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CONTINUOUS AND COMPACT IMBEDDINGS OF WEIGHTED SOBOLEV SPACES II

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This paper is a direct continuation of [2], where the fundamental concepts and their notation were introduced. Continuous and compact imbeddings of weighted Sobolev spaces into weighted Lebesgue spaces for power-type weights and a bounded domain $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$, are investigated. The weight functions may have singularities or degenerations only on the boundary $\partial\Omega$ of Ω.

7. Domains of the type $\mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$

We will consider domains $\Omega \in \mathscr{C}^{0,\kappa}$. Let us recall here the definition of the class $\mathscr{C}^{0,\kappa}$, which is based on the method of local coordinates (cf. [3]).

7.1. Definition. Let $0 < \kappa \leqq 1$. A *bounded* domain Ω is said to be of class $\mathscr{C}^{0,\kappa}$ (notation $\Omega \in \mathscr{C}^{0,\kappa}$) if the following conditions are fulfilled:

(i) There exists a finite number m of coordinate systems

(7.1) $$(y_i', y_{iN}), \quad y_i' = (y_{i1}, y_{i2}, \dots, y_{iN-1})$$

and the same number of functions $a_i = a_i(y_i')$ defined on the closures of the $(N-1)$-dimensional cubes

(7.2) $$\Delta_i = \{y_i';\ |y_{ij}| < \delta \text{ for } j = 1, 2, \dots, N-1\}\,.$$

$(i = 1, 2, \dots, m)$ so that for each point $x \in \partial\Omega$ there is at least one $i \in \{1, 2, \dots, m\}$ such that

(7.3) $$x = (y_i', y_{iN}) \quad \text{and} \quad y_{iN} = a_i(y_i')\,.$$

(ii) The functions a_i satisfy the Hölder condition on $\bar{\Delta}_i$ with the exponent κ and a constant A.

(iii) There exists a positive number $\lambda < 1$ such that the sets

(7.4) $$B_i = \{(y_i', y_{iN}),\ y_i' \in \Delta_i,\ a_i(y_i') - \lambda < y_{iN} < a_i(y_i') + \lambda\}$$

satisfy

(7.5) $$U_i = B_i \cap \Omega = \{(y_i', y_{iN});\ y_i' \in \Delta_i,\ a_i(y_i') - \lambda < y_{iN} < a_i(y_i')\}$$

and

(7.6) $$\Gamma_i = B_i \cap \partial\Omega = \{(y_i', y_{iN});\ y_i' \in \Delta_i,\ y_{iN} = a_i(y_i')\} \quad (i = 1, 2, \dots, m)\,.$$

7.2. Partition of unity. Let $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$ and let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of domains such that

$$(7.7) \qquad \Omega_n \in \mathscr{C}^{0,1}, \quad \left\{x \in \Omega;\ d(x) > \frac{1}{n}\right\} \subset \Omega_n \subset \left\{x \in \Omega;\ d(x) > \frac{1}{n+1}\right\}.$$

By the symbol Ω^n we mean the set $\operatorname{int}(\Omega \setminus \Omega_n)$.

There exists a number $n_0 \in \mathbb{N}$ such that the system

$$(7.8) \qquad \{B_1, \ldots, B_m\},$$

where B_i are the sets given by the formula (7.4), forms a covering of the closure $\overline{\Omega^{n_0}}$ of the domain Ω^{n_0}. Let us denote by

$$\{\Phi_1, \ldots, \Phi_m\}$$

a partition of unity corresponding to the covering (7.8), that is, let

$$(7.9) \qquad \Phi_i \in C^\infty(\mathbb{R}^N), \quad \operatorname{supp} \Phi_i \subset B_i, \quad 0 \leqq \Phi_i(x) \leqq 1,$$

$$\sum_{i=1}^{m} \Phi_i(x) = 1 \quad \text{for} \quad x \in \Omega^{n_0}.$$

7.3. Remark. If the number λ in (7.4) is small enough, then we evidently have

$$7.10) \qquad d(x) = d_i(x), \quad x \in \Omega \cap \operatorname{supp} \Phi_i,$$

where $d_i(x) = \operatorname{dist}(x, \Gamma_i)$, $i = 1, 2, \ldots, m$.

Further, Lemma 4.6 from [3] yields

$$(7.11) \qquad \left(\frac{a_i(y_i') - y_{iN}}{1 + A}\right)^{1/\kappa} \leqq d_i(x) \leqq a_i(y_i') - y_{iN}$$

for $x = (y_i', y_{iN}) \in U_i$, $i = 1, 2, \ldots, m$.

8. Imbeddings of weighted Sobolev spaces with power-type weights – the case $1 \leqq p \leqq q < \infty$

Let us recall the following two theorems which the reader can find in [3] (Theorems 8.2, 8.4).

8.1. Theorem. *Let* $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$. *Let* $1 < p < \infty$ *and*

$$(8.1) \qquad \varepsilon > \kappa(p-1).$$

Then

$$(8.2) \qquad W^{1,p}(\Omega; d^\varepsilon, d^\varepsilon) \hookrightarrow L^p(\Omega; d^\eta),$$

where

$$(8.3) \qquad \eta = \begin{cases} \varepsilon - \kappa p & \text{for} \quad \kappa(p-1) < \varepsilon \leqq \kappa p, \\ (\varepsilon/\kappa) - p & \text{for} \quad \varepsilon > \kappa p. \end{cases}$$

8.2. Theorem. *Let* $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$. *Let* $1 < p < \infty$ *and*

$$(8.4) \qquad \varepsilon \neq \kappa(p-1).$$

Then

(8.5) $$W_0^{1,p}(\Omega; d^\varepsilon, d^\varepsilon) \hookrightarrow L^p(\Omega; d^\eta)$$

where

(8.6) $$\eta = \begin{cases} \kappa(\varepsilon - p) & \text{for } \varepsilon \leqq 0\,, \\ \varepsilon - \kappa p & \text{for } 0 < \varepsilon \leqq \kappa p\,, \\ (\varepsilon/\kappa) - p & \text{for } \varepsilon > \kappa p\,. \end{cases}$$

