f^m in S' is always clear since we have $f, f^m \in \bigcup_{1 \leqq p \leqq \infty} L_p$ (this follows from $n \max(0, (1/p) - (1/m)) \geqq n \max(0, (1/p) - 1)$ and from the embedding

$$F^s_{p,q} \hookrightarrow L_t\,, \quad p \leqq t \leqq \frac{n}{(n/p) - s}\,,$$

cf. H. Triebel [21, 2.7], which implies

$$f^m \in L_t\,, \quad \text{where} \quad t = \frac{n}{m((n/p) - s)} > 1 \Big).$$

Remark 18. The restriction of s to the values larger than $n \max(0, (1/p) - (1/m))$ turns out to be optimal. To see this one can again apply the example of the function f_α (cf. (2.9)). Inequality (3.2) is equivalent to $m((n/p) - s) < n$ and $s > 0$. It follows that (3.2) ensures that $(f_\alpha)^m$ is a distribution. If in (3.2) equality holds and $s > 0$ then $(f_\alpha)^m$ is no longer a tempered distribution and hence by definition it is not contained in any $F^s_{p,q}$, $-\infty < s < \infty$, $0 < p < \infty$, $0 < q \leqq \infty$.

Moreover, as is pointed out in [17], the smoothness s_m is optimal if p is fixed.

3.2. Mappings of type $T_\mu: f \to f^\mu$, $\mu > 1$.

Theorem 5. *Let* $\mu > 1$. *Let* $0 < p < \infty$ *and* $0 < s < n/p$.

(i) *Let*

$$\sigma_p < s_\mu < \mu \tag{3.4}$$

(cf. (3.1)). *Then* $T_\mu : f \to f^\mu$ *is a bounded mapping from*

$$\bigcup_{0<q\leqq\infty} \tilde{F}^s_{p,q} \quad \textit{into} \quad \bigcap_{0<r\leqq\infty} F^{s_\mu}_{p,r}\,.$$

Moreover, for any r, $0 < r \leqq \infty$ *there exists a constant* c *such that*

$$\|f^\mu \mid F^{s_\mu}_{p,r}\| \leqq c\|f \mid F^s_{p,\infty}\|^\mu \tag{3.5}$$

for all $f \in \tilde{F}^s_{p,\infty}$.

(ii) *Let* $\max(1, \sigma_p) < \mu < n/p$ *and*

$$s = 1 + \frac{\mu - 1}{\mu}\frac{n}{p}\,.$$

Then $T_\mu : f \to f^\mu$ *is a bounded mapping from* $\bigcup_{0<q\leqq\infty} \tilde{F}^s_{p,q}$ *into* $F^\mu_{p,\infty}$. *Moreover, there exists a constant* c *such that*

$$\|f^\mu \mid F^\mu_{p,\infty}\| \leqq c\|f \mid F^s_{p,\infty}\|^\mu \tag{3.6}$$

for all $f \in \tilde{F}^s_{p,\infty}$.

Remark 19. An easy calculation shows that (ii) represents the limit case of (i), that is

$$\left(1 + \frac{\mu - 1}{\mu}\frac{n}{p}\right)_\mu = \mu$$

(cf. (3.1)). Let μ be not an integer. For the function $g(x) = \psi(x)\, x_1$, $x = (x_1, \ldots, x_n) \in \in \mathbb{R}_n$ one derives by explicit calculations

$$g \in F^s_{p,q}\,, \qquad -\infty < s < \infty\,,\quad 0 < p < \infty\,,\quad 0 < q \leqq \infty\,,$$

$$g^\mu \in F^s_{p,p}\,, \qquad 0 < p < \infty\,,\quad \sigma_p < s < \mu + (1/p)\,,$$

and

$$g^\mu \notin F^{\mu+(1/p)}_{p,p}\,,\quad 0 < p < \infty\,,\quad \sigma_p < \mu + (1/p)$$

by using the characterizations of $F^s_{p,p}$ in terms of differences, cf. H. Triebel [21, 2.5.12, 2.5.13]. The upper bound $\mu + (1/p)$ reflects the smoothness of the outside function $G(t) = t^\mu$ in the framework of the F-spaces. The reason why μ appears in (3.4) instead of the more natural number $\mu + (1/p)$ is to be found in what follows. We are not able to make use of $G(t) \in F^{\mu+(1/p)-\varepsilon,\mathrm{loc}}_{p,p}(\mathbb{R}_1)$, $\varepsilon > 0$. We only apply here the fact that $G(t) \in C^{\mu,\mathrm{loc}}(\mathbb{R}_1)$. Note in this connection that $C^{\mu,\mathrm{loc}}(\mathbb{R}_1) \not\subset F^{\mu+\varepsilon,\mathrm{loc}}_{p,\infty}(\mathbb{R}_1)$ for any $p, \varepsilon > 0$.

Remark 20. Let us look at the condition $\sigma_p < s_\mu$. In the case $0 < p \leqq 1$ it is equivalent to $\mu((n/p) - s) < n$. In view of our example treated in Remark 18 this is necessary. By virtue of the smoothness of f one can easily derive that $s_\mu = = s - (\mu - 1)((n/p) - s)$ is the best possible value. To see this we apply embedding theorems for the F-spaces (cf. H. Triebel [21, 2.7]). From $f \in \tilde{F}^s_{p,q}$, $s > \sigma_p$ we obtain

$$f \in L_t\,,\quad p \leqq t \leqq n/((n/p) - s) \tag{3.7}$$

and consequently

$$f^{\mu} \in L_t\,, \quad \frac{p}{\mu} \leqq t \leqq \frac{n}{((n/p)-s)\,\mu}\,. \tag{3.8}$$

Since

$$F^{s_\mu+\varepsilon}_{p,\infty} \subset L_t\,, \quad t = n \Big/ \left[\mu\left(\frac{n}{p} - s\right) - \varepsilon\right] \quad \text{for} \quad \varepsilon > 0\,,$$

the optimality of s_μ follows from the fact that (3.7) and so (3.8) cannot be improved.

To prove (3.5), (3.6) we need some preliminaries. As usual,

$$M f(x) = \sup \frac{1}{|Q|} \int_Q |f(y)|\, \mathrm{d}y\,,$$

where the supremum is taken over all balls Q with center x, denotes the Hardy-Littlewood maximal function. Furthermore, we put

$$\Delta_h^1 f(x) = f(x+h) - f(x)\,, \quad x, h \in R_n\,.$$

Now we can introduce the following notation. Let $\mu > 0$. Let $f \in L^{\mathrm{loc}}_{\max(1,\mu)}$. Then we define

$$I_k^\mu f(x) := \int_{|z| \leqq 2^{-k}} |\Delta_z^1 f(x)|^\mu \,\mathrm{d}z\,, \quad x \in R_n\,, \quad k = 0, \pm 1, \pm 2, \ldots\,. \tag{3.9}$$

The crucial step in the proof of Theorem 5 is the following lemma.

