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A CHARACTERIZATION OF FINITE POSETS OF THE WIDTH AT MOST THREE WITH THE FIXED POINT PROPERTY

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T. S. Fofanova [1] has described the family of minimal forbidden retracts for finite posets of the width two to have the fixed point property. In this paper, a set of forbidden retracts will be constructed for the class of all finite posets of the width at most three.

Recall that a poset P has the *fixed point property* if for every order-preserving mapping $f: P \to P$ there exists an element $p \in P$ such that $f(p) = p$. A class $\boldsymbol{Q}$ of posets is a *class of forbidden retracts* for a class of posets $\boldsymbol{P}$ if each poset from $\boldsymbol{Q}$ is a retract of a poset from $\boldsymbol{P}$ and fails to have the fixed point property, and each poset from $\boldsymbol{P}$ either has the fixed point property or has a retract isomorphic to a poset from $\boldsymbol{Q}$. To any class of posets there exists a class of forbidden retracts: it is the class of its elements that have not the fixed point property. To any class of finite posets there exists a class of minimal forbidden retracts: it is the class of all minimal elements in the class of all forbidden retracts ordered by retraction.

DEFINITIONS

Let $(X, \leqq)$ be a poset, and let A and B be non-empty subsets of X. We shall write
$A < B$ if $\exists_{a \in A, b \in B} (a < b)$;
$A \leqq B$ if $A < B$ or $A = B$;
$A \lessdot B$ if $\forall_{a \in A, b \in B} (a < b)$.

Symbols $<, \leqq, \lessdot$ are relational, not operational. A formula $X = A_1 \lhd \ldots \lhd A_n$ is to be read as: the poset X can be decomposed into non-empty subsets $A_1, \ldots, A_n$ such that $A_1 \lhd A_2, \ldots, A_{n-1} \lhd A_n$.

Let $X = A_1 \lhd \ldots \lhd A_n$. We define $A_{pq} = \bigcup_{i=p}^{q} A_i$.

(In the preceding, set $<$ or $\leqq$ or $\lessdot$ for $\lhd$.)

Let $X = A_1 < \ldots < A_n$, where A_i $(i = 1, \ldots, n)$ are antichains. A *block* in X is its subset A_{pq} such that $\neg(A_i \lessdot A_{i+1})$ $(i = p, \ldots, q - 1)$ and it is maximal with this property, i.e. $A_{p-1} \lessdot A_p$ and $A_q \lessdot A_{q+1}$, or $p = 1$ and $A_q \lessdot A_{q+1}$, or $A_{p-1} \lessdot A_p$ and $q = n$, or $p = 1$ and $q = n$.

Let $X = A_1 \lessdot \ldots \lessdot A_n$. Then X is said to be an *ordinal sum* of A_i $(i = 1, \ldots, n)$.

Let $X = A_1 < \ldots < A_n$. Then X is said to be a *linear sum* of A_i $(i = 1, \ldots, n)$.

Denote by $\boldsymbol{P}$ the class of all finite posets, and by $\boldsymbol{P}_3$ the class of all finite posets of the width at most three.

A *section* is

(1) the two-element antichain with the support $S = \mathbf{2} = \{0, 1\}$;

(2) a poset $(S, \precsim)$ with the support $S = \mathbf{3} \times \boldsymbol{n} + \mathbf{1} = \{[i, k] \mid i \in \{0, 1, 2\},\ k \in \{0, 1, \ldots, n\}\}$, where $n \geqq 1$, and an ordering $\precsim$ satisfying:

  (i) $l < k \Rightarrow [i, l] \prec [i, k]$;

  (ii) $[0, k], [1, k], [2, k]$ are pairwise incomparable;

  (iii) $[i, l] \prec [j, k] \Rightarrow [i \oplus 1, l] \prec [j \oplus 1, k]$, where $\oplus$ means $+ \bmod 3$;

  (iv) $\forall_{k \neq 0}\ \exists_{i,j}([i, k - 1] \text{ non } \prec [j, k])$.

