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Ivan Dobrakov

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ON INTEGRATION IN BANACH SPACES, VII

IVAN DOBRAKOV, Bratislava

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INTRODUCTION

In [17] A. Kolmogoroff introduced, using limits of nets of abstract Riemann-type sums, two types of integrability corresponding to finite and countable partitions, respectively. The case of finite partitions (the so called *S-integral*) in our setting was investigated in the last Section 4 of part VI = [10]. In Theorem 1 below we establish the pleasant fact that in the case of countable partitions, the resulting Kolmogoroff integrability (and integral), which we call *S^*-integrability*, of a $\mathscr{P}^{\sim}$-measurable function $f: T \to X$ on the set $F = \{t \in T, f(t) \neq 0\}$ coincides with our integrability of the function f. This S^*-integrability of Kolmogoroff was already thoroughly extended to a very general setting by M. Sion in [19]. At the end of Section 1 we make some remarks to [19].

The first part of Section 2 is concerned with the Beppo Levi property of the measure m, and is related to Section 8 in [3], while in the second part we prove another result on integration by substitution, completing those from part $V = [9]$.

In Theorem 5 in Section 3 we solve affirmatively the problem of measurability of the partial integral from part III = [7]. A simple consequence of Theorem I.15 in [5], Theorem 4, is the key which together with the results from part III yields the desired solution. Further in this section we prove a result on existence of products of measures (Theorem 6), and improve the Tonelli type Theorem 8 of C. Swartz in [21] (Theorems 8 and 9).

Our concept of indirect products of measures introduced by Definition 3 corresponds to the bourbakistic concept of "integration of measures" in Chapter V in [2]. In Section 4 we extend the results of part III = [7] to the case of indirect products of measures. In particular, we prove the General Fubini Theorem (Theorem 11).

We shall use the notation and concepts from the previous parts, which we treat as chapters when referring to them.

Remark. According to Corollary of Lemma 2 below, in our integration theory we may suppose that the semivariation $\hat{m}$ is only σ-finite on P.

Let $E \in \sigma(\mathscr{P})$. By a countable $\mathscr{P}$-partition of E we mean a countable family $\pi(E) = (E_i)$ of pairwise disjoint sets $E_i \in \mathscr{P}$ with $\bigcup_i E_i = E$. If $\pi_1(E) = (E_i)$ and $\pi_2(E) = (F_j)$ are two countable $\mathscr{P}$-partitions of E, then we say that $\pi_2(E)$ is a refinement of $\pi_1(E)$ if for each F_j there is an E_i such that $F_j \subset E_i$. In this case we write $\pi_1(E) \leqq \pi_2(E)$. The set $\Pi(E)$ of all countable $\mathscr{P}$-partitions of E with this partial ordering is a directed set. For more information, see [17].

The next concept of integrability for $Y =$ scalars originates in A. Kolmogoroff's paper [17]. It was thoroughly extended to a very general setting by M. Sion [19] who used summability of series (which in Banach spaces is equivalent to unconditional convergence, see the beginning of Chapter VI in [4]), see pp. 16–17 and pp. 40–41 in [19].

Definition 1. Let $f: T \to X$ and let $E \in \sigma(\mathscr{P})$. We say that the function f is *S^*-integrable on E with respect to a measure $m: \mathscr{P} \to L(X, Y)$* if there exists a $y \in Y$ and for each $\varepsilon > 0$ a countable $\mathscr{P}$-partition $\pi_\varepsilon(E)$ of E such that for any countable $\mathscr{P}$-partition $(E_i) = \pi(E) \geqq \pi_\varepsilon(E)$ and any points $t_i \in E_i$ the series $\sum_i m(E_i) f(t_i)$ is unconditionally convergent in Y and $\left|\sum_i m(E_i) f(t_i) - y\right| < \varepsilon$. In this case we define the S^*-integral ${}_E S^* f \, \mathrm{d}m = y$.

The following simple facts are immediate.

If $N \in \sigma(\mathscr{P})$ is an m-null set ($\hat{m}(N) = 0$), then each function $f: T \to X$ is S^*-integrable on N and ${}_N S^* f \, \mathrm{d}m = 0$.

If both $f, g: T \to X$ are S^*-integrable on $E \in \sigma(\mathscr{P})$, and if a and b are scalars, then $af + bg$ is S^*-integrable on E, and

$$ {}_E S^*(af + bg) \, \mathrm{d}m = a \, . \, {}_E S^* f \, \mathrm{d}m + b \, . \, {}_E S^* g \, \mathrm{d}m \, . $$

If $f: T \to X$ is S^*-integrable on $E \in \sigma(\mathscr{P})$, then

$$ \left|{}_E S^* f \, \mathrm{d}m\right| \leqq \|f\|_E \, . \, \hat{m}(E) \, . $$

A $\mathscr{P}$-measurable function $f: T \to X$ is *S^*-integrable on* $F = \{t \in T, f(t) \neq 0\} \in \sigma(\mathscr{P})$ if and only if f is S^*-integrable on each set $E \in \sigma(\mathscr{P})$.

The assertions of the following lemma are also immediate.

Lemma 1. 1) *Let $f: T \to X$ be S^*-integrable on $E \in \sigma(\mathscr{P})$ and let (E_i) be a countable $\sigma(\mathscr{P})$-partition of E. Then f is S^*-integrable on each E_i, and*

$$ {}_E S^* f \, \mathrm{d}m = \sum_i {}_{E_i} S^* f \, \mathrm{d}m \, , $$

where the series converges unconditionally in Y.

2) *Let $f \in \overline{S(\mathscr{P}^\sim, X)}$ = the closure of $S(\mathscr{P}^\sim, X)$ in the norm $\|\cdot\|_T$ in the Banach space of all bounded X-valued functions on T, let $E \in \sigma(\mathscr{P})$ and $\hat{m}(E) < +\infty$.*

Then f is S^*-integrable on E, $f \,.\, \chi_E$ is integrable, and

$$_E S^* f \,\mathrm{d}m = \int_E f \,\mathrm{d}m \,.$$

The following useful lemma is a direct corollary of Theorem I.15, see [5].

Lemma 2. *Let $f\colon T \to X$ be a $\mathscr{P}$-measurable function, let $F = \{t \in T,\ f(t) \neq 0\}$, and let there be sets $F_k \in \sigma(\mathscr{P})$, $k = 1, 2, \ldots$ such that $F_k \nearrow F$ and $f \,.\, \chi_{F_k}$ is integrable for each $k = 1, 2, \ldots$. Then the function f is integrable if and only if there is a countably additive vector measure $\gamma\colon \sigma(\mathscr{P}) \to Y$ such that $\gamma(E \cap F_k) = \int_E f \,.\, \chi_{F_k} \,\mathrm{d}m$ for each $k = 1, 2, \ldots$ and each $E \in \sigma(\mathscr{P})$. In this case $\gamma(E) = \int_E f \,\mathrm{d}m$ for each $E \in \sigma(\mathscr{P})$.*

Corollary. *Let $m\colon \mathscr{P}_0 \to L(X, Y)$ be countably additive in the strong operator topology and let its semivariation $\hat{m}$ be σ-finite on $\mathscr{P}_0$. Then $S(\mathscr{P}_0, X) \subset \mathfrak{J}$, hence by Theorem I.16, see [5], we may suppose in our integration theory that the semivariation $\hat{m}$ is only σ-finite on $\mathscr{P}$.*

In what follows we suppose that the semivariation $\hat{m}$ is σ-finite on $\mathscr{P}$.

A relation between integrability and S^*-integrability is given by

Theorem 1. *A $\mathscr{P}^{\sim}$-measurable function $f\colon T \to X$ is integrable if and only if it is S^*-integrable on $F = \{t \in T,\ f(t) \neq 0\}$ (equivalently, if f is S^*-integrable on each set $E \in \sigma(\mathscr{P})$). In this case we have the equality*

$$\int_E f \,\mathrm{d}m = {}_E S^* f \,\mathrm{d}m \tag{1}$$

for each set $E \in \sigma(\mathscr{P})$.