Lemma 1. *Let* $0 < p < \infty$, $0 < q \leqq \infty$. *Let*

$$\sigma_{p,q} = n \max(0, (1/p) - 1, (1/q) - 1) < s < \mu\,. \tag{3.10}$$

Then there exists a constant c such that

$$\Big\|\Big(\sum_{k=-\infty}^{\infty} 2^{k(n+s)q} |I_k^\mu f(x)|^q\Big)^{1/q} \Big| L_p\Big\| \leqq c \|f \mid F^{s/\mu}_{p\mu,q\mu}\|^\mu \tag{3.11}$$

for all $f \in L^{\mathrm{loc}}_{\max(1,\mu)}$ *with*

$$f = \sum_{j=0}^{\infty} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f] \quad \text{in} \quad L^{\mathrm{loc}}_\mu \tag{3.12}$$

($\{\varphi_j\}_j$ *is defined in* (1.2)*).*

Proof. Step 1. We start with some basic inequalities. Let k be a natural number and m an integer. Let α be a multiindex of length n. Let $\{\varphi_k\}_{k=0}^\infty$ denote the system defined in (1.2). Put $\varphi_k(x) \equiv 0$ if $k = -1, -2, \ldots$. Then for all $x, h \in R_n$, $|h| \leqq 2^{-k}$ and all $m < 0$ we have

$$\begin{aligned} |\Delta_h^1(\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f](x)| &\leqq 2^{-k} \sup_{|x-y| \leqq 2^{-k}} \sum_{|\alpha|=1} |\partial^\alpha(\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(y)| \leqq \\ &\leqq c_a 2^{-k} \sum_{|\alpha|=1} \sup_{y \in R_n} \frac{|(\mathscr{F}^{-1}[\varphi_j \mathscr{F} f])(y)|}{|1 + |2^j y|^a}\,, \quad a > 0\,. \end{aligned} \tag{3.13}$$

In view of this formula we introduce the maximal function

$$(\varphi_j^* f)(x) := \sup_{y \in R_n} \frac{|(\mathscr{F}^{-1}[\varphi_j \mathscr{F} f])(y)|}{1 + |2^j y|^a} \tag{3.14}$$

for any $f \in S'$, where a is a real positive number which will be fixed later on. Using well-known properties of the Fourier transform and a homogeneity argument, we obtain from the estimate (3.13)

$$|\Delta_h^1(\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)| \leqq c'_a 2^{-k} 2^{k+m}(\varphi^*_{k+m}f)(x) . \tag{3.15}$$

Here c' is independent of f, x, k and m.

If $m > 0$ and $|h| \leqq 2^{-k}$ we obtain

$$|\Delta_h^1(\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)| \leqq 2 \sup_{|x-y| \leqq 2^{-k}} |\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f](y)| \leqq \tag{3.16}$$

$$\leqq 2^{ma+1}\left[\sup_{y \in R_n} \frac{|\mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f](y)|}{1 + |2^{k+m}(x-y)|^a}\right] \leqq 2^{ma+1}(\varphi^*_{k+m}f)(x) .$$

Step 2. Let $0 < q < \infty$. By assumption (3.12) we have

$$\|(\sum_{k=-\infty}^{\infty} 2^{k(n+s)q}|I_k^\mu f(x)|^q)^{1/q} | L_p\| = \tag{3.17}$$

$$= \|(\sum_{k=-\infty}^{\infty} 2^{ksq}(\int_{|z| \leqq 1} |\Delta^1_{2^{-k}z}(\sum_{m=-\infty}^{\infty} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)|^\mu \, dz)^q)^{1/q} | L_p\| \leqq$$

$$\leqq c[\|(\sum_{k=-\infty}^{\infty} 2^{ksq}(\sup_{|z| \leqq 1} |\Delta^1_{2^{-k}z}(\sum_{m=-\infty}^{-1} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)|^\mu)^q)^{1/q} | L_p\| +$$

$$+ \|(\sum_{k=-\infty}^{\infty} 2^{ksq}(\int_{|z| \leqq 1} |\Delta^1_{2^{-k}z}(\sum_{m=0}^{\infty} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)|^\mu \, dz)^q)^{1/q} | L_p\| = c[T_1 + T_2] .$$

Substep 2.1. Estimate of T_1. Let $d = \min(1, p\mu, q\mu)$. We shall apply (3.15) in connection with the maximal inequality

$$\|\varphi_j^* f | L_{\bar{p}}(l_{\bar{q}})\| \leqq c\|\mathscr{F}^{-1}[\varphi_j\mathscr{F}f](\cdot) | L_{\bar{p}}(l_{\bar{q}})\| \tag{3.18}$$

for any $f \in S'$, any $a > n/\min(\bar{p}, \bar{q})$ (cf. H. Triebel [21, 2.3.6], in the case $\bar{p} = p\mu$, $\bar{q} = q\mu$). This leads to

$$T_1^d = \|(\sum_{k=-\infty}^{\infty} 2^{k(s/\mu)q\mu}(\sup_{|z| \leqq 1} |\Delta^1_{2^{-k}z}(\sum_{m=-\infty}^{-1} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)|^{q\mu})^{1/q\mu} | L_{p\mu}\|^{\mu d} \leqq \tag{3.19}$$

$$\leqq c(\sum_{m=-\infty}^{-1} 2^{md}\|\sum_{k=-\infty}^{\infty} 2^{k(s/\mu)q\mu}(\varphi^*_{k+m}f)^{q\mu}(x))^{1/q\mu} | L_{p\mu}\|^d)^\mu \leqq$$

$$\leqq c(\sum_{m=-\infty}^{-1} 2^{md(1-s/\mu)}\|2^{js/\mu}\varphi_j^* f | L_{p\mu}(l_{q\mu})\|^d)^\mu \leqq c'\|f | F^{s/\mu}_{p\mu,q\mu}\|^{\mu d}$$

since $s < \mu$, and again c' does not depend on f.

Substep 2.2. Estimate of T_2. Let $d = \min(1, p, q)$. Since

$$s > \sigma_{p,q} = n\left(\frac{1}{\min(1, p, q)} - 1\right) \geqq \frac{n}{\min(p, q)}(1 - \min(1, p, q))$$

we can choose three numbers λ, a and ε such that

$$0 < \lambda < \min(1, p, q) , \tag{3.20}$$

$$a > \frac{n}{\mu \min(p, q)},$$
$$\mu(s/\mu - a(1 - \lambda)) \geqq 2\varepsilon > 0 .$$