A *nice section* is a section $(S, \precsim)$ with

(V) $x \prec y \Rightarrow \exists_{v \in S}\ (x \prec v \text{ and } y \text{ non } \precsim v)$ for any $x, y \in S$;

(W) $x \prec y \Rightarrow \exists_{w \in S}\ (w \prec y \text{ and } w \text{ non } \precsim x)$ for any $x, y \in S$.

A *tower* is an ordinal sum of sections.

A *very nice section* is a section having no proper retract isomorphic to a tower.

SOME PROPERTIES OF TOWERS

Observations. Every section is of the width at most three. Consequently, every tower is of the width at most three. Towers of the width two are exactly towers described by T. S. Fofanova. Note that a section itself is a tower.

Lemma 1. *No section can be expressed as an ordinal sum of two non-empty posets.*

Proof. Let $(S, \precsim)$ be a section. If $S = \mathbf{2}$, the statement is obvious. We must investigate the case $S = \mathbf{3} \times \boldsymbol{n} + \mathbf{1}$. Suppose $S = A \lessdot B$ and $A \neq \emptyset$. Take $[j, k] \in A$. In view of (i) also $[j, 0] \in A$ and by (ii) $[0, 0], [1, 0], [2, 0] \in A$. The rest of the proof can be done by induction: Assume $[0, k - 1], [1, k - 1], [2, k - 1] \in A$. Property (iv) implies $\exists_{i,j}([i, k - 1] \text{ non } \prec [j, k])$, and we may conclude that $[j, k] \in A$. Applying (ii), we obtain $[0, k], [1, k], [2, k] \in A$. Consequently, $S = A$ and $B = \emptyset$. Q.E.D.

Corollary. *Every tower has a unique decomposition into sections.*

Proposition 1. *Every tower has a retract being a tower of very nice sections.*

Proof. Let $T_0 = S_1 \lessdot \ldots \lessdot S_n$ be a tower of sections S_i $(i = 1, \ldots, n)$. Replace sections that are not very nice by corresponding towers, their proper retracts. The resulting tower T_1 is a proper retract of T_0. Repeat the same procedure for T_i $(i = 1, \ldots)$. Since T_0 is finite, and $|T_i| < |T_{i-1}|$, this procedure stops at some T_i being a tower of very nice sections. Q.E.D.

Lemma 2. *If $T = S_1 \lessdot \ldots \lessdot S_n$ is a tower of sections S_i $(i = 1, \ldots, n)$, and g is an order-preserving mapping of T onto a tower, then $g(S_i)$ $(i = 1, \ldots, n)$ are pairwise disjoint towers.*

Proof. Images of sections are pairwise disjoint: Let $x \in g(S_p) \cap g(S_q)$ where $p < q$, say $x = g(s_p) = g(s_q)$, $s_p \in S_p$, $s_q \in S_q$. Then for an arbitrary element $t \in T$ either $s_p \leqq t$ or $t \leqq s_q$, consequently x is a nodus in $g(T)$ and therefore $g(T)$ can not be a tower. Images of sections are towers: Let $g(T) = Q_1 \lessdot \ldots \lessdot Q_m$ be the unique decomposition of the tower $g(T)$ into sections. Since $g(T) = g(S_1) \lessdot \ldots \lessdot g(S_n)$, it holds $Q_r = Q_r \cap g(T) = (Q_r \cap g(S_1)) \lessdot \ldots \lessdot (Q_r \cap g(S_n))$. We obtain $Q_r \subseteq g(S_p)$ for some $p \in \{1, \ldots, n\}$ using lemma 1. What remains to show is trivial. Q.E.D.