Proof. Let $f\colon T \to X$ be a $\mathscr{P}^{\sim}$-measurable function, and take a sequence of $\mathscr{P}^{\sim}$-simple functions $f_n\colon T \to X$, $n = 1, 2, \ldots$ such that $f_n(t) \to f(t)$ and $|f_n(t)| \nearrow |f(t)|$ for each $t \in T$. Put $X_1 = \overline{sp}\{\bigcup_{n=1}^{\infty} f_n(T)\}$. Then it is clear that in our theorem we may replace the measure m by the measure $m_1 = m\colon \mathscr{P} \to L(X_1, Y)$ $(\hat{m}_1(E) \leqq \hat{m}(E)$ for each $E \in \sigma(\mathscr{P}))$. Now, since X_1 is a separable Banach space, by Theorem III.13.-1), see [7], there is a countably additive measure $\lambda_F\colon F \cap \sigma(\mathscr{P}) \to [0, 1]$ such that $N \in F \cap \sigma(\mathscr{P})$ and $\lambda_F(N) = 0$ implies $m_1(N) = 0$. Owing to the Egoroff-Lusin theorem, see Section 1.4 in part $I = [5]$, there is a set $N \in F \cap \sigma(\mathscr{P})$ and a sequence of sets $F_k \in F \cap \mathscr{P}^{\sim}$, $k = 1, 2, \ldots$ such that $\lambda_F(N) = 0$, $F_k \nearrow F - N$, and on each F_k, $k = 1, 2, \ldots$ the sequence f_n, $n = 1, 2, \ldots$ converges uniformly to f. It is obvious that in our theorem we may replace the function f by the function $f \,.\, \chi_{F-N}$. Since by assumption the semivariation $\hat{m}$ is σ-finite on $\mathscr{P}$, without loss of generality we may suppose that $\hat{m}(F_k) < +\infty$ for each $k = 1, 2, \ldots$. But then for each $k = 1, 2, \ldots$ the function $f \,.\, \chi_{F_k}$, hence also the function $f \,.\, \chi_{F_{k+1} - F_k}$, is integrable; it is S-integrable on F, see Definition 1 and Lemma 2 in part VI $= [10]$; it is S^*-integrable on F by assertion 2) of Lemma 1 above, and all three integrals are equal on each set $E \in \sigma(\mathscr{P})$.

Suppose first that the function $f \,.\, \chi_{F-N}$ is S^*-integrable on $F - N$. Since by assertion 1) of Lemma 1 the indefinite S^*-integral $E \to {}_E S^* f \,\mathrm{d}m$, $E \in \sigma(\mathscr{P})$, is a countably additive vector measure, $f \,.\, \chi_{F-N}$ is integrable and (1) holds by Lemma 2.

Conversely, let the function $f \,.\, \chi_{F-N}$ be integrable. Take $\varepsilon > 0$ and let $k \in \{1, 2, \ldots\}$ be fixed. Then by the S- and S^*-integrability of the function $f \,.\, \chi_{F_{k+1}-F_k}$ on $F_{k+1} - F_k$ there is a finite $\mathscr{P}$-partition $\pi_\varepsilon(F_{k+1} - F_k)$ such that for any countable $\mathscr{P}$-partition $(E_{k,i}) = \pi(F_{k+1} - F_k) \geqq \pi_\varepsilon(F_{k+1} - F_k)$, any points $t_{k,i} \in E_{k,i}$, and any $E \in \sigma(\mathscr{P})$, the series $\sum_i m(E \cap E_{k,i}) f(t_{k,i})$ converges unconditionally, and

$$(*) \qquad \Big|\sum_i m(E \cap E_{k,i}) f(t_{k,i}) - \int_{E\cap(F_{k+1}-F_k)} f \,\mathrm{d}m\Big| < \frac{\varepsilon}{2^{k+1}}\,.$$

Put

$$\pi_\varepsilon(F - N) = \sum_{k=1}^{\infty} \pi_\varepsilon(F_{k+1} - F_k)\,,$$

and let us have a countable $\mathscr{P}$-partition $\pi(F - N) \geqq \pi_\varepsilon(F - N)$. Then

$$\pi(F - N) = \sum_{k=1}^{\infty} \pi(F_{k+1} - F_k)\,,$$

where

$$\pi(F_{k+1} - F_k) = (E_{k,i}) \geqq \pi_\varepsilon(F_{k+1} - F_k)$$

is a countable $\mathscr{P}$-partition for each $k = 1, 2, \ldots$. Take any points $t_{k,i} \in E_{k,i}$. We assert that the series $\sum_{k,i} m(E_{k,i})\, f(t_{k,i})$ is unconditionally convergent in Y. According to Lemma 4 in part III = [7] this is equivalent to the following two assertions:

a) for each $k = 1, 2, \ldots$ the series $\sum_i m(E_{k,i}) f(t_{k,i})$ is unconditionally convergent in Y, which holds by the S^*-integrability of the function f on $F_{k+1} - F_k$, and

b) for any sequence of subsets $I_k \subset \{1, 2, \ldots\}$, $k = 1, 2, \ldots$ the series

$$\sum_{k=1}^{\infty} \sum_{i\in I_k} m(E_{k,i}) f(t_{k,i})$$

is unconditionally convergent in Y. This is also true, since for a fixed sequence I_k, $k = 1, 2, \ldots$, the inequality $(*)$ implies

$$\Big|\sum_{i\in I_k} m(E_{k,i}) f(t_{k,i}) - \int_{(\bigcup_{i\in I_k} E_{k,i})\cap(F_{k+1}-F_k)} f\,\mathrm{d}m\Big| < \frac{\varepsilon}{2^{k+1}}$$

for each $k = 1, 2, \ldots$, and the series $\sum_{k=1}^{\infty} \big(\int_{(\bigcup_{i\in I_k} E_{k,i})\cap(F_{k+1}-F_k)} f\,\mathrm{d}m\big)$ is unconditionally convergent in Y by the countable additivity of the integral $E \to \int_E f \,.\, \chi_{F-N}\,\mathrm{d}m$, $E \in \sigma(\mathscr{P})$.

Now using $(*)$ we have

$$\Big|\sum_{k,i} m(E_{k,i}) f(t_{k,i}) - \int_{E-N} f\,\mathrm{d}m\Big| < \varepsilon\,.$$

Thus $f \,.\, \chi_{F-N}$ is S^*-integrable on $F - N$ by Definition 1 and (1) holds. The theorem is proved.

This theorem enables us to connect our results with the results of M. Sion from [19]. However, let us make the following remarks:

a) As the following two simple examples demonstrate, the concepts (1) – "f behaves almost as a bounded function", and (2) – "λ behaves as a bounded finitely additive measure" of Definition 4.1 on p. 61 in [19] are misleading.

Example 1. We construct a bounded measurable function f which does not "behave almost as a bounded function". In our notation, let $T = N = \{1, 2, \ldots\}$, $\mathscr{P} = 2^T$, X = real l_1, $Y = R$ – reals, and let $m: \mathscr{P} \to c_0 \subset l_\infty = X^* = L(X, R)$ be defined by countable additivity from the following elementary values:

$$m(\{k\}) = \frac{1}{k} e_k = \frac{1}{k} (\underbrace{0, \ldots, 0, 1}_{\kappa}, 0, \ldots) \in c_0 .$$

Now the function $f: T \to X$ defined by the equalities $f(k) = e_k \in l_1$, $k \in N = T$, is the required one. Note that in the setting of this example the requirement "f behaves almost as a bounded function" means $\sum_{k=1}^{\infty} |f(k)| < +\infty$!

Example 2. The concept (2) – "λ behaves as a bounded finitely additive measure" in our setting and terminology means: $m: \mathscr{P} \to L(X, Y)$ is countably additive in the uniform operator topology and has finite semivariation $\hat{m}$ on $\mathscr{P}$.However, there are measures $m: \mathscr{P} \to L(X, Y)$, $\mathscr{P}$ being a σ-ring countably additive in the uniform operator topology, hence necessarily bounded, see Corollary IV.10.2 in [14], which do not have finite semivariation $\hat{m}$ on P. The following is a simple one, see Example 5 in part I = [5]: $T = N$, $\mathscr{P} = 2^T$, X = real l_2, $Y = R$, and $m: \mathscr{P} \to l_2^* = l_2$ is defined by countable additivity from the elementary values $m(\{k\}) = (1/k) . e_k \in l_2$. Hence we have a countably additive bounded measure which does not "behave as a bounded finitely additive measure".