We start our estimate of T_2 by using (3.16):

$$\int_{|z| \leqq 1} |\Delta^1_{2^{-k}z}(\sum_{m=0}^{\infty} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)|^\mu \, dz \leqq$$
$$\leqq \sum_{m=0}^{\infty} 2^{m\varepsilon} \int_{|z| \leqq 1} |\Delta^1_{2^{-k}z} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f](x)|^\mu \, dz \leqq$$
$$\leqq \sum_{m=0}^{\infty} 2^{m\varepsilon} (\int_{|z| \leqq 1} |\Delta^1_{2^{-k}z} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f](x)|^{\mu\lambda} \, dz \, .$$
$$. \sup_{|z| \leqq 1} |\Delta^1_{2^{-k}z} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f](x)|^{\mu(1-\lambda)}) \leqq$$
$$\leqq c \sum_{m=0}^{\infty} 2^{m\varepsilon} (M \mid \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f]^{\mu\lambda})(x) \, 2^{ma(1-\lambda)\mu} (\varphi_{k+m} * f)^{\mu(1-\lambda)}(x) \, .$$

Repeated use of Hölder's inequality with

$$\frac{1}{q\mu} = \frac{\lambda}{q\mu} + \frac{1-\lambda}{q\mu}, \quad \frac{1}{p\mu} = \frac{\lambda}{p\mu} + \frac{1-\lambda}{p\mu}$$

yields

$$T_2^d \leqq c \sum_{m=0}^{\infty} 2^{md(\varepsilon + a(1-\lambda)\mu)} \| \{ (M \mid 2^{ks/\mu} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f]|^{\mu\lambda})(\cdot) \, .$$
$$. (2^{ks/\mu} \varphi_{k+m} * f)(\cdot) \}_{k=-\infty}^{\infty} \mid L_p(l_q) \|^d \leqq$$
$$\leqq c \sum_{m=0}^{\infty} 2^{md(\varepsilon - \mu(s/\mu - a(1-\lambda)))} \| \{ (M \mid 2^{ks/\mu} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f]|^{\mu\lambda})(\cdot) \}_{k=-\infty}^{\infty} \mid L_{p/\lambda}(l_{q/\lambda}) \|^d \, .$$
$$. \| \{ 2^{ks/\mu} \varphi_k * f \}_{k=-\infty}^{\infty} \mid L_{p\mu}(l_{q\mu}) \|^{(1-\lambda)\mu d} \, .$$

Since $\min(p/\lambda, q/\lambda) > 1$ and $a > n/\min(p\mu, q\mu)$ (cf. (3.19)) we can apply the vector-valued Hardy-Littlewood maximal inequality (cf. Ch. Fefferman, E. M. Stein [5]) to $L_{p/\lambda}(l_{q/\lambda})$ and the maximal inequality (3.18) with $\bar{p} = p\mu$ and $\bar{q} = q\mu$. This combined with (3.19) leads to

$$(3.21) \quad T_2 \leqq c' \| (2^{ks/\mu} \mathscr{F}^{-1}[\varphi_k \mathscr{F}f]^{\mu\lambda} \mid L_{p/\lambda}(l_{q/\lambda}) \| \; \| 2^{ks/\mu} \mathscr{F}^{-1}[\varphi_k \mathscr{F}f] \mid L_{p\mu}(l_{q\mu}) \|^{(1-\lambda)\mu} \leqq$$
$$\leqq c' \| f \mid F^{s/\mu}_{p\mu,q\mu} \|^{\lambda\mu} \; \| f \mid F^{s/\mu}_{p\mu,q\mu} \|^{(1-\lambda)\mu} \leqq c' \| f \mid F^{s/\mu}_{p\mu,q\mu} \|^\mu \, .$$

Putting (3.21) and (3.19) into (3.17) proves the lemma in the case $0 < q < \infty$. The case $q = \infty$ follows by obvious modifications.

The limiting case $s = \mu$ is also of some interest. Looking for the proof we easily see that $s < \mu$ is needed only in Substep 2.1 to derive formula (3.19). The counterpart of (3.19) is now given by

$$\| \sup_{k=0,\pm 1,\ldots} 2^{k\mu} \sup_{|z| \leqq 1} |\Delta^1_{2^{-k}z}(\sum_{m=-\infty}^{-1} \mathscr{F}^{-1}[\varphi_{k+m}\mathscr{F}f])(x)|^\mu \, |L_p\| \leqq$$
$$\leqq c \| \sup_{k=0,\pm 1,\ldots} \sum_{m=-\infty}^{-1} 2^k \, 2^m (\varphi^*_{k+m} f)(x) \mid L_{p\mu} \|^\mu \leqq$$
$$\leqq c' \| 2^j (\varphi^*_j f) \mid L_{p\mu}(l_1) \|^\mu \leqq c'' \| f \mid F^1_{p\mu,1} \|^\mu \, .$$

Hence, we have proved another lemma:

Lemma 2. *Let $0 < p < \infty$ and $\sigma_p < \mu$. Then there exists a constant c such that*

$$\| \sup_{k=0,\pm 1,\dots} 2^{(\mu+n)k} |I_k^\mu f(x)| \, |L_p\| \leqq c \|f \mid F^1_{p\mu,1}\|^\mu \tag{3.22}$$

for all $f \in L^{\text{loc}}_{\max(1,\mu)}$ with $f = \sum_{j=0}^{\infty} \mathscr{F}^{-1}[\varphi_j \mathscr{F} f]$ in L^{loc}_μ.

Remark 21. The use of maximal functions of the type (3.14) is standard in the modern theory of function spaces provided $p < 1$. They were introduced by Ch. Fefferman, E. M. Stein [6], J. Peetre [13], and afterwards extensively used by H. Triebel, cf. [21]. Furthermore, in our proof we have sometimes used the ideas of H. Triebel [21, 2.5.10 and 2.5.12]. So, the basic formulas (3.15) and (3.16) may be found in his paper.

Remark 22. The importance of $I_k^\mu f$ in connection with the mapping properties of the superposition operators was pointed out by Th. Runst [16]. There he gave estimates of $I_k^\mu f$ in the case of bounded functions.

Proof of Theorem 5. Step 1. We collect some preliminaries. Let μ have the decomposition $\mu = N + \tau$, where N is a positive integer and $0 < \tau \leqq 1$. Then $G(t) = t^\mu$ has the Taylor expansion

$$t^\mu = \sum_{j=0}^{N-1} \frac{\mu(\mu-1)\dots(\mu-j+1)}{j!} w^{\mu-j}(t-w)^j + \\ + \frac{1}{(N-1)!} \int_w^t (v-w)^{N-1} \mu(\mu-1)\dots(\mu-N+1) v^\tau \, dv$$

for any pair $(t, w) \in \mathbb{R}_1 \times \mathbb{R}_1$.