Proposition 2. *A tower of very nice sections has no proper retract being a tower.*

Proof. Let $T = S_1 \lessdot \ldots \lessdot S_n$ be a tower of very nice sections S_i $(i = 1, \ldots, n)$, and let r, e be a retraction and the corresponding coretraction respectively such that $r(T)$ is a tower. Denote by $v_i = \max \{k \mid e.r(S_i) \cap S_k \neq \emptyset\}$ and $v = \min \{v_i \mid v_i \leqq i\}$. Then $v \neq 1$ would imply $v_{i-1} > i - 1$, hence $i - 1 < v_{i-1} \leqq v_i \leqq i$, which would yield $v_{i-1} = i$ and so $e . r(S_i) \subseteq S_i$, i.e. the section S_i would have a retract $r(S_i)$ being a tower in virtue of lemma 2. As S_i is a very nice section, it follows that $e . r(S_i) = S_i$, which contradicts the assumption that $v_{i-1} = i$. We must conclude that $v = 1$ and $e . r(S_1) = S_1$, $e . r(S_2 \lessdot \ldots \lessdot S_n) \subseteq S_2 \lessdot \ldots \lessdot S_n$. Repeating the construction just described we obtain $e . r(S_i) = S_i$ $(i = 1, \ldots, n)$, the retract $r(T)$ is not proper. Q.E.D.

Proposition 3. *Towers have not the fixed point property.*

Proof. Let $T_0 = S_1 \lessdot \ldots \lessdot S_n$ be a tower of sections S_i $(i = 1, \ldots, n)$. Define $f: T \to T$ by $f|_{S_i} = (0 \mapsto 1,\ 1 \mapsto 0)$ for $S_i = \mathbf{2}$, $f|_{S_i} = ([i, k] \mapsto [i \oplus 1, k])$ for $S_i = \mathbf{3} \times \boldsymbol{n} + \mathbf{1}$. Obviously, f is an order-preserving mapping and it has no fixed point. Q.E.D.

MINIMAL FORBIDDEN RETRACTS FOR P_3 ARE TOWERS

In the following, $\lhd$ should be replaced by $<$ or $>$.

Lemma 3. *Let $x \in \boldsymbol{P}$, and let $f: X \to X$ be an order-preserving mapping.* If A and B are f-cycles, and $A \lhd B$, then $\forall_{x \in A} \exists_{y \in B} (x \lhd y)$.

Proof. Let $A \lhd B$. By definition, there exist elements $a \in A$, $b \in B$ such that $a \lhd b$. Take an arbitrary element $x \in A$. There exists a positive integer k such that $x = f^k(a)$. Define $y = f^k(b)$. It is easy to see that $x \lhd y$. Q.E.D.

Lemma 4. *Let $X \in \boldsymbol{P}$, and let $f: X \to X$ be an order-preserving mapping that fails to have a fixed point. Then every f-cycle is an antichain, and its cardinality is at least two.*

Proof. Let x be an element of an f-cycle. If it would be comparable with another element of the same f-cycle, say $x \lhd f^k(x)$ where $k \geqq 1$, then $x \lhd f^k(x) \lhd$ or $= \dots$ $\dots \lhd$ or $= f^{n.k}(x) = x$, where n is the cardinality of the f-cycle. If an f-cycle would contain only one element, this would be a fixed point of f. Q.E.D.

Lemma 5. *Let $X \in \boldsymbol{P}$, and let $f: X \to X$ be an order-preserving mapping that fails to have a fixed point. Then the union of all f-cycles X^f forms a retract in X, and the corresponding restriction of f is an automorphism of X^f that has no fixed point.*

Proof. Let $n = \Pi\{t_x \mid x \in X, t_x = \min\{t \geqq 1 \mid f^t(x) \in \{x, f(x), \dots, f^{t-1}(x)\}\}$. Then $f^n: X \to X$ maps X onto X^f: Let x be an arbitrary element of X. We obtain $f^{t_x}(x) = f^i(x)$ for some $i \in \{0, \dots, t_x - 1\}$ and consequently $f^{t_x - i}(f^{t_x}(x)) = f^{t_x - i}(f^i(x)) = f^{t_x}(x)$. Hence $f^{t_x}(x)$ lies in an f-cycle and so does $f^n(x) = f^{n-t_x}(f^{t_x}(x))$. Surjectivity is obvious. Now, let $x \in X^f$. Then $f^{t_x}(x) = x$ and it follows that $f^n(x) = x$. We have proved that X^f is a retract in X. It is obvious that $f|_{X^f}$ is an automorphism of X^f that has no fixed point. The inverse of $f|_{X^f}$ is $f^{n-1}|_{X^f}$. Q.E.D.