2. TWO SUPPLEMENTS TO PART V

A. The Beppo Levi property. According to Theorem II.4, see [6], the L_1-pseudonorm satisfies $\hat{m}(f, E) = \hat{m}(|f|, E) = \sup_{|y^*| \leq 1} \int_E |f| \, dv(y^*m, \cdot)$ for each $\mathscr{P}$-measurable function $f: T \to X$ (or $f: T \to [0, +\infty)$) and each set $E \in \sigma(\mathscr{P}) \cup T$. Hence by the classical Fatou property of the integral we immediately see that $\hat{m}(\cdot, E)$ has the Fatou property, i.e., if $f, f_n: T \to [0, +\infty)$, $n = 1, 2, \ldots$ are $\mathscr{P}$-measurable and $f_n \nearrow f$, then $\hat{m}(f_n, E) \nearrow \hat{m}(f, E)$ for each $E \in \sigma(\mathscr{P})$, hence also for $E = T$. From this we easily see that for a given $\mathscr{P}$-measurable function $f: T \to X$ (or $f: T \to [0, +\infty)$) the set function $\hat{m}(f, \cdot): \sigma(\mathscr{P}) \to [0, +\infty)$ is continuous if and only if it is exhaustive i.e., if $\hat{m}(f, E_n) \to 0$ for any sequence of pairwise disjoint sets $E_n \in \sigma(\mathscr{P})$, $n = 1, 2, \ldots$.

Definition 2. We say that a measure $m: \mathscr{P} \to L(X, Y)$ (countably additive in the strong operator with σ-finite semivariation $\hat{m}$ on $\mathscr{P}$) has the *Beppo Levi property*, if the conditions: $f_n \in \mathscr{L}_1(m)$, $n = 1, 2, \ldots$, $f: T \to X$ $\mathscr{P}$-measurable, $|f_n(t)| \nearrow |f(t)|$ for each $t \in T$, and $\hat{m}(f, T) = \lim_{n \to \infty} \hat{m}(f_n, T) < +\infty$ imply $f \in \mathscr{L}_1(m)$.

Let us note two immediate facts:

1) If $c_0 \not\subset Y$, i.e., if Y does not contain a subspace isomorphic to the space c_0, for example if Y is weakly sequentially complete, see pp. 160–161 in [1], then any m considered has the Beppo Levi property, see Theorem V.2 in [9].

2) If m has the Beppo Levi property, then the conditions: $f_n \in \mathscr{L}_1(m)$, $n = 1, 2, \ldots$, $f_n(t) \to f(t) \in X$ and $|f_n(t)| \nearrow |f(t)|$ for each $t \in T$, and $\hat{m}(f, T) < +\infty$ imply that the sequence f_n, $n = 1, 2, \ldots$ converges to f in $\mathscr{L}_1(m)$, i.e., that $\hat{m}(f - f_n, T) \to 0$ as $n \to \infty$. This assertion is an immediate consequence of the Lebesgue dominated convergence theorem in $\mathscr{L}_1(m)$, see Theorem II.17 in [6].

Using Theorem 1 in [20] one can easily see that the next theorem contains the result of Theorem 8.8 in [3]. Note also the close relationship between [3] and our part II = [6] with some de facto overlaps.

Before stating the theorem let us recall some notions from part II. By Definition II.4 $f \in \mathscr{L}_1\mathscr{M}(m)$ ($f \in \mathscr{L}_1\mathfrak{J}(m)$) if $f: T \to X$ is $\mathscr{P}$-measurable (integrable), and $\hat{m}(f, T) < +\infty$. Further, $\mathscr{P}^\sim$ denotes the greatest δ-subring of $\mathscr{P}$ on which the semivariation $\hat{m}$ is continuous, see the end of p. 686 in [6].

Theorem 2. *The following conditions are equivalent*:

a) $\mathscr{L}_1\mathscr{M}(m) = \mathscr{L}_1\mathfrak{J}(m)$,

b) $f_n \in \mathscr{L}_1\mathscr{M}(m)$, $n = 1, 2, \ldots$, $|f_n| \cdot |f_k| = 0$ *for* $n \neq k$, $n, k = 1, 2, \ldots$, *and* $\sup_N \hat{m}(\sum_{n=1}^N f_n, T) = \lim_{N\to\infty} \hat{m}(\sum_{n=1}^N |f_n|, T) = \hat{m}(\sum_{n=1}^\infty |f_n|, T) < +\infty$ *imply* $\hat{m}(f_n, T) \to 0$, *and*

c) $\mathscr{L}_1\mathscr{M}(m) = \mathscr{L}_1(m)$,

and if they hold, then m has the Beppo Levi property.

If $\mathscr{P} \subset \sigma(\mathscr{P}^\sim)$ and m has the Beppo Levi property, then, conversely, $\mathscr{L}_1\mathscr{M}(m) = \mathscr{L}_1(m)$.

Proof. a) $\Rightarrow$ b) Suppose the contrary. Then there is an $\varepsilon > 0$ and a sequence f_n, $n = 1, 2, \ldots$ satisfying the assumptions of b) such that $\hat{m}(f_n, T) > \varepsilon$ for each $n = 1, 2, \ldots$. But then by the definition of the L_1-pseudonorm there are $g_n \in S(\mathscr{P}, X)$, $n = 1, 2, \ldots$ such that $|g_n| \leqq |f_n|$ and $|\int_{F_n} g_n \, dm| > \varepsilon$ for each $n = 1, 2, \ldots$, where $F_n = \{t \in T, f_n(t) \neq 0\}$. Put $g = \sum_{n=1}^\infty g_n$. Then clearly $g: T \to X$ is $\mathscr{P}$-measurable, and $\hat{m}(g, T) \leqq \hat{m}(\sum_{n=1}^\infty |f_n|, T) < +\infty$. Hence g is integrable by a), $\int_{F_n} g \, dm = \int_{F_n} g_n \, dm \to 0$ by the countable additivity of the integral (F_n, $n = 1, 2, \ldots$ are pairwise disjoint), a contradiction.

b) $\Rightarrow$ c) Suppose there is an $f \in \mathscr{L}_1\mathscr{M}(m) - \mathscr{L}_1(m)$. Then $\hat{m}(f, \cdot)$ is not continuous, equivalently not exhaustive, on $\sigma(\mathscr{P})$. Hence there is an $\varepsilon > 0$ and a sequence of pairwise disjoint sets $E_n \in \sigma(\mathscr{P})$, $n = 1, 2, \ldots$ such that $\hat{m}(f, E_n) = \hat{m}(f \cdot \chi_{E_n}, T) > \varepsilon$ for each $n = 1, 2, \ldots$. Taking $f_n = f \cdot \chi_{E_n}$, $n = 1, 2, \ldots$ we see that b) does not hold.

c) $\Rightarrow$ a), since $\mathscr{L}_1\mathscr{M}(m) \supset \mathscr{L}_1\mathscr{I}(m) \supset \mathscr{L}_1(m)$ in general.

Clearly c) implies the Beppo Levi property of m.

If $\mathscr{P} \subset \sigma(\mathscr{P}^{\sim})$, then each $\mathscr{P}$-measurable function $f: T \to X$ is $\mathscr{P}^{\sim}$-measurable. Hence for any $\mathscr{P}$-measurable function $f: T \to X$ there is a sequence $f_n \in S(\mathscr{P}^{\sim}, X) \subset \subset \mathscr{L}_1(m)$, $n = 1, 2, \ldots$ such that $f_n(t) \to f(t)$ and $|f_n(t)| \nearrow |f(t)|$ for each $t \in T$. Thus obviously the Beppo Levi property of m implies c). The theorem is proved.

B. A theorem on integration by substitution. Before proceeding to the next theorem let us note that if $v(m, \cdot): \mathscr{P} \to [0, +\infty]$ is σ-finite, and if $f: T \to X$ and $g: T \to \to L(Y, Z)$ are $\mathscr{P}$-measurable functions, then there is a δ-subring $\mathscr{P}' \subset \mathscr{P}$ such that $\mathscr{P} \subset \sigma(\mathscr{P}')$, $v(m, \cdot)$ is finite valued on $\mathscr{P}'$, and both the functions f and g are bounded on each set $E \in \mathscr{P}'$. (Put $\mathscr{P}' = \bigcup_{k=1}^{\infty} \{t \in T, |f(t)| + |g(t)| \leqq k\} \cap \mathscr{P}''$, where $\mathscr{P}'' \subset \mathscr{P}$ is such a δ-ring that $v(m, \cdot)$ is finite valued on it.)