For a function $f \in \tilde{F}^s_{p,q}$, $s > \sigma_p$ there exists a set E of Lebesgue-measure zero such that $|f(x)| < \infty$ for all $x \in \mathbb{R}_n \setminus E$. Since we can change our function f on a set of measure zero without changing the distribution f we suppose $|f(x)| < \infty$ for any $x \in \mathbb{R}_n$. Hence, for any pair $(x, y) \in \mathbb{R}_n \times \mathbb{R}_n$ we have

$$f(x)^\mu = \sum_{l=0}^{N-1} \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \frac{\mu(\mu-1)\dots(\mu-l+1)}{l!} f(y)^{\mu-l} f(x)^j f(y)^{l-j} + \\ + \frac{1}{(N-1)!} \int_{f(y)}^{f(x)} (v - f(y))^{N-1} \mu(\mu-1)\dots(\mu-N+1) v^\tau \, dv \, .$$

Recall that $\{\varphi_k\}_{k=0}^\infty$ is the system defined in (1.2). The following identity is the starting point of our calculations.

$$\mathscr{F}^{-1}[\varphi_k \mathscr{F} f](y) = \tag{3.23}$$

$$= (2\pi)^{-n/2} \int_{R^n} (\mathscr{F}^{-1}\varphi_k)(y-x) \left\{ \sum_{l=0}^{N-1} \sum_{j=0}^{l} \binom{l}{j} (-1)^{l-j} \frac{\mu(\mu-1)\dots(\mu-l+1)}{l!} f(y)^{\mu-j} f(x)^j + \right.$$

$$+ \frac{1}{(N-1)!} \int_{f(y)}^{f(x)} (v - f(y))^{N-1} \mu(\mu-1)\dots(\mu-N+1) v^{\tau} \, dv \Big\} \, dx =$$

$$= (2\pi)^{-n/2} \Big(\sum_{l=0}^{N-1} \sum_{j=0}^{l} T_{1,k,l,j}(y) + T_{2,k}(y) \Big), \quad y \in R_n .$$

Step 2. Estimate of $T_{1,k,l,j}$. Let $r > 0$. Let $s_\mu < \mu$. Then we choose a number $\bar{s}$ with $s_\mu < \bar{s} < \min(\mu, s_N)$ and define

$$\bar{p} = \frac{n}{\bar{s} - s_\mu - n/p}$$

This ensures the continuous embeddings

$$F^{\bar{s}}_{\bar{p},\infty} \hookrightarrow \bigcap_{0<r\leqq\infty} F^{s_\mu}_{p,r} \tag{3.24}$$

and

$$F^{s_j}_{p,\infty} \hookrightarrow \bigcap_{0<r\leqq\infty} F^{\bar{s}}_{p_2,r} \quad \text{with} \quad p_2 = \frac{n}{(n/\bar{p}) - (\mu - j)((n/p) - s)}, \quad j = 1, \dots, N \tag{3.25}$$

(cf. H. Triebel [21, 2.7]).

Let $0 < j \leqq N$. Obviously $s_\mu < s$ and hence $\sigma_p < s$. By (3.8) we can apply Hölder's inequality with

$$\frac{1}{\bar{p}} = \frac{(\mu-j)((n/p)-s)}{n} + \frac{(n/\bar{p}) - (\mu-j)((n/p)-s)}{n} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{3.26}$$

This yields

$$\|\{2^{k\bar{s}} T_{1,k,l,j}\}_{k=0}^{\infty} \mid L_{\bar{p}}(l_\infty)\| \leqq \tag{3.27}$$

$$\leqq c \|f^{\mu-j} \mid L_{p_1}\| \cdot \|\{2^{k\bar{s}} \int (\mathscr{F}^{-1}\varphi_k)(y-x) f(x)^j \, dx\}_{k=0}^{\infty} \mid L_{p_2}(l_\infty)\| .$$

Observe that the embedding $F^s_{p,\infty} \hookrightarrow L_{n/((n/p)-s)}$ is continuous and hence

$$\|f^{\mu-j} \mid L_{p_1}\| \leqq c \|f \mid F^s_{p,\infty}\|^{\mu-j} . \tag{3.28}$$

On the other hand, by Theorem 4 and the continuous embedding (3.25) we have

$$\|\{2^{ks}\mathscr{F}^{-1}[\varphi_k \mathscr{F} f^j]\}_{k=0}^{\infty} \mid L_{p_2}(l_\infty)\| = \|f^j \mid F^{\bar{s}}_{p_2,\infty}\| \leqq c \|f \mid F^s_{p,\infty}\|^j . \tag{3.29}$$

Now, (3.28) and (3.29) together with (3.27) yield

$$\|\{2^{k\bar{s}} T_{1,k,l,j}\}_{k=0}^{\infty} \mid L_{\bar{p}}(l_\infty)\| \leqq c \|f \mid F^s_{p,\infty}\|^\mu \quad \text{if} \quad 0 < j \leqq N . \tag{3.30}$$

In the case $j = 0$ we have to use some properties of our system $\{\varphi_k\}_{k=0}^{\infty}$ in connection with the well-known formula $c\mathscr{F} 1 = \delta$ (Dirac distribution). We obtain

$$\int (\mathscr{F}^{-1}\varphi_k)(y-x) \, dx = 0 \quad \text{if} \quad k = 1, 2, \dots$$

and

$$\int (\mathscr{F}^{-1}\varphi_0)(y-x) \, dx = c_{\varphi_0} .$$

Moreover, observe that $p \leqq \mu\bar{p}$ for an appropriate choice of $\bar{s}$ in a small neighbourhood of s_μ. This together with $s_\mu > \sigma_p$ and $F^s_{p,\infty} \hookrightarrow L_t$, $p \leqq t \leqq n/((n/p) - s)$ (cf. H. Triebel [21, 2.7]) ensures the continuous embedding $F^s_{p,\infty} \hookrightarrow L_{\bar{p}\mu}$. Applying this one obtains

$$(3.31)\qquad \|\{T_{1,k,l,0}\}_{k=0}^{\infty} \mid L_{\bar{p}}(l_\infty)\| = c_{\varphi 0}\|f^\mu \mid L_{\bar{p}}\| = \\ = \|f \mid L_{\bar{p}\mu}\|^\mu \leqq c\|f \mid F^s_{p,\infty}\|^\mu .$$

Step 3. Estimate of $T_{2,k}$. We follow Th. Runst [16]. For any τ, $0 < \tau \leqq 1$ we have

(3.32)

$$\int_{f(y)}^{f(x)} v^\tau (v - f(y))^{N-1}\, dv = f(y)^\tau \frac{(f(x) - f(y))^N}{N} + \int_{f(y)}^{f(x)} (v - f(y))^{N-1}(v^\tau - f(y)^\tau)\, dv .$$

Note that

$$(3.33)\qquad |v^\tau - w^\tau| \leqq 2|v - w|^\tau$$

holds for arbitrary $v, w \in \mathbb{R}_1$. Applying both (3.32) and (3.33), we obtain

$$(3.34)\qquad |T_{2,k}(y)| \leqq c\{|\int (\mathscr{F}^{-1}\varphi_k)(y - x) f(y)^\tau (f(x) - f(y))^N\, dx| + \\ + \int |(\mathscr{F}^{-1}\varphi_k)(y - x)|\, |f(x) - f(y)|^\mu\, dx\} .$$