8.3. Remark. In the proof of Theorem 8.1 (or 8.2) the *classical Hardy inequality* was used (see [3]), i.e.

(8.7) $$\int_0^\infty |u(t)|^p\, t^{\varepsilon - p}\, dt \leqq \left(\frac{p}{|\varepsilon - p + 1|}\right)^p \int_0^\infty |u'(t)|^p\, t^\varepsilon\, dt\,, \quad 1 < p < \infty\,,^{*)}$$

which holds for every absolutely continuous function u on $(0, \infty)$ such that

(i) $$\lim_{t\to\infty} u(t) = 0 \quad \text{and} \quad \varepsilon > p - 1\,,$$

or

(ii) $$\lim_{t\to 0^-} u(t) = 0 \quad \text{and} \quad \varepsilon < p - 1\,,$$

or

(iii) $$\lim_{t\to 0^-} u(t) = \lim_{t\to\infty} u(t) = 0 \quad \text{and} \quad \varepsilon \neq p - 1\,.$$

Instead of (8.7) it is possible to use another inequality of Hardy type

(8.8) $$\int_0^b |u(t)|^p\, t^\alpha\, dt \leqq c \int_0^b |u'(t)|^p\, t^\beta\, dt\,, \quad 1 \leqq p < \infty$$

($c > 0$ is a suitable constant independent of u), which holds for every function u defined on $(0, b)$ $(0 < b < \infty)$ and such that eithter

(i') $$u \in AC((0, b))\,, \quad \lim_{t\to b^-} u(t) = 0 \quad \text{and}$$
$$\beta \leqq p - 1\,, \quad \alpha > -1 \vee \beta > p - 1\,, \quad \alpha \geqq \beta - p\,;$$

or

(ii') $$u \in AC((0, b))\,, \quad \lim_{t\to 0^+} u(t) = 0 \quad \text{and} \quad \beta < p - 1\,, \quad \alpha \geqq \beta - p\,;$$

or

(iii') $$u \in AC((0, b))\,, \quad \lim_{t\to 0^+} u(t) = \lim_{t\to b^-} u(t) = 0 \quad \text{and}$$
$$\beta \neq p - 1\,, \quad \alpha \geqq \beta - p \vee \beta = p - 1\,, \quad \alpha > -1\,.$$

Then we obtain the following assertions (which generalize Theorems 8.1 and 8.2).

8.4. Theorem. *Let* $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$, $1 \leqq p < \infty$. *Then*

(8.9) $$W^{1,p}(\Omega; d^\varepsilon, d^\varepsilon) \hookrightarrow L^p(\Omega; d^\eta)$$

if either

(8.10) $$\kappa(p - 1) < \varepsilon \leqq \kappa p\,, \quad \eta \geqq \varepsilon - \kappa p\,,$$

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*) It is possible to show that the inequality (8.7) takes place also if $p = 1$.

or

(8.11) $$\kappa p < \varepsilon\,, \quad \eta \geqq (\varepsilon/\kappa) - p\,.$$

Further,

(8.12) $$W^{1,p}(\Omega;\, d^{\varepsilon}, d^{\varepsilon}) \hookrightarrow\hookrightarrow L^{p}(\Omega;\, d^{\eta})$$

if

(8.13) $$\varepsilon \leqq \kappa(p-1)\,, \quad \eta > -\kappa\,.$$

8.5. Theorem. *Let* $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$, $1 \leqq p < \infty$. *Then*

(8.14) $$W_0^{1,p}(\Omega;\, d^{\varepsilon}, d^{\varepsilon}) \hookrightarrow L^{p}(\Omega;\, d^{\eta})$$

if

(8.15) $$\varepsilon \neq \kappa(p-1)$$

and

(8.16) $$\begin{aligned} &\eta \geqq \kappa(\varepsilon - p) && \text{for} \quad \varepsilon \leqq 0\,, \\ &\eta \geqq \varepsilon - \kappa p && \text{for} \quad 0 < \varepsilon \leqq \kappa p \\ &\eta \geqq (\varepsilon/\kappa) - p && \text{for} \quad \varepsilon > \kappa p\,. \end{aligned}$$

Further,

(8.17) $$W_0^{1,p}(\Omega;\, d^{\varepsilon}, d^{\varepsilon}) \hookrightarrow\hookrightarrow L^{p}(\Omega;\, d^{\eta})$$

if

(8.18) $$\varepsilon = \kappa(p-1)\,, \quad \eta > -\kappa\,.$$

Proofs of Theorems 8.4 and 8.5 are left to the reader. In the case (8.13) (or (8.18)) the compactness of the imbedding (8.12) (or (8.17), respectively) is due to the strictness of the inequality $\eta > -\kappa$ (cf. part b) of the proof of Theorem 9.3).

We will generalize the above theorems. Our aim is to find conditions under which the imbeddings

(8.19) $$W^{1,p}(\Omega;\, d^{\beta}, d^{\beta}) \hookrightarrow L^{q}(\Omega;\, d^{\alpha})\,,$$

(8.20) $$W_0^{1,p}(\Omega;\, d^{\beta}.\, d^{\beta}) \hookrightarrow L^{q}(\Omega;\, d^{\alpha})\,,$$

(8.21) $$W^{1,p}(\Omega;\, d^{\beta}, d^{\beta}) \hookrightarrow\hookrightarrow L^{q}(\Omega;\, d^{\alpha})\,,$$

(8.22) $$W_0^{1,p}(\Omega;\, d^{\beta}, d^{\beta}) \hookrightarrow\hookrightarrow L^{q}(\Omega;\, d^{\alpha})$$

take place.

Let us recall the results of Example 5.1 from [2]. Suppose that

(8.23) $$1 \leqq p \leqq q < \infty\,, \quad 1/N \geqq 1/p - 1/q \quad (\text{or } 1/N > 1/p - 1/q)\,.$$

Then

(8.24) $$W^{1,p}(\Omega;\, d^{\gamma - p}, d^{\gamma}) \hookrightarrow L^{q}(\Omega;\, d^{\alpha})$$

(or

(8.25) $$W^{1,p}(\Omega;\, d^{\gamma - p}, d^{\gamma}) \hookrightarrow\hookrightarrow L^{q}(\Omega;\, d^{\alpha}))$$

if and only if

(8.26) $$N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\gamma}{p}+1 \geqq 0$$

(or

(8.27) $$N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\gamma}{p}+1>0\,,$$

respectively).