Lemma 6. *Let $X \in \boldsymbol{P}_3$, and let $f: X \to X$ be an order-preserving mapping that fails to have a fixed point. The set of all f-cycles is linearly ordered by the induced relation $\leqq$.*

Proof. Let A, B, and C be arbirary f-cycles.

Obviously, $A \leqq A$. Reflexivity is proved.

$A \leqq B$ and $B \leqq A$ implies $A = B$, or $A < B$ and $B > A$. In the latter case, there would exist, by lemma 3, $a \in A$, $b \in B$, and $c \in A$ such that $a < b < c$. Since A is an antichain in virtue of lemma 4, we conclude that $A = B$. Antisymmetry is proved.

$A \leqq B$ and $B \leqq C$ implies $A = B \leqq C$, or $A \leqq B = C$, or $A < B < C$. In the latter case, there exist, by lemma 3, elements $a \in A$, $b \in B$, and $c \in C$ such that $a < b < c$. Hence $a < c$ and therefore $A < C$. Transitivity is proved.

Take $A \neq B$. Since A and B are disjoint and $|A| \geqq 2$, $|B| \geqq 2$ by lemma 4, it follows that $|A \cup B| \geqq 4$. Hence $A \cup B$ is not an antichain. We have $a \in A$ and $b \in B$ such that $a < b$ or $b < a$ and consequently $A < B$ or $B < A$.

Thence $\leqq$ is a linear ordering on the set of all f-cycles. Q.E.D.

Lemma 7. *Let $X \in \boldsymbol{P}_3$, and let $f: X \to X$ be an order-preserving mapping that fails to have a fixed point. If A, B are f-cycles such that $|A| = 2$, $|B| = 3$ and $A \lhd B$, then $\forall_{x \in A, y \in B}(x \lhd y)$.*

Proof. By definition, there exist elements $a \in A$, $b \in B$ such that $a \lhd b$. Let $x \in A$, $y \in B$ be arbitrary elements. Then $x = f^i(a)$, $y = f^{i+j}(b)$ for some positive integers i and j, which yields $x = f^{i+4j}(a)$, $y = f^{i+4j}(b)$. Consequently, $x \lhd y$.

Q.E.D.

Lemma 8. *Let $X \in \boldsymbol{P}_3$, and let $f: X \to X$ be an automorphism. Then $f^6(x) = x$ for any $x \in X$.*

Proof. For, $X = X^f$ and f-cycles are two-element or three-element antichains, which yields $f^2(x) = x$ or $f^3(x) = x$, and so $f^6(x) = x$. Q.E.D.

Lemma 9. *Let $X \in \boldsymbol{P}_3$, and let $f\colon X \to X$ be an automorphism that fails to have a fixed point. Let A and B be f-cycles such that $A \lhd B$, and let*

$$(*) \qquad \exists_{a\in A, b\in B}\ \forall_{u\in X^f}\ (a \lhd u \Rightarrow b \lhd u \text{ or } b = u)\,.$$

Then $a \lhd b$ and the mapping $g = (f^k(a) \mapsto f^k(b),\ x \mapsto x$ for $x \notin A)$ is a retraction.