The first summand can be handled as in Step 2. For the estimate of the second we apply the formula $\varphi_k(\cdot) = \varphi_1(2^{-k+1}\,\cdot)$, $k = 1, 2, \dots$, cf. (1.2). Since $\varphi_1 \in S$ this yields

$$|(\mathscr{F}^1\varphi_{k+1})(z)| = 2^{kn}|(\mathscr{F}^{-1}\varphi_1)(2^k z)| \leqq c_M(1 + |2^k z|)^{-M}\, 2^{kn},$$

where M is an arbitrary positive number and c_M an appropriate constant independent of $k = 0, 1, \dots$ and $z \in \mathbb{R}_n$. By means of this inequality we find

$$(3.35)\qquad \int |(\mathscr{F}^{-1}\varphi_k)(z)|\, |f(y) - f(z + y)|^\mu\, dz \leqq \\ \leqq c_M\Big(\int_{|z| \leqq 2^{-k}} 2^{kn}|f(y) - f(z + y)|^\mu\, dz + \\ + \sum_{l=0}^{\infty} \int_{2^{-k+l} \leqq |z| \leqq 2^{-k+l+1}} 2^{kn - lM}|f(y) - f(z + y)|^\mu\, dz\Big) \leqq \\ \leqq c'_M \sum_{l=0}^{\infty} 2^{kn - lM}(I^\mu_{k-l} f)(y) .$$

Let $d = \min(1, \bar{p})$. With the help of Lemma 1 the following estimate is derived from (3.35):

$$(3.36)\qquad \|\{2^{k\bar{s}} \int |(\mathscr{F}^{-1}\varphi_k)(z)|\, |f(y) - f(y + z)|^\mu\, dz\}_{k=0}^{\infty} | L_{\bar{p}}(l_\infty)\|^d \leqq \\ \leqq c \sum_{l=0}^{\infty} 2^{-lMd}\|\{2^{k(n+\bar{s})} I^\mu_{k-l} f\}_{k=0}^{\infty} \mid L_{\bar{p}}(l_\infty)\|^d \leqq \\ \leqq c \sum_{l=0}^{\infty} 2^{-ld(M - n - \bar{s})}\|\{2^{k(n+\bar{s})} I^\mu_k f\}_{k=-\infty}^{\infty} \mid L_{\bar{p}}(l_\infty)\|^d \leqq c'\|f \mid F^{\bar{s}/\mu}_{\bar{p}\mu,\infty}\|^{\mu d} ,$$

since M can be chosen arbitrarily large. It remains to check whether the assumptions of Lemma 1 can be satisfied. Since $s_\mu > \sigma_p$ we have $\mu(n/p - s) < n/\min(1, p)$. By the continuous embedding mentioned at the end of Step 2 we know that $f \in L_{n/((n/p)-s)}$ and hence by the above inequality also $f \in L^{\mathrm{loc}}_\mu$. Moreover, $\mu > 1$ implies $F^{0,\mathrm{loc}}_{\mu,2} = L^{\mathrm{loc}}_\mu$ and the identity (3.12) appears now as the so-called Nikol'skij representation of f in $F^0_{\mu,2}$ (cf. H. Triebel [21, 2.5.2]). Finally, consider (3.10). We have to guarantee

$$\sigma_{\bar{p},\infty} = n \max(0, (1/p) - 1) < \bar{s},$$

but this is equivalent to $s_\mu > \sigma_p$.

To complete the calculations in this step observe the continuous embedding

$$F^s_{p,\infty} \hookrightarrow F^{\bar{s}/\mu}_{\bar{p}\mu,\infty} \tag{3.37}$$

which follows from

$$s - \frac{n}{p} = \frac{1}{\mu}\left(s_\mu - \frac{n}{p}\right) = \frac{\bar{s}}{\mu} - \frac{n}{\bar{p}\mu}.$$

As a consequence of (3.34), (3.36) and (3.37) one obtains

$$\|2^{k\bar{s}} T_{2,k} \mid L_{\bar{p}}(l_\infty)\| \leqq c \|f \mid F^s_{p,\infty}\|^\mu.$$

This combined with (3.30)–(3.31) implies

$$\|f^\mu \mid F^{\bar{s}}_{\bar{p},\infty}\| \leqq c \|f \mid F^s_{p,\infty}\|^\mu, \tag{3.38}$$

whenever s is chosen in a sufficiently small neighbourhood of s_μ. But $F^{\bar{s}}_{\bar{p},\infty} \hookrightarrow$ $\hookrightarrow \bigcap_{0 \leqq r < \infty} F^{s_\mu}_{p,r}$ and so (3.35) follows from (3.38).

Step 4. The proof of (ii) is principally the same as that of part (i). We only have to apply Lemma 2 instead of Lemma 1 since $s_\mu = \mu$. The proof is complete.

Remark 23. We have proved more than is stated in Theorem 5 (i). We have shown in (3.38) that f is contained in a proper subspace of $\bigcap_{0<r\leqq\infty} F^{s_\mu}_{p,r}$ since $\bigcup_{\bar{p}<p} F^{\bar{s}}_{\bar{p},\infty}$, where $\bar{s} > s_\mu$ and

$$\bar{p} = n/(\bar{s} - s_\mu + (n/p))$$

is a proper subspace of $\bigcap_{0<r\leqq\infty} F^{s_\mu}_{p,r}$. One can ask for the best possible choice of $\bar{s}$. Ommitting details we observe that it can be $s = \bar{s}$ under stronger assumptions on s and p. Results in this direction in the case $\mu = m$, $m = 2, 3, \ldots$ may be found in [17]. We return to this problem in Remark 28.

Let us look again at the proof of Theorem 5 and let us collect the properties of $G(t) = t^\mu$, $\mu > 1$ applied there.

First, we have used the fact that G admits a Taylor expansion of order N where N is an integer with $0 < \mu - N \leqq 1$, or equivalently $G\colon \mathbb{R}_1 \to \mathbb{R}_1$ is an N-times continuously differentiable function. Secondly, certain growth behaviour of G and of its derivatives, namely

$$|G^{(l)}(t)| \leqq c_l |t|^{\mu - l}, \quad t \in R_1, \quad l = 0, 1, \ldots, N, \tag{3.39}$$

c_l appropriate constants, was applied in (3.28) and (3.31). Finally, the Hölder continuity of order $\tau = \mu - N$ of the N-th derivative of G,

$$\sup_{t_1, t_0 \in R_1} \frac{|G^{(N)}(t_0) - G^{(N)}(t_1)|}{|t_0 - t_1|^\tau} \leqq c < \infty, \tag{3.40}$$

was used in (3.33). If

$$G^{(l)}(0) = 0\,, \quad l = 0, 1, \ldots, N\,, \tag{3.41}$$

then (3.39) is a consequence of (3.40). Hence, we come to the following result.