Theorem 3. *Let the variation $v(m, \cdot)$ be finite valued on $\mathscr{P}$, and let $f: T \to X$ and $g: T \to L(Y, Z)$ be $\mathscr{P}$-measurable functions bounded on sets from $\mathscr{P}$. For $E \in \mathscr{P}$ put*

$$l_g(E) = \int_E g \, \mathrm{d}m \quad \text{and} \quad n_f(E) = \int_E f \, \mathrm{d}m \,,$$

where in the first equality we consider $m: \mathscr{P} \to L(X, Y) \subsetneq L(L(Y, Z), L(X, Z))$ ($U \in L(X, Y)$, $U \subsetneq \dot{U} \in L(L(Y, Z), L(X, Z))$, $\dot{U}V = VU$ for $V \in L(Y, Z)$). Then $l_g: \mathscr{P} \to L(X, Z)$ is countably additive in the uniform operator topology with $v(l_g, E) \leqq \|g\|_E \cdot v(m, E) < +\infty$, and $n_f: \mathscr{P} \to Y \subsetneq L(L(Y, Z), Z)$ is countably additive in the uniform operator topology of $L(L(Y, Z), Z)$ with $v(n_f, E) \leqq \|f\|_E \cdot . v(m . E) < +\infty$ for each $E \in \mathscr{P}$.

Further, the function f is integrable with respect to the measure l_g if and only if the function g is integrable with respect to the measure n_f. In this case

$$\int_E f \, \mathrm{d}l_g = \int_E f \, \mathrm{d}(\int_. g \, \mathrm{d}m) = \int_E g \, \mathrm{d}(\int_. f \, \mathrm{d}m) = \int_E g \, \mathrm{d}n_f$$

for each $E \in \sigma(\mathscr{P})$.

Proof. The first assertions are immediate from Theorem I.6, see [5].

Suppose f is integrable with respect to l_g, and let $f_n: T \to X$, $n = 1, 2, \ldots$ be a sequence of $\mathscr{P}$-simple functions such that $f_n(t) \to f(t)$ for each $t \in T$ and $\int_E f \, \mathrm{d}l_g = = \lim_{n \to \infty} \int_E f_n \, \mathrm{d}l_g$ for each $E \in \sigma(\mathscr{P})$. Put $F = \bigcup_{n=1}^{\infty} \{t \in T, f_n(t) \neq 0\} \in \sigma(\mathscr{P})$. According to the Egoroff-Lusin theorem, see Section 1.4 in [5], which obviously remains valid for the σ-finite measure $v(m, \cdot): \sigma(\mathscr{P}) \to [0, +\infty]$, there is a set $N \in \sigma(\mathscr{P}) \cap F$ and a sequence of sets $F_k \in \mathscr{P}$, $k = 1, 2, \ldots$ such that $v(m, N) = 0$, $F_k \nearrow F - N$, and on each F_k, $k = 1, 2, \ldots$ the sequence f_n, $n = 1, 2, \ldots$ converges uniformly to the function f.

Put $\mathscr{P}' = \bigcup_{k=1}^{\infty} (F_k \cup N) \cap \mathscr{P}$. Then $\mathscr{P}' \subset \mathscr{P}$ is a δ-subring, and clearly $\sigma(\mathscr{P}') = = F \cap \sigma(\mathscr{P})$. Hence we may replace $\mathscr{P}$ by $\mathscr{P}'$. But for $E \in \mathscr{P}'$ we have

$$n_{f_n}(E) = \int_E f_n \, \mathrm{d}m \to \int_E f \, \mathrm{d}m = n_f(E) \,,$$

and

$$\sup_n v(n_{f_n}, E) \leqq \sup_n \|f_n\|_E \,.\, v(m, E) < +\infty\,.$$

Further, since f_n, $n = 1, 2, \ldots$ are $\mathscr{P}$-simple functions, clearly

$$\int_E f_n \, \mathrm{d}l_g = \int_E f_n \, \mathrm{d}(\int_. g \, \mathrm{d}m) = \int_E g \, \mathrm{d}(\int_. f_n \, \mathrm{d}m) = \int_E g \, \mathrm{d}n_{f_n} \to \int_E f \, \mathrm{d}l_g$$

for each $E \in \sigma(\mathscr{P}') = F \cap \sigma(\mathscr{P})$.

Thus by Theorem IV.1 in [8] the function g is integrable with respect to the measure n_f: $\mathscr{P}' \to Y \mathbin{\text{Q}} L(L(Y, Z), Z)$, and

$$\int_E g \, \mathrm{d}n_f = \lim_{n \to \infty} \int_E g \, \mathrm{d}n_{f_n} = \int_E f \, \mathrm{d}l_g$$

for each $E \in \sigma(\mathscr{P}') = F \cap \sigma(\mathscr{P})$.

The converse assertion may be proved similarly. The theorem is proved.

3. MEASURABILITY OF THE PARTIAL INTEGRAL AND OTHER SUPPLEMENTS TO PART III

The following simple consequence of Lemma 2, hence of the basic Theorem I.15 in [5], is the key to the affirmative solution of the problem of measurability of the partial integral from Section 2 in part III = [7].

Theorem 4. *Let $\mathscr{P}' \subset \mathscr{P}$ be a δ-subring, and let the function f': $T \to X$ be integrable with respect to the measure m: $\mathscr{P} \to L(X, Y)$ and $\mathscr{P}'$-measurable. Then f' is integrable with respect to the restricted measure $m' = m$: $\mathscr{P}' \to L(X, Y)$, and*

$$\int_E f' \, \mathrm{d}m' = \int_E f' \, \mathrm{d}m$$

for each $E \in \sigma(\mathscr{P}')$.

Proof. Let f'_n: $T \to X$, $n = 1, 2, \ldots$ be a sequence of $\mathscr{P}'$-simple functions such that $f'_n(t) \to f'(t)$ for each $t \in T$, and put $X_1 = \overline{sp}\,\{\bigcup_{n=1}^{\infty} f'_n(T)\}$. Obviously we may replace the measure m by the measure $m_1 = m$: $\mathscr{P} \to L(X_1, Y)$. Since X_1 is a separable Banach space, according to Theorem III.13-1), there is a countably additive measure $\lambda_{F'}$: $F' \cap \sigma(\mathscr{P}) \to [0, 1]$, where $F' = \bigcup_{n=1}^{\infty} \{t \in T,\ f'_n(t) \neq 0\} \in \sigma(\mathscr{P}')$, such that $N \in F' \cap \sigma(\mathscr{P})$ and $\lambda_{F'}(N) = 0$ implies $m_1(N) = 0$. Now by the Egoroff-Lusin theorem, see Section 1.4 in [5], there is a set $N' \in F' \cap \sigma(\mathscr{P}')$ and a sequence of sets $F'_k \in \mathscr{P}'$, $k = 1, 2, \ldots$ such that $\lambda_{F'}(N') = 0$, $F'_k \nearrow F' - N'$, and on each set F'_k, $k = 1, 2, \ldots$ the sequence f'_n, $n = 1, 2, \ldots$ converges uniformly to the function f'. Since the semivariation $\hat{m}$ is σ-finite on $\mathscr{P}$, without loss of generality we may suppose that $\hat{m}(F'_k) < +\infty$ for each $k = 1, 2, \ldots$. But then also $\hat{m}'_1(F'_k) \leqq \hat{m}_1(F'_k) \leqq \hat{m}(F'_k) < +\infty$ for each $k = 1, 2, \ldots$. Now the assertion of the theorem immediately follows from Lemma 2 above.

The following theorem solves affirmatively the measurability problem for the partial integral from Section 2 in part III. Obviously we may suppose that the semivariation $\hat{m}$ is only σ-finite on $\mathscr{P}$. Hence in the Fubini theorem, i.e., in Theorem III.15 the assumption of l-essential $\mathscr{Q}$-measurability of g_E is superfluous and, moreover, we may suppose that the semivariations $\hat{m}$ and $\hat{l}$ are only σ-finite on $\mathscr{P}$ and $\mathscr{Q}$, respectively. Up to the end of this section let us use the notation from part III.