8.6. Theorem. *Let* $1 \leqq p \leqq q < \infty$, $1/N \geqq 1/p - 1/q$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$. *Then*

(8.28) $$W^{1,p}(\Omega; d^{\beta}, d^{\beta}) \hookrightarrow L^{q}(\Omega; d^{\alpha})$$

if either

(8.29) $$\beta \leqq \kappa(p-1)\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\kappa(p-1)}{p}+\kappa>0\,,$$

or

(8.30) $$\kappa(p-1) < \beta \leqq \kappa p\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{p}+\kappa \geqq 0\,,$$

or

(8.31) $$\kappa p < \beta\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{\kappa p}+1 \geqq 0\,.$$

Proof. By Theorem 8.4 we have for $u \in W^{1,p}(\Omega; d^{\beta}, d^{\beta})$:

(8.32) $$\|u\|_{p,\Omega,d^{\gamma-p}} \leqq c\|u\|_{1,p,\Omega,d^{\beta},d^{\beta}}$$

(with a constant $c > 0$ independent of u), where

(8.33) $$\gamma = \beta - \kappa p + p \quad \text{if} \quad \kappa(p-1) < \beta \leqq \kappa p\,,$$
$$\gamma = \beta/\kappa \quad \text{if} \quad \kappa p < \beta\,.$$

In both cases we have $\beta \leqq \gamma$ in (8.33) and so

(8.34) $$\|\,|\nabla u|\,\|_{p,\Omega,d^{\gamma}} \leqq \left(\frac{\operatorname{diam}\Omega}{2}\right)^{(\gamma-\beta)/p} \|\,|\nabla u|\,\|_{p,\Omega,d^{\beta}}\,.$$

It follows from (8.32) and (8.34) that

(8.35) $$W^{1,p}(\Omega; d^{\beta}, d^{\beta}) \hookrightarrow W^{1,p}(\Omega; d^{\gamma-p}, d^{\gamma})\,.$$

Using the embedding (8.24) we get the imbedding (8.28) provided the condition (8.30) or (8.31) is fulfilled.

Now, let $\beta \leqq \kappa(p-1)$. Denote

$$\beta_{\omega} = \kappa(p-1) + \omega\,, \quad \text{where} \quad 0 < \omega \leqq \kappa\,.$$

Then

$$\kappa(p-1) < \beta_{\omega} \leqq \kappa p\,,$$

and by (8.30) we have

(8.36) $$W^{1,p}(\Omega; d^{\beta_{\omega}}, d^{\beta_{\omega}}) \hookrightarrow L^{q}(\Omega; d^{\alpha})$$

if

$$N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta_\omega}{p}+\kappa \geqq 0\,.$$

As $\beta_\omega > \beta$, we also have

$$W^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow W^{1,p}(\Omega; d^{\beta_\omega}, d^{\beta_\omega})\,. \tag{8.37}$$

By (8.36) and (8.37) we get the imbedding (8.28) if there exists $\omega \in (0, \kappa\rangle$ such that

$$N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\kappa(p-1)+\omega}{p}+\kappa \geqq 0\,.$$

Consequently, (8.28) takes place if the condition (8.29) is satisfied.

8.7. Theorem. *Let* $1 \leqq p \leqq q < \infty$, $1/N \geqq 1/p - 1/q$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$. *Then*

$$W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow L^q(\Omega; d^\alpha) \tag{8.38}$$

if either

$$\beta \leqq 0\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\kappa\frac{\beta}{p}+\kappa \geqq 0\,, \tag{8.39}$$

or

$$0 < \beta \leqq \kappa p\,, \quad \beta \neq \kappa(p-1)\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{p}+\kappa \geqq 0\,, \tag{8.40}$$

or

$$\kappa p < \beta\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{\kappa p}+1 \geqq 0\,, \tag{8.41}$$

or

$$\beta = \kappa(p-1)\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\kappa(p-1)}{p}+\kappa > 0\,. \tag{8.42}$$

Proof is analogous to that of the previous theorem and is left to the reader.

8.8. Theorem. *Let* $1 \leqq p \leqq q < \infty$, $1/N > 1/p - 1/q$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$. *Then*

$$W^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d^\alpha) \tag{8.43}$$

if either

$$\beta \leqq \kappa(p-1)\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\kappa(p-1)}{p}+\kappa > 0\,, \tag{8.44}$$

or

$$\kappa(p-1) < \beta \leqq \kappa p\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{p}+\kappa > 0\,, \tag{8.45}$$

or

$$\kappa p < \beta\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{\kappa p}+1 > 0\,. \tag{8.46}$$

Proof. The assertion follows from (8.25) and (8.27) by the same argument as in the proof of Theorem 8.6. The details are left to the reader.

For the sake of completeness we formulate a theorem which is the analogue of Theorem 8.7 for the compact imbedding.

8.9. Theorem. *Let* $1 \leqq p \leqq q < \infty$, $1/N > 1/p - 1/q$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$. *Then*

$$W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d^\alpha) \tag{8.47}$$

if either

$$\beta \leqq 0\,, \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \kappa\frac{\beta}{p} + \kappa > 0\,, \tag{8.48}$$

or

$$0 < \beta \leqq \kappa p\,, \quad \beta \neq \kappa(p-1)\,, \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + \kappa > 0\,, \tag{8.49}$$

or

$$\kappa p < \beta\,, \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{\kappa p} + 1 > 0\,, \tag{8.50}$$

or

$$\beta = \kappa(p-1)\,, \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\kappa(p-1)}{p} + \kappa > 0\,. \tag{8.51}$$

Theorems 8.6–8.9 give only sufficient conditions for the existence of the corresponding imbeddings. The question arises whether these conditions are also necessary.

We restrict ourselves only to the case $\kappa = 1$. First we present an auxiliary assertion.