Theorem 6. *Let N be a natural number. Let $G\colon \mathbb{R}_1 \to \mathbb{R}_1$ be an N-times continuously differentiable function satisfying (3.40) and (3.41) with some τ, $0 < \tau \leqq 1$. Put $\mu = N + \tau$. Then Theorem 5 remains valid with $T_G\colon f \mapsto G(f)$ instead of T_μ.*

Remark 24. The smoothness of G and of its derivatives ensures the Lebesgue measurability of $G(f)$. Moreover, since $F^s_{p,\infty} \subset L_{n/((n/p)-s)}$ and $s_\mu > \sigma_p$, we have $G(f) \in \bigcup\limits_{1 \leqq p \leqq \infty} L_p \subset S'$. So we can interpret T_G as a mapping of a subspace of S' into S'.

Remark 25. The restriction to real-valued functions appears in connection with (3.33). This inequality holds no longer true if one replaces v, $w \in \mathbb{R}_1$ by z_0, $z_1 \in \mathbb{C}$. However, in the case of the function $G(z) = |z|^\mu$, $z \in \mathbb{C}$ all the above considerations are meaningful, the counterpart of (3.33) holds true also in the complex case, and so Theorem 6 is valid for the whole space $F^s_{p,q}$ instead of its real-valued part.

Remark 26. Of course, Theorem 5 or 6 can be extended also to the values $s > n/p$. Results in this direction may be found for instance in Th. Runst [16] and the references given there. Mappings of the type $f \to |f|^\mu$, $\mu > 0$ are studied also in M. Marcus, V. J. Mizel [8], ($\mu \geqq 1$), H. Triebel [22], D. E. Edmunds, H. Triebel [4] and Th. Runst [16]. The very interesting case $\mu = 1$ has been considered only in spaces of smoothness $s \leqq 1$. Nonetheless, in view of G. Bourdaud's result stated in Remark 15 it is at least meaningful to ask whether the mapping $f \to |f|$ is bounded in spaces with smoothness $s < 1 + 1/p$.

3.3. Proof of Theorem 1. In order to make the proof more transparent we prove at first a weaker assertion than that stated in Theorem 1. Recall that

$$\varrho = \frac{n/p}{(n/p) - s + 1}\,.$$

Lemma 3. *Let $0 < p < \infty$, $0 < q \leqq \infty$ and*

$$1 < s < n/p\,. \tag{3.42}$$

Furthermore, let

$$\varrho > \sigma_p\,. \tag{3.43}$$

Let $G\colon \mathbb{R}_1 \to \mathbb{R}_1$ be a function with $G(0) = 0$ and $G \in C^\infty(\mathbb{R}_1)$. Then $T_G\colon f \to G(f)$ is a bounded mapping of $\tilde{F}^s_{p,q}$ into $F^\varrho_{p,\infty}$. Moreover, there exists a constant c such that

$$\|G(f) \mid F^\varrho_{p,\infty}\| \leqq c(\|f \mid F^s_{p,q}\| + \|f \mid F^s_{p,q}\|^\varrho) \tag{3.44}$$

for all $f \in \tilde{F}^s_{p,q}$.

Proof. Step 1. First we shall study the mapping properties of T_H, where

$$H(t) = G(t) - \sum_{j=1}^{N} \frac{G^{(j)}(0)}{j!} t^j , \quad t \in \mathbb{R}_1 ,$$

and N is a positive integer which will be fixed later on. Let s be fixed. Then we choose

$$\mu = \varrho = \frac{n/p}{(n/p) - s + 1} .$$

This implies

$$s = 1 + \frac{\mu - 1}{\mu} \frac{n}{p} .$$

Consequently, by the assumptions (3.42) and (3.43) we obtain

$$\max(1, \sigma_p) < \mu < n/p . \tag{3.45}$$

Now we choose N such that $0 < \mu - N \leqq 1$.

Obviously, H fulfils the assumptions of Theorem 6 for μ. Hence we obtain

$$\|H(f) \mid F^{\mu}_{p,\infty}\| \leqq c\|f \mid F^{s}_{p,\infty}\|^{\mu} \tag{3.46}$$

by applying the analogue of Theorem 5 (ii).

Step 2. Let us consider the remainder. It is a polynomial of order N. Therefore we make use of Theorem 4. Assumption (3.43) implies (3.2) for any m, $m \leqq N$, because (3.2) is equivalent to $m((n/p) - s) < n$, $s > 0$, and

$$\sigma_p < \varrho \ \text{ iff } \ \varrho((n/p) - s) < \frac{n}{\min(1, p)}$$

(use $\varrho = n/p - \varrho((n/p) - s)$). This implies

$$\begin{aligned} \|G(f) - H(f) \mid F^{\varrho}_{p,\infty}\| &= \left\| \left(\sum_{j=1}^{N} \frac{G^{(j)}(0)}{j!} t^j \right)(f) \mid F^{\varrho}_{p,\infty} \right\| \leqq \\ &\leqq c \sum_{j=1}^{N} \frac{|G^{(j)}(0)|}{j!} \|f^j \mid F^{\varrho}_{p,\infty}\| \leqq c' \sum_{j=1}^{N} \|f^j \mid F^{sj}_{p,\infty}\| \leqq \\ &\leqq c''(\|f \mid F^{s}_{p,\infty}\| + \|f \mid F^{s}_{p,\infty}\|^N) . \end{aligned} \tag{3.47}$$

Now, (3.46) and (3.47) yield (3.44).

Remark 27. If one compares Lemma 3 and Theorem 1 then the fact that the lower bound 1 for s in (3.42) is better than that in (2.2), at least if case $p < 1$, is of certain interest.

Proof of Theorem 1. Our assumptions ensure

$$\frac{\partial}{\partial x_j} G(f(x)) = G^{(1)}(f(x)) \frac{\partial f}{\partial x_j}(x) \quad \text{in} \quad S' , \quad f \in \tilde{F}^{s}_{p,\infty} , \tag{3.48}$$

since the right-hand side belongs to L_1^{loc} and $G \in C^1(\mathbb{R}_1)$ (cf. M. Marcus, V. J. Mizel [7]).

Now, we use a lifting property of the underlying spaces. Since $\varrho > \sigma_p$, we have for any $q > 0$

$$G(f) \in F^{\varrho}_{p,q} \quad \text{iff} \quad G(f) \in F^{\varrho}_{p,\infty} \quad \text{and} \quad \frac{\partial}{\partial x_j} G(f) \in F^{\varrho-1}_{p,q}\,, \quad j = 1, \ldots, n$$

(this is a small modification of Theorem 2.3.8 combined with formula (2.5.9/37) of H. Triebel [21]), and also the appropriate quasi-norms are equivalent.

Applying Lemma 3 to $G^{(1)}(f(x))$ we obtain $G^{(1)}(f) \in \tilde{F}^{\varrho}_{p,\infty}$. Moreover, since $G \in C^\infty(\mathbb{R}_1)$, we have $G^{(1)}(f) \in \tilde{F}^{\varrho}_{p,\infty} \cap L_\infty$. Thus, the right-hand side of (3.48) can be interpreted as a product of type $h \,.\, g$, where $h \in \tilde{F}^{\varrho}_{p,\infty} \cap L_\infty$, $0 < \varrho < n/p$ and $g \in F^{s-1}_{p,q}$, $\sigma_p < s - 1 < n/p$.