Theorem 5. *Let $f: T \times S \to X$ be a $\mathscr{P} \otimes \mathscr{D}$-measurable function, $\mathscr{D} \subset 2^S$ being an arbitrary δ-ring, and let for each $s \in S$ the function $f(\cdot, s): T \to X$ be integrable with respect to the measure m. Then for each set $E \in \sigma(\mathscr{P} \otimes \mathscr{D})$ the function $g_E: S \to \to Y$, $g_E(s) = \int_{E^s} f(\cdot, s)\,\mathrm{d}m$, $s \in S$, is $\mathscr{D}$-measurable.*

Proof. Let $E \in \sigma(\mathscr{P} \otimes \mathscr{D})$. Since the function $f \cdot \chi_E$ is $\mathscr{P} \otimes \mathscr{D}$-measurable, according to Lemma III.3 there exists a sequence $A_n \in \mathscr{P}$, $n = 1, 2, \ldots$ such that $f \cdot \chi_E$ is $\delta(\{A_n\}_{n=1}^{\infty}) \otimes \mathscr{D}$-measurable. But then by Theorem 4 for each $s \in S$ the function $(f \cdot \chi_E)(\cdot, s)$ is integrable with respect to the restricted measure $m' = = m: \delta(\{A_n\}_{n=1}^{\infty}) \to L(X, Y)$, and

$$g_E(s) = \int_{E^s} f(\cdot, s)\,\mathrm{d}m = \int_{E^s} f(\cdot, s)\,\mathrm{d}m' = g'_E(s)\,.$$

Since $\delta(\{A_n\}_{n=1}^{\infty})$ is countably generated, $g_E = g'_E$ is $\mathscr{D}$-measurable by Theorem III.12-2) in [7].

Concerning the existence of products of operator valued measures, see Theorem III.1 in [7], we add the following result.

Theorem 6. *Let $m: \mathscr{P}_0 \to L(X, Y)$ be countably additive in the strong operator topology and let $v(m(\cdot)\,x, A) < +\infty$ for each $x \in X$ and each $A \in \mathscr{P}_0$ (for example, if Y is the space of scalars and $m: \mathscr{P}_0 \to X^*$ is countably additive in the X-topology of X^*). Let further $l: \mathscr{Q}_0 \to L(Y, Z)$ be countably additive in the uniform operator topology. (We do not suppose that the semivariations $\hat{m}$ and $\hat{l}$ are σ-finite on $\mathscr{P}_0$ and $\mathscr{Q}_0$, respectively.) Then the product measure $l \otimes m: \mathscr{P}_0 \otimes \mathscr{Q}_0 \to L(X, Z)$ exists. If the semivariation $\hat{l}$ is finite on a δ-subring $\mathscr{Q} \subset \mathscr{Q}_0$ then, moreover,*

$$(1) \qquad (l \otimes m)(E)\,x = \int_S m(E^s)\,x\,\mathrm{d}l$$

for each $E \in \mathscr{P}_0 \otimes \mathscr{Q}$ and each $x \in X$.

Proof. For $x \in X$ define $\dot{\mu}_x: \mathscr{P}_0 \to L(L(Y, Z), Z)$ by the equality $\dot{\mu}_x(A)\,U = = U\,m(A)\,x$, where $U \in L(Y, Z)$ and $A \in \mathscr{P}_0$. Then $\dot{\mu}_x: \mathscr{P}_0 \to L(L(Y, Z), Z)$ is evidently countably additive in the strong operator topology and $\hat{\dot{\mu}}_x(A) = v(\dot{\mu}_x, A) = = v(m(\cdot)\,x, A) < +\infty$ for each $A \in \mathscr{P}_0$. Hence $\hat{\dot{\mu}}_x = v(m(\cdot)\,x, \cdot)$ is a finite countably additive measure on $\mathscr{P}_0$. Thus by Theorem III.3 in [7] the product measure $\dot{\mu}_x \otimes l$: $\mathscr{Q}_0 \otimes \mathscr{P}_0 \to Z$ exists, and by Theorem III.1 we have

$$(\dot{\mu}_x \otimes l)(E) = \int_T l(E_t)\,\mathrm{d}\dot{\mu}_x$$

for each $E \in \mathscr{Q}_0 \otimes \mathscr{P}_0$. Since $\dot{\mu}_x(A)\,l(B) = l(B)\,m(A)\,x$ for each $A \in \mathscr{P}_0$, $B \in \mathscr{Q}_0$ and $x \in X$, Lemma III.1 implies that the product measure $l \otimes m: \mathscr{P}_0 \otimes \mathscr{Q}_0 \to$

$\to L(X, Z)$ exists and $(l \otimes m)(E)\, x = (\dot{\mu}_x \otimes l)(E)$ for each $E \in \mathscr{P}_0 \otimes \mathscr{Q}_0 = \mathscr{Q}_0 \otimes \mathscr{P}_0$ and each $x \in X$. Finally, if the semivariation $\hat{l}$ is finite on a δ-subring $\mathscr{Q} \subset \mathscr{Q}_0$, then Theorem III.1 implies (1). The theorem is proved.

The following theorem is related to Theorem III.6.

Theorem 7. *Let $f: T \times S \to X$ be a $\mathscr{P} \otimes \mathscr{D}$-measurable function, let $\mathscr{P}^\vee \subset \mathscr{P}$ be a δ-subring such that $A \in \mathscr{P}$, $B \in \mathscr{P}^\vee$ imply $A \cap B \in \mathscr{P}^\vee$, and let the function $f(\cdot, s)$ be $\mathscr{P}^\vee$-measurable for each $s \in S$. Then the function f is $\mathscr{P}^\vee \otimes \mathscr{D}$-measurable.*

Proof. First we prove the theorem for $\mathscr{P} \otimes \mathscr{D}$-simple functions. However, this is clearly equivalent to proving the theorem for the characteristic function $\chi_E: T \times \times S \to \{0, 1\}$ of each set $E \in \mathscr{P} \otimes \mathscr{D}$. If $E \in \mathscr{R}$, where $\mathscr{R}$ is the ring of all finite unions of pairwise disjoint rectangles $A \times B$, $A \in \mathscr{P}$, $B \in \mathscr{D}$, then obviously χ_E is $\mathscr{P}^\vee \otimes \mathscr{D}$-measurable if $\chi_E(\cdot, s)$ is $\mathscr{P}^\vee$-measurable for each $s \in S$. Denote by $\mathscr{M}$ the class of all sets $E \in \sigma(\mathscr{P} \otimes \mathscr{D})$ for which the theorem is true for χ_E. Then clearly $\mathscr{M}$ is a monotone ring, hence $\mathscr{M} = \sigma(\mathscr{P} \otimes \mathscr{D})$ by Theorem B in § 6 in [15].

Next we show that if $u: T \to X$ is $\mathscr{P}^\vee$-measurable, $v: T \to X$ is $\mathscr{P}$-measurable, and $|v(t)| \leqq |u(t)|$ for each $t \in T$, then v is $\mathscr{P}^\vee$-measurable. Given such functions u and v, take a sequence of $\mathscr{P}^\vee$-simple functions $u_n: T \to X$, $n = 1, 2, \ldots$ such that $u_n(t) \to u(t)$ and $|u_n(t)| \nearrow |u(t)|$ for each $t \in T$, and similarly take a sequence of $\mathscr{P}$-simple functions $v_n: T \to X$, $n = 1, 2, \ldots$ such that $v_n(t) \to v(t)$ and $|v_n(t)| \nearrow |v(t)|$ for each $t \in T$, see Section 1.2 in part I. Now clearly

$$w_n = \frac{|u_n| \wedge |v_n|}{|v_n|}\, v_n\,, \quad n = 1, 2, \ldots$$

is a sequence of $\mathscr{P}^\vee$-simple functions such that $w_n(t) \to v(t)$ for each $t \in T$, hence v is $\mathscr{P}^\vee$-measurable.