8.10. Lemma. *Let* $1 \leqq p, q < \infty$, $\alpha, \beta \in \mathbb{R}$, *let* Ω *be a domain in* $\mathbb{R}^N$. *Let*

$$W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow L^q(\Omega; d^\alpha) \tag{8.52}$$

(*or*

$$W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d^\alpha))\,. \tag{8.53}$$

Then

$$1/N \geqq 1/p - 1/q\,, \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 \geqq 0 \tag{8.54}$$

(*or*

$$1/N > 1/p - 1/q\,, \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0 \Bigg)\,. \tag{8.55}$$

Proof. Let G be a domain such that $G \subset \bar{G} \subset \Omega$. Using the fact that the functions d^β, d^α are bounded from above and from below on G by positive constants we get from (8.52) (or (8.53)) the imbedding

$$W_0^{1,p}(G) \hookrightarrow L^q(G)$$

(or

$$W_0^{1,p}(G) \subset\subset L^q(G)) .$$

Now, Theorem 6.2 (or Theorem 6.4) from [2] (with $\Omega = G$ and $w = v \equiv 1$ on G) implies

$$1/N \geqq 1/p - 1/q$$

(or

$$1/N > 1/p - 1/q) .$$

Further, Theorem 6.2 (or Theorem 6.4) (with Ω, $w = d^\alpha$, $v = d^\beta$) implies that the space $W_0^{1,p}(\Omega; d^\beta, d^\beta)$ is not continuously (or compactly) imbedded in the space $L^q(\Omega; d^\alpha)$ if

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 < 0$$

(or

$$N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 \leqq 0) .$$

The lemma is proved.

8.11. Theorem. *Let us suppose that* $1 \leqq p \leqq q < \infty$, $\Omega \in C^{0,1}$, $\beta > p - 1$ (*or* $\beta \neq p - 1$). *Then*

(8.56) $$W^{1,p}(\Omega; d^\beta, d^\beta) \subset L^q(\Omega; d^\alpha)$$

(*or*

(8.57) $$W_0^{1,p}(\Omega; d^\beta, d^\beta) \subset L^q(\Omega; d^\alpha))$$

if and only if

(8.58) $$1/N \geqq 1/p - 1/q , \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 \geqq 0 .$$

Proof. The "if" part follows from Theorem 8.6 (or 8.7, respectively). The "only if" part is a consequence of Lemma 8.10.

8.12. Theorem. Let $1 \leqq p \leqq q < \infty$, $\Omega \in C^{0,1}$, $\beta > p - 1$ (or $\beta \neq p - 1$). *Then*

(8.59) $$W^{1,p}(\Omega; d^\beta, d^\beta) \subset\subset L^q(\Omega; d^\alpha)$$

(*or*

(8.60) $$W_0^{1,p}(\Omega; d^\beta, d^\beta) \subset\subset L^q(\Omega; d^\alpha))$$

if and only if

(8.61) $$1/N > 1/p - 1/q , \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0 .$$

Proof is analogous to that of Theorem 8.11 and is left to the reader.

9. Imbeddings of weighted Sobolev spaces with power-type weights – the case $1 \leqq q < p < \infty$

Throughout this section we suppose that

$$1 \leqq q < p < \infty .$$

In the proof of the imbedding

$$W^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow L^q(\Omega; d^\alpha)$$

the Hardy inequality (8.8) played the principal role. Now the inequality (9.1) from the next lemma will be of similar importance for the imbedding

$$W^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow L^q(\Omega; d^\alpha) .$$

9.1. Lemma. Let $0 < b < \infty$. *Then there exists a positive constant c such that the inequality*

(9.1) $$\left(\int_0^b |u(t)|^q t^\varepsilon \, dt\right)^{1/q} \leqq c\left(\int_0^b |u'(t)|^p t^\eta \, dt\right)^{1/p}$$

holds for all functions

(9.2) $$u \in T_1(0, b) = \{f \in AC((0, b));\ \lim_{x \to 0^+} f(x) = 0\}$$

or

(9.3) $$u \in T_2(0, b) = \{f \in AC((0, b));\ \lim_{x \to b^-} f(x) = 0\}$$

or

(9.4) $$u \in T(0, b) = T_1(0, b) \cap T_2(0, b)$$

if and only if

(9.5) $$\eta < p - 1 , \quad \varepsilon > \eta \frac{q}{p} - \frac{q}{p'} - 1$$

or

(9.6) $$\eta \leqq p - 1 , \ \varepsilon > -1 \ \vee \ \eta > p - 1 , \ \varepsilon > \eta \frac{q}{p} - \frac{q}{p'} - 1$$

or

(9.7) $$\eta \in \mathbb{R} , \quad \varepsilon > \eta \frac{q}{p} - \frac{q}{p'} - 1 ,$$

respectively.

Proof. In the cases (9.2) and (9.3) the assertion follows from [4], Section 1.3.2. In the case (9.4) the assertion follows from [6].

9.2. Remark. From the proof of necessity of the condition (9.5) or (9.6) or (9.7) for the inequality (9.1) to hold on the class $T_1(0, b)$ or $T_2(0, b)$ or $T(0, b)$, respectively, one can see that the condition (9.5) or (9.6) or (9.7) is necessary for the validity of the inequality (9.1) even on the (smaller) class of functions $T_1^*(0, b)$ or $T_2^*(0, b)$ or

$T^*(0, b)$, respectively, where

$$T_1^*(0, b) = \{u \in C^\infty(\langle 0, b\rangle);\ 0 \notin \operatorname{supp} u\},$$
$$T_2^*(0, b) = \{u \in C^\infty(\langle 0, b\rangle);\ b \notin \operatorname{supp} u\},$$
$$T^*(0, b) = T_1^*(0, b) \cap T_2^*(0, b).$$

9.3. Theorem. *Let* $1 \leqq q < p < \infty$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$. *Then*

$$W^{1,p}(\Omega; d^\beta, d^\beta) \subset\subset L^q(\Omega; d^\alpha) \tag{9.8}$$

if either

$$\beta \leqq \kappa(p-1), \quad \alpha > -\kappa \text{ *)} \tag{9.9}$$

or

$$\kappa(p-1) < \beta \leqq \kappa(p-1) + \kappa\frac{p}{q}, \quad \kappa\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + \kappa > 0 \tag{9.10}$$

or

$$\kappa(p-1) + \kappa\frac{p}{q} < \beta, \quad \left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{\kappa p} + 1 > 0. \tag{9.11}$$

Proof. a) First we will prove that

$$W^{1,p}(\Omega; d^\beta, d^\beta) \subset L^q(\Omega; d^\alpha) \tag{9.12}$$

if the assumptions of Theorem 9.3 are satisfied.