Such products have been studied in [17]. Theorem 7 of this paper leads to

$$\left\| G^{(1)}(f) \frac{\partial f}{\partial x_j} \,\middle|\, F^{\varrho-1}_{p,q} \right\| \leqq c(\|G^{(1)}(f) \mid L_\infty\| +$$
$$+ \|G^{(1)}(f) \mid L_\infty\|^{1-p(s-1)/n} \|G^{(1)}(f) \mid F^{\varrho}_{p,\infty}\|^{p(s-1)/n}) \left\| \frac{\partial f}{\partial x_j} \,\middle|\, F^{s-1}_{p,\infty} \right\|$$

for any q, $0 < q \leqq \infty$.

In view of (3.44) and of the above stated equivalence this implies

$$\|G(f) \mid F^{\varrho}_{p,q}\| \leqq c\left(\|G(f) \mid F^{\varrho}_{p,\infty}\| + \sum_{j=1}^{n} \left\| \frac{\partial}{\partial x_j} G(f) \,\middle|\, F^{\varrho-1}_{p,q} \right\|\right) \leqq$$
$$\leqq c'(\|f \mid F^{s}_{p,\infty}\| + \|f \mid F^{s}_{p,\infty}\|^{\varrho} +$$
$$+ \sum_{j=1}^{n} (\|f \mid F^{s}_{p,\infty}\| + \|f \mid F^{s}_{p,\infty}\|^{\varrho})^{p(s-1)/n} \|f \mid F^{s}_{p,\infty}\|) \leqq$$
$$\leqq c''(\|f \mid F^{s}_{p,\infty}\| + \|f \mid F^{s}_{p,\infty}\|^{\varrho})$$

since $\varrho(s-1)(p/n) + 1 = \varrho$ and $1 < 1 + (s-1)\,p/n < \varrho$. The proof is complete.

Remark 28. Similarly as in Remark 23, even in the case of operators $T_G: f \to G(f)$, $G \in C^\infty(\mathbb{R}_1)$ one can look for an improvement of Lemma 3 or Theorem 1. We know that for $f \in \tilde{F}^{s}_{p,\infty}$ the best possible space with first lower index p containing the composition $G(f)$ is $\bigcap_{0<r\leqq\infty} F^{\varrho}_{p,r}$. Now, it is meaningful to ask whether it is possible to find $G(f)$ in a space $F^{\bar{s}}_{\bar{p},q}$, where $\bar{s} > \varrho$ and $\bar{s} - (n/p) = \varrho - (n/p)$ (cf. H. Triebel [21, 2.7]). Consequently, a necessary condition for such an improvement is $G(f) \in L_{\bar{p}}$. However, this is not true in general. One needs further assumptions. The easest way is to assume $\operatorname{supp} f$ to be compact. Then one can follow the method described in this section respecting this new point of view. The crucial point is an improvement of Theorem 4. For $m = 2$ this can be found in [17]. The result is as follows. Let $0 < p < \infty$, $0 < q \leqq \infty$ and $1 < s < n/p$. Let $s((n/p) - s) < n$. Let $G \in C^\infty(\mathbb{R}_1)$ and $G(0) = 0$.

Let $f \in \tilde{F}^s_{p,\infty}$ and let $\operatorname{supp} f$ be compact. Then $G(f) \in F^s_{t,\infty}$ and

$$(3.49) \qquad \|G(f) \mid F^s_{t,\infty}\| \leqq c(\|f \mid F^s_{p,\infty}\| + \|f \mid F^s_{p,\infty}\|^s), \quad t = \frac{n}{s((n/p) - s + 1)}.$$

However, the constant c depends on the volume of $\operatorname{supp} f$.

3.4. Proof of Theorems 2 and 3. Proof of Theorem 2. As remarked above our counterexample goes back to B. E. J. Dahlberg [3] and was later used also by G. Bourdaud [2] and Th. Runst [16]. The proof of Theorems 2 and 3 is a refinement of G. Bourdaud's proof, added here more or less for completeness.

Step 1. Properties of the outside function G. There exist real numbers a, b, A with $0 \leqq a < b < T$ and $A > 0$ such that $|G^{(m+1)}(t)| \geqq A$ if $t \in [a, b]$. By the mean-value theorem we have $|G^m(t_0) - G^{(m)}(t)| \geqq A|t - t_0|$ if $t, t_0 \in [a, b]$. Since G is T-periodic we obtain

$$(3.50) \qquad |G^{(m)}(t_0) - G^{(m)}(t)| \geqq A|t - t_0|$$

if there exists an integer j with $t, t_0 \in [a + jT, b + jT]$.

We put $a_j = a + jT$ and $b_j = b + jT$.

Step 2. Construction of the inside part of the superposition. Let u be a real-valued, infinitely differentiable and compactly supported function with

$$(3.51) \qquad \begin{aligned} u(x_1, \ldots, x_n) = u(x) &= x_1 \quad \text{if} \quad |x| \leqq 1, \\ u(x) &= 0 \quad \text{if} \quad |x| \geqq 2. \end{aligned}$$

Let $\alpha, \beta > 0$. Let $\{z^j\}_{j=1}^{\infty}$ be a sequence of points in $\mathbb{R}_n$ with

$$(3.52) \qquad \inf_k |z^j - z^k| = |z^j - z^{j+1}| = \frac{1}{j^\alpha}, \quad j = 1, 2, \ldots.$$

We put

$$(3.53) \qquad f(x) = \sum_{j=1}^{\infty} j^\beta u(j^\alpha(x - z^j)), \quad x \in \mathbb{R}_n.$$

Applying some known properties of the dilatation operator (cf. H. Triebel [21, Proposition 3.4.1]), we obtain $f \in F^s_{p,\infty}$ for α, β satisfying

$$(3.54) \qquad \beta + \alpha(s - (n/p)) < -1/\min(1, p)$$

at least if $s > \sigma_p$.

Step 3. Construction of the cubes $P_{k,j}$. As usual, we denote the components of $x \in \mathbb{R}_n$ by $x_1, \ldots, x_n$. Let us define two sequences of cubes by

$$(3.55) \qquad P_{k,j} = \Big\{ x \mid x \in \mathbb{R}_n, \ |x_r - z^k_r| < \frac{1}{2\sqrt{(n)}\, k^\alpha}, \quad r = 2, \ldots, n,$$
$$a_j k^{-(\alpha+\beta)} < x_1 - z^k_1 < b_j k^{-(\alpha+\beta)} \Big\}, \quad k, j = 1, 2, \ldots$$

and

$$P^*_{k,j} = \left\{ x \mid x \in \mathbb{R}_n,\ |x_r - z^k_r| < \frac{1}{2\sqrt{(n)}\,k^\alpha},\quad r = 2, \ldots, n\,, \right. \tag{3.56}$$

$$\left. a_j k^{-(\alpha+\beta)} < x_1 - z^k_1 < \frac{a_j + b_j}{2}\, k^{-(\alpha+\beta)} \right\},\quad k, j = 1, 2, \ldots .$$

We list some properties of these cubes.