Finally, let $f: T \times S \to X$ be $\mathscr{P} \otimes \mathscr{D}$-measurable, and let $f(\cdot, s)$ be $\mathscr{P}^\vee$-measurable for each $s \in S$. Take a sequence of $\mathscr{P} \otimes \mathscr{D}$-simple functions $f_n: T \times S \to X$, $n = = 1, 2, \ldots$ such that $f_n(t, s) \to f(t, s)$ and $|f_n(t, s)| \nearrow |f(t, s)|$ for each $(t, s) \in T \times S$. By the preceding paragraph each function $f_n(\cdot, s)$, $n = 1, 2, \ldots$, $s \in S$, is $\mathscr{P}^\vee$-measurable. Hence by the first paragraph of the proof each f_n, $n = 1, 2, \ldots$ is $\mathscr{P}^\vee \otimes \mathscr{D}$-measurable. Thus f is $\mathscr{P}^\vee \otimes \mathscr{D}$-measurable, and the theorem is proved.

The next Tonelli type result is an improvement of Theorem 8 in [21].

Theorem 8. *Let $v(m, \cdot): \mathscr{P} \to [0, +\infty)$ be σ-finite, let $f: T \times S \to X$ be $\mathscr{P} \otimes \mathscr{Q}$-measurable, and let*

$$q(z^*) = \int_{T \times S} |f(\cdot, s)|\, \mathrm{d}(v(m, \cdot) \otimes v(z^* l, \cdot)) < +\infty$$

for each $z^* \in Z^*$. Then $f(\cdot, s) \in \mathscr{L}_1(m)$ for l-almost every $s \in S$; in particular, $f(\cdot, s)$ is integrable with respect to m for l-almost every $s \in S$.

Proof. By the classical Fubini-Tonelli theorem, see Theorem III.11.14 in [14], since $v(z^*l, \cdot)$ is finite valued on a δ-subring $\mathcal{Q}_1 \subset \mathcal{Q}$ such that $\mathcal{Q} \subset \sigma(\mathcal{Q}_1)$, we have

$$\int_{S\times T} |f(\cdot, s)| \, dv(m, \cdot) \, dv(z^*l, \cdot) = q(z^*) < +\infty$$

for each $z^* \in Z^*$. Hence by Theorem II.4 in [6] and the uniform boundedness principle we have

$$a = \hat{l}(\int_T |f(\cdot, s)| \, dv(m, \cdot), S) = \sup_{|z^*| \leqq 1} q(z^*) < +\infty$$

Now according to Theorem III.6 in [7] the function $h'_{T\times S}: S \to [0, +\infty]$, $h'_{T\times S}(s) = \int_T |f(\cdot, s)| \, dv(m, \cdot)$, $s \in S$, is $\mathcal{Q}$-measurable. Hence by the Tschebyscheff inequality, see Corollary of Theorem II.1 in [6],

$$\hat{l}(\{s \in S, \ h'_{T\times S}(s) = +\infty\}) \leqq \hat{l}(\{s \in S, \ h'_{T\times S}(s) \geqq n\}) \leqq \frac{\hat{l}(h'_{T\times S}, S)}{n} = \frac{a}{n}$$

for each $n = 1, 2, \ldots$. Thus $f(\cdot, s) \in \mathscr{L}_1(v(m, \cdot)) \subset \mathscr{L}_1(m)$ for l-almost every $s \in S$; in particular, $f(\cdot, s)$ is integrable with respect to m for l-almost every $s \in S$, see Lemma II.1 in [6]. The theorem is proved.

The next corollary is immediate.

Corollary. *Let $y^* \in Y^*$ and let $f: T \times S \to X$ be a $\mathscr{P} \otimes \mathcal{Q}$-measurable function. Suppose that*

$$\int_{T\times S} |f(t, s)| \, d(v(y^*m, \cdot) \otimes v(z^*l, \cdot)) < +\infty$$

for each $z^* \in Z^*$. Then the function $f(\cdot, s)$ is integrable with respect to the measure y^*m for l-almost every $s \in S$ (the l-null set depending on y^*).

The next theorem, which may be proved similarly as Theorem 8, shows that in the case $c_0 \not\subset Y$, see pp. 160–161 in [1], the assumptions of the preceding theorem can be weakened.

Theorem 9. *Let $f: T \times S$ be a $\mathscr{P} \otimes \mathcal{Q}$-measurable function, let the function $h_{T\times S}: S \to [0, +\infty]$, $h_{T\times S}(s) = \hat{m}(f(\cdot, s), T)$, $s \in S$, be $\mathcal{Q}$-measurable, and let $\int_S h_{T\times S} \, dv(z^*l, \cdot) < +\infty$ for each $z^* \in Z^*$. Then $\hat{m}(f(\cdot, s), T) < +\infty$ for l-almost every $s \in S$; in particular, the function $f(\cdot, s)$ is weakly integrable,* (see the paragraph before Example on p. 533 in [5]) *for l-almost every $s \in S$. Hence if $c_0 \not\subset Y$, then $f(\cdot, s) \in \mathscr{L}_1(m)$ for l-almost every $s \in S$.* (see Theorem II.10 in [6]); *in particular, $f(\cdot, s)$ is integrable with respect to the measure m for l-almost every $s \in S$.*

Concerning the $\mathcal{Q}$-measurability of the function $h_{T\times S}$ see Theorem III.7 in [7].

4. INDIRECT PRODUCTS OF OPERATOR VALUED MEASURES AND THE GENERAL FUBINI THEOREM

Let $(T, \mathscr{P})$, $(S, \mathcal{Q})$ or $(S, \mathscr{D})$, X, Y, and Z be as in part III, let $l: \mathcal{Q} \to L(Y, Z)$ be countably additive in the strong operator topology, and let its semivariation $\hat{l}$ be σ-finite on $\mathcal{Q}$. Our basic new assumptions are the following:

$(A_0)/(A_0b)/$: For each $s \in S$ an operator valued measure $m(s, \cdot): \mathscr{P} \to L(X, Y)$

is given which is countably additive in the strong operator topology, and the function $s \to m(s, A)\, x \,.\, \chi_B(s)$, $s \in S$, is /bounded/ $\mathcal{Q}$-measurable (or $\mathcal{D}$-measurable).

Throughout this section we assume (A_0). From the well known theorem of Pettis, see Theorem 3.5.5 in [16], we immediately have that for separable Banach spaces X the function $s \to m(s, A) \,.\, \chi_B(s) \in L(X, Y)$, $s \in S$, is $\mathcal{Q}$- (or $\mathcal{D}$-) measurable for each $A \in \mathcal{P}$ and $B \in \mathcal{Q}$ (or $B \in \mathcal{D}$).

We shall also use the following assumption

(A_1): For each $A \in \mathcal{P}$, $B \in \mathcal{Q}$ and $x \in X$ the function $s \to m(s, A)\, x \,.\, \chi_B(s)$, $s \in S$, is integrable with respect to l.

Theorem 10. *Assume* (A_1) *and for* $A \in \mathcal{P}$, $B \in \mathcal{Q}$ *and* $x \in X$ *put*

$$B(A, B)\, x = \beta_x(A, B) = \int_B m(\cdot, A)\, x \mathrm{d}l\,.$$

Then $\beta_x: \mathcal{P} \times \mathcal{Q} \to Z$ *is separately countably additive, i.e., it is a vector bimeasure in the sense of Definition* VIII.1 *in* [11], *for each* $x \in X$. *If either* $(A_0 b)$ *holds, or* X *is separable, then* $B(A, B) \in L(X, Z) = L^{(2)}(X, K; Z)$. *Hence in these cases* $B: \mathcal{P} \times \mathcal{Q} \to L^{(2)}(X, K; Z)$ *is an operator valued bimeasure separately countably additive in the strong operator topology*, (see Definition VIII.1). *If either* $\sup_{s \in B} \hat{m}(s, A) < +\infty$ *for each* $A \in \mathcal{P}$ *and* $B \in \mathcal{Q}$, *or the function* $s \to \hat{m}(s, A) \,.\, \chi_B(s)$, $s \in S$, *is finite valued and* $\mathcal{Q}$*-measurable for each* $A \in \mathcal{P}$ *and* $B \in \mathcal{Q}$, *then the semivariation* $\hat{B}: \mathcal{P} \times \mathcal{Q} \to [0, +\infty]$, (see Definition VIII.3), *is locally* σ*-finite*, (see Definition VIII.5).

Proof. The first assertion of the theorem may be proved similarly as Lemma III.5 in [7].