In virtue of Lemma 3.1 from [2] it is sufficient to verify that

$$\lim_{n\to\infty} \sup_{\|u\|_X \leqq 1} \|u\|_{q,\Omega^n,d^\alpha} = \mathscr{A} < \infty, \tag{9.13}$$

where we put $X = W^{1,p}(\Omega; d^\beta, d^\beta)$ (the sets Ω^n were introduced in Section 7.2).

We shall use the result on density of smooth functions in weighted Sobolev spaces (see [1]):

$$W^{1,p}(\Omega; d^\beta, d^\beta) = \overline{\mathscr{V}}^{\|\cdot\|_{1,p,\Omega,d^\beta,d^\beta}}, \tag{9.14}$$

where

$$\mathscr{V} = \{u \in C^\infty(\Omega);\ \|u\|_{1,p,\Omega,d^\beta,d^\beta} < \infty\}.$$

Take $u \in \mathscr{V}$. Let us introduce the local coordinates (y_i', y_{iN}) $(i = 1, 2, \ldots, m)$ from Definition 7.1. For $x \in \Omega^n$ $(n > n_0)$ we have

$$u(x) = u(x)\sum_{i=1}^m \Phi_i(x) = \sum_{i=1}^m u_i(x), \tag{9.15}$$

where we set $u_i = u\Phi_i$ (the functions Φ_i and the number n_0 were introduced in Section 7.2). The relations (9.15) and (7.10) imply

$$\|u\|_{q,\Omega^n,d^\alpha} \leqq \sum_{i=1}^m \|u_i\|_{q,\Omega^n,d^\alpha} = \sum_{i=1}^m \|u_i\|_{q,\Omega^n\cap\operatorname{supp}\Phi_i,d_i^\alpha} = \sum_{i=1}^m \|u_i\|_{q,U_i,d_i^\alpha}. \tag{9.16}$$

*) Instead of $\alpha > -\kappa$ it is possible to write

$$\kappa\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \kappa\frac{p-1}{p} + \kappa > 0.$$

Let us now estimate the norm

(9.17)

$$\|u_i\|^q_{q,U_i,d_i^\alpha} = \int_{U_i} |u_i(x)|^q\, d_i^\alpha(x)\,\mathrm{d}x = \int_{\Delta_i} \mathrm{d}y_i' \int_{a_i(y_i')-\lambda}^{a(y_i')} |u_i(y_i', y_{iN})|^q\, d_i^\alpha(y_i', y_{iN})\,\mathrm{d}y_{iN}\,.$$

If $\alpha \geqq 0$ then (7.11) and (9.17) yield

$$\|u_i\|^q_{q,U_i,d_i^\alpha} \leqq \int_{\Delta_i} \mathrm{d}y_i' \int_0^\lambda |u_i(y_i', a_i(y_i') - t)|^q\, t^\alpha\,\mathrm{d}t\,. \tag{9.18}$$

If $\alpha < 0$ then by (7.11) and (9.17) we obtain

$$\|u_i\|^q_{q,U_i,d_i^\alpha} \leqq (1 + A)^{-\alpha/\kappa} \int_{\Delta_i} \mathrm{d}y_i' \int_0^\lambda |u_i(y_i', a_i(y_i') - t)|^q\, t^{\alpha/\kappa}\,\mathrm{d}t\,. \tag{9.19}$$

Now, we estimate the inner integral on the right hand side of the inequalities (9.18) and (9.19) using the Hardy inequality (9.1). If the numbers ε and η satisfy the condition (9.6), then Lemma 9.1 implies

(9.20)

$$\int_{\Delta_i} \mathrm{d}y_i' \left(\int_0^\lambda |u_i(y_i', a_i(y_i') - t)|^q\, t^\varepsilon\,\mathrm{d}t\right) \leqq c^q \int_{\Delta_i} \mathrm{d}y_i' \left(\int_0^\lambda \left|\frac{\mathrm{d}}{\mathrm{d}t}\, u_i(y_i', a_i(y_i') - t)\right|^p t^\eta\,\mathrm{d}t\right)^{q/p}.$$

Let $\eta < 0$. Using the Hölder inequality and the relations (7.11) and (7.10) we get

$$\int_{\Delta_i} \mathrm{d}y_i' \left(\int_0^\lambda \left|\frac{\mathrm{d}}{\mathrm{d}t}\, u_i(y_i', a_i(y_i') - t)\right|^p t^\eta\,\mathrm{d}t\right)^{q/p} \leqq \tag{9.21}$$

$$\leqq |\Delta_i|^{(p-q)/p} \left(\int_{\Delta_i} \mathrm{d}y_i' \int_0^\lambda \left|\frac{\partial}{\partial y_{iN}}\, u_i(y_i'.\, a(y_i') - t)\right|^p t^\eta\,\mathrm{d}t\right)^{q/p} \leqq c_0 \|u\|^q_{1,p,\Omega,d^\eta,d^\eta}\,.$$

Similarly, for $\eta \geqq 0$ we obtain

$$\int_{\Delta_i} \mathrm{d}y_i' \left(\int_0^\lambda \left|\frac{\mathrm{d}}{\mathrm{d}t}\, u_i(y_i', a_i(y_i') - t)\right|^p t^\eta\,\mathrm{d}t\right)^{q/p} \leqq c_1 \|u\|^q_{1,p,\Omega,d^{\kappa\eta},d^{\kappa\eta}}\,. \tag{9.22}$$

(The constants $c_0, c_1 > 0$ are independent of the function $u \in \mathscr{V}$.)