Proof. Choose elements a, b satisfying $(*)$. They satisfy $a \lhd b$ by lemma 3. Clearly $X^f = X$. Observe that g is a correctly defined mapping by lemma 7. It remains to show that g is order-preserving. Let $x \lhd y$. If $x \notin A$ and $y \notin A$, then $g(x) = x \lhd y = g(y)$. If $x \in A$ and $y \notin A$, we obtain $x = f^k(a) \lhd f^k(b)$ for some positive integer $k \leqq 3$, and $a = f^{6-k}(x) \lhd f^{6-k}(y)$ by lemma 8, whence $b \lhd$ $\lhd f^{6-k}(y)$ or $b = f^{6-k}(y)$. In summary, $g(x) = f^k(b) \lhd$ or $= f^k f^{6-k}(y) = f^6(y) =$ $= y = g(y)$. If $x \notin A$ and $y \in A$, we obtain $g(x) = x \lhd y = f^k(a) \lhd f^k(b) = g(y)$ for some positive integer $k \leqq 6$. It is obvious that g is a retraction, since $g|_{g(X)}$ is the identical mapping by definition. Q.E.D.

Corollary. *Let $X \in \boldsymbol{P}_3$ have no proper retract without the fixed point property, and let $f\colon X \to X$ be an automorphism that fails to have a fixed point. Then to any two f-cycles A and B such that $A \lhd B$ and arbitrary elements $a \in A$ and $b \in B$ there exists an element $u \in X$ such that $a \lhd u$ and b* non$(\lhd$ or $=)$ *u.*

In the proof of lemma 9 using the relation $\lhd$ instead of $<$ is legitimate, since a mapping is order-preserving if and only if it is dual-order-preserving.

Proposition 4. *Let $X \in \boldsymbol{P}_3$ have not the fixed point property. Then X has a retract isomorphic to a tower of nice sections.*

Proof. The set of all retracts in X that fail to have the fixed point property contains at least one minimal element with respect to set inclusion as it is finite. Denote this minimal retract R, and the corresponding retraction r. Let $f\colon R \to R$ be one of the mappings that have no fixed point. This mapping must be surjective by lemma 5. In view of lemma 6, R can be represented as a linear sum of f-cycles: $R = A_1 < \dots$ $\dots < A_n$.

Let $R = X_1 \lessdot \dots \lessdot X_m$ be the finest ordinal decomposition of R. Then X_s $(s = 1, \dots, m)$ are blocks in R: Let $x \in X_s \cap A_i$. Then $A_i \subseteq X_s$ as it is an antichain by lemma 4. Let $A_i < A_j < A_k$, $A_i \subseteq X_s$, $A_k \subseteq X_s$. Then $A_j \subseteq X_s$ since $\leqq$ is a linear ordering on the set of f-cycles. If $A_i \subseteq X_s$ and $A_j \subseteq X_s$, then A_i non $\lessdot A_j$ as $\lessdot$ is the finest ordinal decomposition of R. It follows that all cycles included in the same block X_s have the same cardinality (by lemma 7), either two, or three. If this cardinality is two, we may conclude, by lemma 9, that X_s is formed by a unique two-element f-cycle, i.e. X_s is isomorphic to a nice section. Let us turn our attention to the case when this cardinality is three. It is clear that X_s can not be formed by a unique f-cycle; it could be replaced by a two-element antichain, and the mapping defined by $(x \mapsto x$ for $x \notin X_s$, $x \mapsto 0$ for one chosen element of X_s, $x \mapsto 1$ for the remaining two elements of $X_s)$ would be a retraction of R to a proper retract not having the fixed point property. Hence X_s includes at least two f-cycles. For instance, let $X_s = A_{pq}$. Choose $a_l \in A_l$ $(l = p, \dots, q)$ such that $a_l < a_k$ whenever $l < k$. This is possible by lemma 3. Now, construct a mapping $z: X_s \to S$, where $S = \mathbf{3} \times \boldsymbol{q} - \boldsymbol{p} + \mathbf{1}$, by prescriptions $z(a_{p+l}) = [0, l]$, $z(f(a_{p+l})) = [1, l]$, $z(f^2(a_{p+l})) = [2, l]$ $(l = 0, \dots, q - p)$. It is obviously a bijective mapping, thus an ordering $\precsim$ can be defined on S that corresponds with that on X_s (i.e. $[i, l] \precsim [j, k] :\Leftrightarrow f^i(a_{p+l}) \leqq f^j(a_{p+k})$). It remains to show that $(S, \precsim)$ is a nice section:

(i): Let $l < k$. Then $a_l < a_k$, $f(a_l) < f(a_k)$ and $f^2(a_l) < f^2(a_k)$. Thus $[0, l] \prec [0, k]$, $[1, l] \prec [1, k]$, $[2, l] \prec [2, k]$.

(ii): $a_k, f(a_k), f^2(a_k)$ form a three-element antichain, therefore $[0, k]$, $[1, k]$, $[2, k]$ also form an antichain.

(iii): Let $[i, l] \prec [j, k]$, i.e. $f^i(a_l) < f^j(a_k)$; then $f^{i \oplus 1}(a_l) = ff^i(a_l) < ff^j(a_k) = f^{j \oplus 1}(a_k)$, which yields $[i \oplus 1, l] \prec [j \oplus 1, k]$.

(iv): As A_{pq} is a block in R, it holds A_{p+k-1} non $\lessdot$ A_{p+k} $(k = 1, \dots, q - p)$. There exist elements $b \in A_{p+k-1}$ and $c \in A_{p+k}$ such that $b \nleqq c$. We can write $b = f^i(a_{k-1})$, $c = f^j(a_k)$ for suitable $i, j \in \{0, 1, 2\}$. It is obvious that $[i, k - 1]$ non $\prec$ non $\prec$ $[j, k]$.

(V): Let $x \prec y$, say $[i, l] \prec [j, k]$. Then $l < k$ and $f^i(a_l) < f^j(a_k)$. Since R has no proper retract without the fixed point property, and f is an automorphism without fixed points, there exists, by lemma 9 and its corollary, an element $u \in R$ such that $f^i(a_l) < u$ and $f^j(a_k) \nleqq u$. It follows that $u \in A_{pq}$ and it can be expressed as $u = f^g(a_h)$. Consequently, it holds $x = [i, l] \prec [g, h]$ and $y = [j, k]$ non $\precsim [g, h]$.
(W) can be proved analogously.
We may conclude that $(S, \precsim)$ is a nice section. Q.E.D.

Theorem. *For $X \in \boldsymbol{P}_3$, the following assertions are equivalent:*
(NFP) *X has not the fixed point property;*
(T) *X has a retract isomorphic to a tower of sections;*
(TN) *X has a retract isomorphic to a tower of nice sections;*
(TVN) *X has a retract isomorphic to a tower of very nice sections.*

Proof.

(T) yields (TVN) by proposition 1;
(TVN) yields (T) by definition;
(TN) yields (T) by definition;
(T) yields (NFP) by proposition 3;
(NFP) yields (TN) by proposition 4. Q.E.D.

We have shown that every forbidden retract for $\boldsymbol{P}_3$ has a retract isomorphic to a tower of very nice sections. It means that minimal forbidden retracts for $\boldsymbol{P}_3$ are towers of very nice sections. Conversely, by proposition 2 and proposition 4 together, every tower of very nice sections is a minimal forbidden retract for $\boldsymbol{P}_3$. We have a corollary:

Corollary. *Minimal forbidden retracts for* $\boldsymbol{P}_3$ *are exactly* (*up to isomorphism*) *towers of very nice sections.*

It is true that any very nice section is nice as it has a retract isomorphic to a tower of nice sections. The converse remains open.

Problem. Are there nice sections that are not very nice? Characterize very nice sections.

Reference

[1] *T. S. Fofanova:* Characterization of finite posets of the width two with the fixed point property. Summer School on Ordered Sets and Universal Algebra, Donovaly 1985.

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