To the second assertion of the theorem: If X is separable, then by (A_0) and the Pettis theorem, see Theorem 3.5.5 in [16], the function $s \to m(s, A) \,.\, \chi_B(s) \in L(X, Y)$, $s \in S$, is $\mathcal{Q}$-measurable for each $A \in \mathcal{P}$ and $B \in \mathcal{Q}$. Let $A \in \mathcal{P}$ and $B \in \mathcal{Q}$ be fixed. Since by assumption the semivariation $\hat{l}$ is σ-finite on $\mathcal{Q}$, there is a sequence $B'_n \in \mathcal{Q}$, $n = 1, 2, \ldots$ such that $B'_n \nearrow B$ and $\hat{l}(B'_n) < +\infty$ for each $n = 1, 2, \ldots$. For $n = 1, 2, \ldots$ put $B''_n = \{s \in B, |m(s, A)| \leqq n\}$, and let $\mathcal{Q}^* = \bigcup_{n=1}^{\infty} (B'_n \cap B''_n) \cap \mathcal{Q}$. Then $\mathcal{Q}^*$ is a δ-subring of $\mathcal{Q}$ such that $\sigma(\mathcal{Q}^*) \subset \mathcal{Q}$, and $B(A, B) \in L(X, Z)$ for each $B^* \in \mathcal{Q}^*$. Now $B(A, B) \in L(X, Z)$ by Theorem 1 in [12].

If $(A_0 b)$ holds, then we put $\mathcal{Q}^* = \bigcup_{n=1}^{\infty} B'_n \cap \mathcal{Q}$.

The last assertions of the theorem are evident from the definitions and the properties of the integral.

The next definition generalizes Definition III.1 in [7].

Definition 3. Assuming (A_1), we say that the indirect product of the measures $m(s, \cdot): \mathcal{P} \to L(X, Y)$, $s \in S$, and the measure $l: \mathcal{Q} \to L(Y, Z)$ exists, if there is a necessarily unique $L(X, Z)$ valued measure on $\mathcal{P} \otimes \mathcal{Q}$ countably additive in the strong operator topology, which we denote by $l \otimes m(s, \cdot)$, such that

$$(l \otimes m(s, \cdot))(A \times B)\, x = \int_B m(s, A)\, x \, \mathrm{d}l$$

for each $A \in \mathscr{P}$, $B \in \mathscr{Q}$ and $x \in X$.

We now successively check the validity of the results from part III in this general setting of indirect products. When the situation is clear, we omit the formulation of the corresponding theorem, lemma etc.. We will simply write G-Theorem III (G for generalized) and, if necessary, indicate its proof.

G-Lemma III.1 holds – by virtue of Theorem 1 from [12] the proof is evident.

We shall need the following assumptions:

(A_2) $/(A_2b)/$: There is a δ-subring $\mathscr{P}_2 \subset \mathscr{P}$ such that the measure $m(s, \cdot)\colon \mathscr{P}_2 \to L(X, Y)$ is countably additive in the uniform operator topology for each $s \in S$, and the function $s \to \|m(s, \cdot)\|\,(A) \,.\, \chi_B(s)$, $s \in S$, is $\mathscr{D}$-measurable /and bounded/ for each $A \in \mathscr{P}_2$ and $B \in \mathscr{D}$.

(A_3) $/(A_3b)/$: There is a δ-subring $\mathscr{P}^\sim \subset \mathscr{P}$ such that the semivariation $\hat{m}(s, \cdot)\colon \mathscr{P}^\sim \to [0, +\infty)$ is continuous for each $s \in S$, and the function $s \to \hat{m}(s, A) \,.\, \chi_B(s)$, $s \in S$, is $\mathscr{D}$-measurable /and bounded/ for each $A \in \mathscr{P}^\sim$ and $B \in \mathscr{D}$.

G-Lemma III.2, 1) Assume (A_0) $/(A_0b)/$. Then for each $E \in \mathscr{P} \otimes \mathscr{D}$ and each $x \in X$ the function $s \to m(s, E^s)\, x$, $s \in S$, is $\mathscr{D}$-measurable /and bounded/.

2) Assume (A_2) $/(A_2b)/$. Then for each $E \in \mathscr{P}_2 \otimes \mathscr{D}$ the function $s \to \|m(s, \cdot)\|\,(E^s)$, $s \in S$, is $\mathscr{D}$-measurable /and bounded/.

3) Assume (A_3) $/(A_3b)/$. Then for each $E \in \mathscr{P}^\sim \otimes \mathscr{D}$ the function $s \to \hat{m}(s, E^s)$, $s \in S$, is $\mathscr{D}$-measurable /and bounded/.

G-Theorem III.1 holds under the assumption (A_1). The necessity part may be proved as in Theorem III.1, while the sufficiency part may be proved similarly as Lemma III.5 in [7].

G-Theorems III.2 holds.

G-Theorem III.3 holds if we successively assume $(A_1b) = (A_0b) \cap (A_1)$, (A_2b) and (A_3b).

G-Theorem III.4 is valid if $\sup_{s \in B} \hat{m}(s, A) < +\infty$ for each $A \in \mathscr{P}$ and $B \in \mathscr{Q}$.

G-Theorem III.5 is valid if both $\sup_{s \in S} \hat{m}(s, T)$ and $\hat{l}(S)$ are finite.

G-Theorem III.6 holds under the assumption (A_3).

G-Theorem III.7 is valid if the following three conditions are satisfied:

a) Y has a countable norming set,

b) $v(y^* m(s, \cdot), A) < +\infty$ for each $A \in \mathscr{P}$, $s \in S$, and each $y^* \in Y^*$, and

c) the function $s \to v(y^* m(s, \cdot), A) \,.\, \chi_B(s)$, $s \in S$, is $\mathscr{D}$-measurable for each $A \in \mathscr{P}$, $B \in \mathscr{D}$, and each $y^* \in Y^*$.

G-Theorem III.8 holds for the function g_E for each $E \in \mathscr{P}^\sim \otimes \mathscr{Q}$ if (A_3) holds.

G-Theorem III.9 is valid under the assumption (A_3).

We shall need another assumption:

(A_4): For each couple $(A, B) \in \sigma(\mathscr{P}) \times \sigma(\mathscr{D})$ there are $(A_n, B_n) \in \mathscr{P} \times \mathscr{D}$, $n =$

$= 1, 2, \ldots$ such that $A_n \nearrow A$, $B_n \nearrow B$, and $\sup_{s \in B_n} \hat{m}(s, A_n) < +\infty$ for each $n = 1, 2, \ldots$

G-Theorem III.10 is valid under the assumption (A_4).

G-Theorem III.11 holds under the assumption (A_4) in the sense that the set of all finite sums of the form $\sum_{j=1}^{r} m(s, A_j) x_j$, $A_j \in A_0 \cap \mathscr{P}$, $x_j \in X_0$, $s \in B_0$ and $r = 1, 2, \ldots$ is dense in the subset $\{\int_A f(\cdot, s) \, dm(s, \cdot), A \in \sigma(\mathscr{P}), s \in S\}$ of Y, where $A_0 \in \sigma(\mathscr{P})$ and $B_0 \in \sigma(\mathscr{D})$ are such that $\{(t, s) \in T \times S, f(t, s) \neq 0\} \subset A_0 \times B_0$.

G-Corollary of Theorem III.11 is valid under the assumption (A_4) and the assumption that the set $\{m(s, A) x, A \in A_0 \cap \mathscr{P}, s \in B_0\}$ is separable for each $x \in X$ for some $A_0 \in \sigma(\mathscr{P})$, $B_0 \in \sigma(\mathscr{D})$ such that $\{(t, s) \in T \times S, f(t, s) \neq 0\} \subset A_0 \times B_0$.

G-Theorem III.12 holds under the assumption (A_4). (Take A_0 and B_0 as above. If $\mathscr{P} = \delta(\mathscr{R})$, where $\mathscr{R}$ is a countable ring, then the function $s \to m(s, R) x \,.\, \chi_{B_0}(s)$, $s \in S$, being $\mathscr{D}$-measurable by (A_0), is separable valued for each $R \in \mathscr{R}$ and each $x \in X$.)

G-Theorem 5 (VII.5) is valid under the assumption (A_4).

G-Lemma III.5 holds under the assumption (A_4).

For the general Fubini theorem below we shall need our last assumption

(A_5): For each couple $(A, B) \in \mathscr{P} \times \mathscr{Q}$ and each $x \in X$ there is a countably additive measure $\lambda_{A,B,x}: A \cap \mathscr{P} \to [0, 1]$ such that $C \in A \cap \mathscr{P}$, $\lambda_{A,B,x}(C) = 0$ implies $m(s, C) x = 0$ for each $s \in B$.