If $0 \leqq b_j \leqq k^\beta/(2\sqrt{n})$ then $x \in P_{k,j}$ implies $|x - z^k| < 1/(2k^\alpha)$, and hence by (3.52)

$$P_{k,j} \cap P_{r,l} = \emptyset \tag{3.57}$$

if $k \neq r$, $0 \leqq (j + 1)\,T \leqq k^\beta/(2\sqrt{n})$, and $0 \leqq (l + 1)\,T \leqq r^\beta/(2\sqrt{n})$. For brevity we introduce the notation

$$C_k := \frac{k^\beta}{2\sqrt{(n)}\,T} - 1\,.$$

Using (3.57) we find

$$f(x) = k^{\alpha+\beta}(x_1 - z^k_1) \quad \text{if} \quad x \in P_{k,j}\,,\quad j \leqq C_k\,. \tag{3.58}$$

The advantage of the sequence $\{P^*_{k,j}\}_{k,j}$ consists in the fact that for $h = (h_1, 0, \ldots, 0)$ with $0 < h_1 < \frac{1}{2}(b_j - a_j)\,k^{-(\alpha+\beta)}$ we have the implication

$$x \in P^*_{k,j} \Rightarrow x + h \in P_{k,j}\,. \tag{3.59}$$

We put $D_h := \left(\frac{1}{2}(b - a)\,h_1^{-1}\right)^{1/(\alpha+\beta)}$ in view of the condition on h_1.

Step 4. Estimate of $(\partial/\partial x_1)^m\,G(f)$. To this end we make some use of a lifting property of the underlying function spaces. The assumption $G(f) \in F^{\varrho+\varepsilon}_{p,\infty}$ necessarily implies $(\partial/\partial x_1)^m\,G(f) \in F^{\varrho+\varepsilon-m}_{p,\infty}$ and since $\sigma_p < \varrho + \varepsilon - m < 1$, by choosing $\varepsilon > 0$ small enough we obtain

$$\sum_{|\alpha|=m} \sup_{h \neq 0} |h|^{m-\varrho-\varepsilon} \left\| \Delta^1_h \left(\left(\frac{\partial}{\partial x_1}\right)^m G(f) \right) \,\Big|\, L_p \right\| < \infty \tag{3.60}$$

(cf. H. Triebel [21, 2.3.2, 2.5.12]).

Our aim is to show that (3.60) is impossible. Let us look at $\Delta^1_h(\partial/\partial x_1)^m\,G(f)$. Restricting these functions to $P^*_{k,j}$ and using (3.58), (3.59) we find

$$\left(\Delta^1_h \left(\frac{\partial}{\partial x_1}\right)^m G(f) \right)(x) =$$

$$= k^{(\alpha+\beta)m}\left(G^{(m)}(k^{\alpha+\beta}(x_1 - z^k_1 + h_1)) - G^{(m)}(k^{\alpha+\beta}(x_1 - z^k_1))\right),$$

at least if $h = (h_1, 0, \ldots, 0)$, $k < D_h$ and $j \leqq C_k$. By (3.50), (3.55)–(3.57) this leads to

$$\left(\int \left| \left(\Delta^1_h \left(\frac{\partial}{\partial x_1}\right)^m G(f) \right)(x) \right|^p \mathrm{d}x \right)^{1/p} \geqq \tag{3.61}$$

$$\geqq A\Big(\sum_{k=1}^{D_h-1} \sum_{j=1}^{C_k} \int_{P_{k,j^*}} k^{(\alpha+\beta)(m+1)p} h_1^p\,\mathrm{d}x\Big)^{1/p} \geqq c h_1 \Big(\sum_{k=1}^{D_h-1} k^{(\alpha+\beta)(m+1)p-\alpha n}\Big)^{1/p} \geqq$$

$$\geqq c' h_1^{-m+\frac{\alpha}{\alpha+\beta}\frac{n}{p}-\frac{1}{\alpha+\beta}\frac{1}{p}}$$

if

$$(3.62) \qquad (\alpha + \beta)(m + 1)p - \alpha n > -1 .$$

Note that, c' is independent of $h_1 > 0$. By (3.60) and in view of (3.61)

$$(3.63) \qquad \frac{\alpha}{\alpha + \beta}\frac{n}{p} - \frac{1}{\alpha + \beta}\frac{1}{p} - m \geqq \varrho + \varepsilon - m .$$

If β tends to

$$-\frac{1}{\min(1, p)} + \alpha\left(\frac{n}{p} - s\right)$$

from below (cf. (3.54)) then

$$\frac{\alpha}{\alpha + \beta}\frac{n}{p} - \frac{1}{\alpha + \beta}\frac{1}{p} \to \frac{1}{p}\frac{\alpha n - 1}{\alpha\left(\frac{n}{p} - s + 1\right) - \frac{1}{\min(1, p)}} .$$

By the foregoing considerations, for any α, β satisfying

$$\frac{1}{((n/p) - s)\min(1, p)} < \alpha < \infty , \quad \beta > 0$$

and (3.54) we have $f \in \tilde{F}^s_{p,\infty}$. Letting α tend to infinity we obtain

$$(3.64) \qquad \frac{\frac{n}{p} - \frac{1}{\alpha}}{\frac{n}{p} - s + 1 - \frac{1}{\alpha}\frac{1}{\min(1, p)}} \to \varrho .$$

Hence, for any sufficiently small $\varepsilon > 0$ we can find a pair α, β such that (3.54) is satisfied while (3.63) is not. For small $\varepsilon > 0$ and consequently for any $\varepsilon > 0$ this contradicts our assumption $G(f) \in F^{\varrho+\varepsilon}_{p,\infty}$.

We have to add two remarks. First, we must look at (3.62). The last inequality in (2.13) implies

$$\left(\frac{n}{p} - s + 1\right)(m + 1)p > n$$

and if α is large enough then also

$$\alpha\left(\frac{n}{p} - s + 1\right)(m + 1)p - \frac{(m + 1)p}{\min(1, p)} > \alpha n - 1 .$$

Now, if we take sufficiently large β satisfying (3.54) the condition (3.62) is fulfiled. The second remark concerns the compactness of supp f. However, this is an easy consequence of $\alpha > 1$ and (3.51). The proof is complete.

Proof of Theorem 3. The proof follows the same lines as above and is easier, since one has to replace any sequence $\{P_{k,j}\}_j$ by a single cube P_k only. This is exactly the case considered in the papers mentioned at the beginning of the proof of Theorem 2.

Acknowledgement. The author wishes to express his sincere thanks to Prof. J. Nečas and Prof. A. Kufner, who have given him the possibility to speak about the subject of the paper at their seminars in Prague, and to Dr. V. Šverák, who has drawn the author's attention to the possibilities mentioned in Remark 28.

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