We have to distinguish four cases:

(i) Let $\alpha \geqq 0$, $\beta \geqq 0$. Then we put $\varepsilon = \alpha$, $\eta = \beta/\kappa$ and by (9.18), (9.20) and (9.22) we have

$$\|u_i\|^q_{q,U_i,d_i^\alpha} \leqq K^q \|u\|^q_{1,p,\Omega,d^\beta,d^\beta} \tag{9.23}$$

(where $K = K_1 = cc_1^{1/q}$) provided

$$\beta \leqq \kappa(p - 1) \vee \beta > \kappa(p - 1)\,, \quad \alpha > \frac{\beta}{\kappa}\frac{q}{p} - \frac{q}{p'} - 1\,. \tag{9.24}$$

(ii) Let $\alpha < 0$, $\beta \geqq 0$. Then we set $\varepsilon = \alpha/\kappa$, $\eta = \beta/\kappa$ and by (9.19), (9.20) and (9.22) we arrive at (9.23) (with $K = K_2 = c(1 + A)^{-\alpha/\kappa q}\, c_1^{1/q}$) provided

$$\beta \leqq \kappa(p - 1)\,, \ \alpha > -\kappa \ \vee \ \beta > \kappa(p - 1)\,, \ \alpha > \beta\frac{q}{p} - \kappa\left(\frac{q}{p'} + 1\right). \tag{9.25}$$

(iii) Let $\alpha \geqq 0$, $\beta < 0$. Then we set $\varepsilon = \alpha$, $\eta = \beta$ and the inequalities (9.18), (9.20) and (9.21) immediately imply (9.23) (with $K = K_3 = cc_0^{1/q}$).

(iv) Let $\alpha < 0$, $\beta < 0$. Putting $\varepsilon = \alpha/\kappa$, $\eta = \beta$ and using (9.19), (9.20) and (9.21) we get (9.23) (with $K = K_4 = (1 + A)^{-\alpha/\kappa q} c c_0^{1/q}$) provided

(9.26) $$\alpha > -\kappa .$$

From (i)–(iv) we conclude that (9.23) takes place (with $K = \max\{K_1, K_2, K_3, K_4\}$) if α and β satisfy the condition (9.9) or (9.10) or (9.11). Under these conditions we have by (9.16) and (9.23)

$$\|u\|_{q,\Omega^n,d^\alpha} = mK\|u\|_{1,p,\Omega,d^\beta,d^\beta}$$

and by (9.14) the same estimate holds for all functions $u \in W^{1,p}(\Omega; d^\beta, d^\beta)$. Thus the condition (9.13) is fulfilled (with $\mathscr{A} \leqq mK$), which completes the proof of (9.12).

b) It is now easy to prove Theorem 9.3. Let the numbers β and α satisfy for example the condition (9.10) (the proof in the other cases is analogous). In virtue of the strictness of the inequality

$$\kappa\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + \kappa > 0$$

there exists $\varepsilon > 0$ such that the numbers β and $\bar{\alpha} = \alpha - \varepsilon$ satisfy (9.10) (with $\bar{\alpha}$ instead of α). Then part a) of our proof implies

$$W^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow L^q(\Omega; d^{\bar{\alpha}}) .$$

Hence there is a positive constant K such that

(9.27) $$\|u\|_{q,\Omega,d^{\bar{\alpha}}} \leqq K\|u\|_{1,p,\Omega,d^\beta,d^\beta}, \quad u \in W^{1,p}(\Omega; d^\beta, d^\beta) .$$

Let $u \in W^{1,p}(\Omega; d^\beta, d^\beta)$. Then by (7.7) and (9.27) we get

$$\|u\|^q_{q,\Omega^n,d^\alpha} = \int_{\Omega^n} |u(x)|^q d^\alpha(x)\,dx = \int_{\Omega^n} |u(x)|^q d^\varepsilon(x)\, d^{\bar{\alpha}}(x)\,dx \leqq$$
$$\leqq \frac{1}{n^\varepsilon}\|u\|^q_{q,\Omega^n,d^{\bar{\alpha}}} \leqq \frac{K^q}{n^\varepsilon}\|u\|^q_{1,p,\Omega,d^\beta,d^\beta},$$

hence

$$\lim_{n\to\infty} \sup_{u\in X, \|u\|_X \leqq 1} \|u\|_{q,\Omega^n,d^\alpha} = 0$$

and the proof of Theorem 9.3 is completed by using Remark 3.2 from [2].

9.4. Theorem. *Let* $1 \leqq q < p < \infty$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$. *Then*

(9.28) $$W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d^\alpha)$$

if either

(9.29) $$\beta \leqq 0, \quad \kappa\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \kappa\frac{\beta}{p} + \kappa > 0$$

or

(9.30) $$0 < \beta \leqq \kappa(p - 1) + \kappa\frac{p}{q}, \quad \kappa\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + \kappa > 0$$

or

$$(9.31)\qquad \kappa(p-1)+\kappa\frac{p}{q}<\beta\,,\quad \left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{\kappa p}+1>0\,.$$

Proof is analogous to that of Theorem 9.3 and is left to the reader.

9.5. Theorem. *Let* $1 \leqq q < p < \infty$, $\Omega \in \mathscr{C}^{0,1}$. *Then*

$$(9.32)\qquad W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d^\alpha)$$

if and only if

$$(9.33)\qquad \beta \in \mathbb{R}\,,\quad \left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{p}+1>0\,.$$

Proof. If the condition (9.33) is fulfilled then the imbedding (9.32) follows from Theorem 9.4.

Conversely, let us suppose that the condition (9.33) is not fulfilled for some α, β, i.e.

$$\alpha \leqq \beta\frac{q}{p}-\frac{q}{p'}-1\,.$$

Let us introduce the local coordinates (y_i', y_{iN}) and the numbers λ and δ from Definition 7.1. By Lemma 9.1 and Remark 9.2 (with $b=\lambda$, $\varepsilon=\alpha$, $\eta=\beta$) there exists a sequence of functions $\{u_n\}_{n=1}^\infty \subset C_0^\infty((0,\lambda))$ such that

$$(9.34)\qquad \int_0^\lambda |u_n'(t)|^p\, t^\beta\, dt = 1\,,\quad n\in\mathbb{N}\,,$$

$$(9.35)\qquad \int_0^\lambda |u_n(t)|^q\, t^\alpha\, dt \to \infty \quad \text{for} \quad n\to\infty\,.$$

Let $\Phi \in C_0^\infty(\mathbb{R}^{N-1})$ be a function satisfying

$$(9.36)\qquad \begin{cases} \Phi(z)=1 & \text{for } |z|<\delta/2\,,\\ \Phi(z)=0 & \text{for } |z|>3\delta/4\,,\\ 0\leqq\Phi(z)\leqq 1 & \text{for } z\in\mathbb{R}^{N-1}\,. \end{cases}$$

For $n \in \mathbb{N}$ and $x \in \Omega$ let us define

$$(9.37)\qquad v_n(x)=\begin{cases} v_n(y_1', y_{1N}) = \Phi(y_1')\,u_n(a_1(y_1')-y_{1N}) & \text{if } x=(y_1', y_{1N})\in U_1\,,\\ 0 & \text{if } x\in\Omega\setminus U_1\,. \end{cases}$$

Fix $n \in \mathbb{N}$. Then

$$\operatorname{supp} v_n \subset U_1$$

and there exists a domain G_n such that

$$(9.38)\qquad \operatorname{supp} v_n \subset G_n \subset \bar{G}_n \subset U_1\,.$$

The function a_1 is Lipschitzian on $\bar{\Delta}_1$ and consequently

$$v_n \in W_0^{1,p}(G_n)\,.$$

This together with (9.38) yields

$$(9.39)\qquad v_n \in W_0^{1,p}(\Omega; d^\beta, d^\beta)\,.$$

By (9.37), (9.36), (7.10), (7.11), Remark 8.3 (the inequality (8.8) with (iii')) and (9.34) we have

$$(9.40)\qquad \|v_n\|_{1,p,\Omega,d^\beta,d^\beta} = \left(\int_\Omega |v_n(x)|^p\, d^\beta(x)\, dx + \int_\Omega |\nabla v_n(x)|^p\, d^\beta(x)\, dx\right)^{1/p} \leqq$$

$$\leqq c_0\Big(\int_{U_1 \cap \operatorname{supp}\Phi} |u_n(a_1(y_1') - y_{1N})|^p\, d^\beta(y_1', y_{1N})\, dy_1'\, dy_{1N} +$$

$$+ \int_{U_1} |\nabla_{y_1}[u_n(a_1(y_1') - y_{1N})\, \Phi(y_1')]|^p\, d^\beta(y_1', y_{1N})\, dy_1'\, dy_{1N}\Big)^{1/p} \leqq$$

$$\leqq c_0 \Bigg[\Bigg(1 + \max_{y_1' \in \mathbf{R}^{N-1}} \sum_{i=1}^{N-1} \left|\frac{\partial \Phi(y_1')}{\partial y_{1i}}\right|^p\Bigg) \int_{U_1 \cap \operatorname{supp}\Phi} |u_n(a_1(y_1') - y_{1N})|^p\, d^\beta(y_1', y_{1N})\, dy_1' dy_{1N} +$$

$$+ \int_{U_1 \cap \operatorname{supp}\Phi} |u_n'(a_1(y_1') - y_{1N})|^p\, d^\beta(y_1', y_{1N})\, dy_1'\, dy_{1N}\Bigg] \leqq$$

$$\leqq c_1\left(\int_0^\lambda |u_n(t)|^p\, t^\beta\, dt + \int_0^\lambda |u_n'(t)|^p\, t^\beta\, dt\right)^{1/p} \leqq c_2\left(\int_0^\lambda |u_n'(t)|^p\, t^\beta\, dt\right)^{1/p} \leqq c_2$$

(the constants c_0, c_1 and c_2 are independent of n).

Further, by (7.10), (7.11), (9.36) and (9.37) we obtain

$$(9.41)\qquad \|v_n\|_{q,\Omega,d^\alpha}^q = \int_{U_1} |\Phi(y_1')\, u_n(a_1(y_1') - y_{1N})|^p\, d^\alpha(y_1', y_{1N})\, dy_1'\, dy_{1N} \geqq$$

$$\geqq \left(\frac{1}{1+A}\right)^\alpha \int_{U_1 \cap \operatorname{supp}\Phi} |\Phi(y_1')|^p\, |u_n(a_1(y_1') - y_{1N})|^p\, (a_1(y_1') - y_{1N})^\alpha\, dy_1'\, dy_{1N} \geqq$$

$$\geqq \left(\frac{1}{1+A}\right)^\alpha \left(\frac{\delta}{2}\right)^{N-1} \int_0^\lambda |u_n(t)|^p\, t^\alpha\, dt\,.$$

Now, (9.39), (9.40), (9.41) and (9.35) imply that the space $W_0^{1,p}(\Omega; d^\beta, d^\beta)$ is not continuously imbedded in the space $L^q(\Omega; d^\alpha)$ and the theorem is proved.

9.6. Theorem. *Let* $1 \leqq q < p < \infty$, $\Omega \in \mathscr{C}^{0,1}$. *Then*

$$(9.42)\qquad W^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d^\alpha)$$

if and only if either

$$(9.43)\qquad \beta \leqq -1\,, \quad \left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0$$

or

$$(9.44)\qquad -1 < \beta \leqq p - 1\,, \quad \alpha > -1$$

or

$$(9.45)\qquad \beta > p - 1\,, \quad \left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0\,.$$

Proof. In the cases (9.44) and (9.45) the proof is analogous to that of Theorem 9.5 and is left to the reader.

For $\beta \leqq -1$ we have (see [3], Remark 11.12 (ii))

$$W^{1,q}(\Omega; d^\beta, d^\beta) = W_0^{1,p}(\Omega; d^\beta, d^\beta)$$

and the result follows from Theorem 9.6.

9.7. Remarks. (i) In the case $0 < \kappa < 1$ it is possible to find (by the same method as in the proof of Theorem 9.5) necessary conditions for the validity of the imbeddings (9.8) and (9.28) (and the imbeddings (8.28), (8.38), (8.43) and (8.47) for $1 \leqq q < p < \infty$) but these conditions are different from the corresponding sufficient ones (due to the inequality (7.11)).

(ii) If $\Omega \in \mathscr{C}^{0,1}$ then it follows from the proof of Theorem 9.5 that either

$$W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d^\alpha)$$

(or $W^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow\hookrightarrow L^q(\Omega; d^\alpha)$), or the space $W_0^{1,p}(\Omega; d^\beta, d^\beta)$ (or the space $W^{1,p}(\Omega; d^\beta, d^\beta)$, respectively) is not continuously imbedded in $L^q(\Omega; d^\beta)$.

(iii) When proving the necessity of the condition (9.43) it is impossible to proceed in the same way as in the proof of necessity of the conditions (9.33), (9.44) and (9.45). In the case $\beta \leqq -1$ we do not know whether the functions v_n defined analogously as in (9.37) are elements of $W^{1,p}(\Omega; d^\beta, d^\beta)$, because the inclusion

$$C^\infty(\bar{\Omega}) \subset W^{1,p}(\Omega; d^\beta, d^\beta)$$

does not hold.

10. *N*-dimensional Hardy inequality with power-type weights

We will deal with the inequality

$$(10.1) \qquad (\textstyle\int_\Omega |u(x)|^q \, d^\alpha(x) \, dx)^{1/q} \leqq c(\int_\Omega |\nabla u(x)|^p \, d^\beta(x) \, dx)^{1/p},$$

where $1 \leqq p, q < \infty$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$, $u \in W_0^{1,p}(\Omega; d^\beta, d^\beta)$ and c is a positive constant independent of u. This inequality will be called the *N-dimensional Hardy inequality* (cf. [5]).

First, let us recall

10.1. Lemma. *If* $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$, $1 \leqq p < \infty$, $-\kappa p/(1-\kappa) < \varepsilon < \kappa(p-1)$,*) *then the norms* $\|\cdot\|_{1,p,\Omega,d^\varepsilon,d^\varepsilon}$ *and* $|||\cdot|||_{1,p,\Omega,d^\varepsilon}$, *where*

$$(10.2) \qquad |||u|||_{1,p,\Omega,d^\varepsilon} = (\textstyle\int_\Omega |\nabla u(x)|^p \, d^\varepsilon(x) \, dx)^{1/p},$$

are equivalent on the space $W_0^{1,p}(\Omega; d^\varepsilon, d^\varepsilon)$.

For the proof see [3], Proposition 9.2, which can be easily extended to the case $p = 1$.

The imbedding theorems (Sections 8, 9) and Lemma 10.1 imply

10.2. Theorem. *Let* $1 \leqq p \leqq q < \infty$, $1/N \geqq 1/p - 1/q$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$, *and either*

$$(10.3) \qquad -\frac{\kappa p}{1-\kappa} < \beta \leqq 0, \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \kappa\frac{\beta}{p} + \kappa \geqq 0,$$

*) For $\kappa = 1$ we put $-\kappa p/(1-\kappa) = -\infty$.

or

$$(10.4)\qquad 0 < \beta < \kappa(p-1)\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{p}+\kappa \geqq 0\,.$$

Then there exists a positive constant c such that the inequality (10.1) *holds for all* $u \in W_0^{1,p}(\Omega; d^\beta, d^\beta)$.

Proof. For $u \in W_0^{1,p}(\Omega; d^\beta, d^\beta)$ we have by Theorem 8.7

$$(10.5)\qquad \|u\|_{q,\Omega,d^\alpha} \leqq c_1 \|u\|_{1,p,\Omega,d^\beta,d^\beta}\,,$$

and by Lemma 10.1

$$(10.6)\qquad \|u\|_{1,p,\Omega,d^\beta,d^\beta} \leqq c_2 |||u|||_{1,p,\Omega,d^\beta}$$

($c_1, c_2 > 0$ are suitable constants independent of u). Combining (10.5) and (10.6) we immediately get (10.1).

10.3. Theorem. *Let* $1 \leqq q < p < \infty$, $\Omega \in \mathscr{C}^{0,\kappa}$, $0 < \kappa \leqq 1$, *and either*

$$(10.7)\qquad -\frac{\kappa p}{1-\kappa} < \beta \leqq 0\,, \quad \kappa\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\kappa\frac{\beta}{p}+\kappa > 0$$

or

$$(10.8)\qquad 0 < \beta < \kappa(p-1)\,, \quad \kappa\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{p}+\kappa > 0\,.$$

Then there exists a positive constant c such that the inequality (10.1) *holds for all* $u \in W_0^{1,p}(\Omega; d^\beta, d^\beta)$.

Proof is analogous to that of the previous theorem and is left to the reader.

If $\Omega \in \mathscr{C}^{0,1}$ then we are able to give necessary and sufficient conditions for the validity of the inequality (10.1).

10.4. Theorem. *Let* $\Omega \in \mathscr{C}^{0,1}$, $1 \leqq p$, $q < \infty$, $\beta < p - 1$. *Then there exists a positive constant c such that the inequality*

$$(10.9)\qquad \left(\int_\Omega |u(x)|^q\, d^\alpha(x)\,dx\right)^{1/q} \leqq c\left(\int_\Omega |\nabla u(x)|^p\, d^\beta(x)\,dx\right)^{1/p}$$

holds for all $u \in W_0^{1,p}(\Omega; d^\beta, d^\beta)$ *if and only if either*

$$(10.10)\qquad 1 \leqq p \leqq q < \infty\,, \quad 1/N \geqq 1/p - 1/q\,, \quad N\left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{p}+1 \geqq 0$$

or

$$(10.11)\qquad 1 \leqq q < p < \infty\,, \quad \left(\frac{1}{q}-\frac{1}{p}\right)+\frac{\alpha}{q}-\frac{\beta}{p}+1 > 0\,.$$

Proof. Let the conditions (10.10) or (10.11) be fulfilled. Then the validity of the inequality (10.9) for functions $u \in W_0^{1,p}(\Omega; d^\beta, d^\beta)$ follows from Theorem 10.2 or 10.3, respectively.

Now, let us suppose that for all functions $u \in W_0^{1,p}(\Omega; d^\beta, d^\beta)$ the inequality (10.9) holds. Then

$$W_0^{1,p}(\Omega; d^\beta, d^\beta) \hookrightarrow L^q(\Omega; d^\alpha) .$$

If $p \leqq q$, then Theorem 8.11 yields

$$1/N \geqq 1/p - 1/q , \quad N\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 \geqq 0 ,$$

i.e., the condition (10.10) is fulfilled.

If $q < p$, then by Remark 9.7 (ii) and by Theorem 9.5 we have

$$\left(\frac{1}{q} - \frac{1}{p}\right) + \frac{\alpha}{q} - \frac{\beta}{p} + 1 > 0 ,$$

i.e., the condition (10.11) is fulfilled.

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