Let us note that if $m(s, \cdot) = m$ for each $s \in S$, then (A_5) obviously holds, since each countably additive vector measure has a control measure.

The following lemma is immediate.

Lemma 3. *Let X be a separable Banach space and assume (A_5). Then for each couple $(A, B) \in \sigma(\mathscr{Q})$ there is a countably additive measure $\lambda_{A,B}: A \cap \mathscr{P} \to [0, 1]$ such that $C \in A \cap \mathscr{P}$, $\lambda_{A,B}(C) = 0$ implies $m(s, C) = 0$ for each $s \in B$.*

For $A \in \mathscr{P}$ let $\text{ca}(A \cap \mathscr{P}, Y)$ denote the Banach space of all countably additive vector measures $\gamma: A \cap \mathscr{P} \to Y$ with the semivariation norm $\|\gamma\| (A)$. If Y is the space of scalars, we write simply $\text{ca}(A \cap \mathscr{P})$.

The next lemma is also immediate.

Lemma 4. *Assumption (A_5) holds if either*

a) *the set of vector measures $m(s, \cdot) x \in \text{ca}(A \cap \mathscr{P}, Y)$, $s \in B$, is separable for each couple $(A, B) \in \mathscr{P} \times \mathscr{Q}$ and each $x \in X$, or*

b) *Y has a countable norming set, for example, if Y is separable or a dual of a separable Banach space,* (see Theorem 2.8.5 *in* [16]), *and the set of scalar measures $y^* m(s, \cdot) x \in \text{ca}(A \cap \mathscr{P})$, $s \in B$, is separable for each couple $(A, B) \in \mathscr{P} \times \mathscr{Q}$, each $x \in X$, and each $y^* \in Y^*$.*

We are now ready to prove

Theorem 11 (General Fubini Theorem). *Assume (A_1) and (A_4). Let the indirect product $l \otimes m(s, \cdot): \mathscr{P} \otimes \mathscr{Q} \to L(X, Z)$ exist, let $f: T \times S \to X$ be a $\mathscr{P} \otimes \mathscr{Q}$-*

measurable function, and let the function $f(\cdot, s)$ *be integrable with respect to the measure* $m(s, \cdot)$ *for each* $s \in S$. *For* $E \in \sigma(\mathscr{P} \otimes \mathscr{Q})$ *let* $g_E: S \to Y$ *be the* $\mathscr{Q}$*-measurable function* $g_E(s) = \int_{E^s} f(\cdot, s)\, \mathrm{d}m(s, \cdot)$, $s \in S$, (see G-Theorem 5), *and consider the following assertions*:

a) f *is integrable with respect to the indirect product measure* $l \otimes m(s, \cdot)$;

b) *the function* g_E *is integrable with respect to the measure* l *for each set* $E \in \sigma(\mathscr{P} \otimes \mathscr{Q})$, *and*

$$\text{(F)} \qquad \int_E f\, \mathrm{d}(l \otimes m(s, \cdot)) = \int_S \int_{E^s} f(\cdot, s)\, \mathrm{d}m(s, \cdot)\, \mathrm{d}l = \int_S g_E\, \mathrm{d}l$$

for each $E \in \sigma(\mathscr{P} \otimes \mathscr{Q})$.

Then b) *implies* a) *and* (F). *If we assume* (A_5) *then, conversely,* a) *implies* b) *and* (F).

Proof. Take a couple $(A_0, B_0) \in \sigma(\mathscr{P}) \times \sigma(\mathscr{Q})$ such that $\{(t, s) \in T \times S, f(t, s) \neq 0\} \subset A_0 \times B_0$. According to assumption (A_4) there are $(A_k, B_k) \in \mathscr{P} \times \mathscr{Q}$, $k = 1, 2, \dots$ such that $A_k \nearrow A_0$, $B_k \nearrow B_0$, and $\sup_{s \in B_k} \hat{m}(s, A_k) < +\infty$ for each $k = 1, 2, \dots$. By σ-finiteness of the semivariation $\hat{l}$ on $\mathscr{Q}$ there are $B'_k \in \mathscr{Q}$, $k = 1, 2, \dots$ such that $B'_k \nearrow B_0$, and $\hat{l}(B'_k) < +\infty$ for each $k = 1, 2, \dots$.

Let $f_n: T \times S \to X$, $n = 1, 2, \dots$ be a sequence of $\mathscr{P} \otimes \mathscr{Q}$-simle functions such that $f_n(t, s) \to f(t, s)$ and $|f_n(t, s)| \nearrow |f(t, s)|$ for each $(t, s) \in T \times S$, see Section 1.2 in [5]. Put $X_1 = \overline{sp}\,\{\bigcup_{n=1}^{\infty} f_n(T \times S)\}$. Obviously we may replace X by X_1. Since X_1 is separable, by Theorem III.13-1) there is a countably additive measure $\lambda_1: (A_0 \times B_0) \cap \sigma(\mathscr{P} \otimes \mathscr{Q}) \to [0, 1]$ such that $N \in (A_0 \times B_0) \cap \sigma(\mathscr{P} \otimes \mathscr{Q})$ and $\lambda_1(N) = 0$ implies $(l \otimes m(s, \cdot))'(N) = 0$, where $(l \otimes m(s, \cdot))' = l \otimes m(s, \cdot): \mathscr{P} \otimes \mathscr{Q} \to L(X_1, Z)$.

b) $\Rightarrow$ a) and (F). According to G-Lemma III.5 the set function

$$E \to \int_S \int_{E^s} f(\cdot, s)\, \mathrm{d}m(s, \cdot)\, \mathrm{d}l\,, \quad E \in (A_0 \times B_0) \cap \sigma(\mathscr{P} \otimes \mathscr{Q})\,,$$

is a countably additive vector measure. Let $\lambda_2: (A_0 \times B_0) \cap \sigma(\mathscr{P} \otimes \mathscr{Q}) \to [0, 1]$ be its control measure. Put $\lambda = \lambda_1 + \lambda_2$. By the Egoroff-Lusin theorem, see Section 1.4 in [5], there is a set $N \in (A_0 \times B_0) \cap \sigma(\mathscr{P} \otimes \mathscr{Q})$ and a sequence of sets $F_k \in \mathscr{P} \otimes \mathscr{Q}$, $k = 1, 2, \dots$ such that $\lambda(N) = 0$, $F_k \nearrow F - N$, where $F = \{(t, s) \in T \times S, f(t, s) \neq 0\}$, and on each set F_k, $k = 1, 2, \dots$ the sequence f_n, $n = 1, 2, \dots$ converges uniformly to the function f. Put $F'_k = (A_k \times (B_k \cap B'_k)) \cap F_k$, $k = 1, 2, \dots$. Now a) and (F) immediately follow from G-Theorem III.4 and Lemma 2.

(A_5) and a) $\Rightarrow$ b) and (F). Since X_1 is separable, by Lemma 3 there is a countably additive measure $\lambda_{A_0,B_0}: A_0 \cap \mathscr{P} \to [0, 1]$ such that $C \in A_0 \cap \mathscr{P}$ and $\lambda_{A_0,B_0}(C) = 0$ implies $m(s, C) = 0$ for each $s \in B_0$. Let $E \in \sigma(\mathscr{P} \otimes \mathscr{Q})$. Since the function g_E is $\mathscr{Q}$-measurable, by Lemma III.6 there is a countably additive measure $\omega_E: \sigma(\mathscr{Q}) \to [0, 1]$ such that $D \in \sigma(\mathscr{Q})$ and $\omega_E(D) = 0$ implies that $g_E \cdot \chi_D$ is integrable with respect to l and $\int_D g_E\, \mathrm{d}l = 0$. It remains to put $\mu_E(G) = \lambda_1(G) + (\omega_E \otimes \lambda_{A_0,B_0})(G)$ for $G \in \sigma(\mathscr{P} \otimes \mathscr{Q})$ and to apply G-Theorem III.4 and Lemma 2 to obtain b) and (F).

Finally, as a corollary we have that G-Theorem III.16 holds if $\sup_{s \in S} \hat{m}(s, T) < +\infty$.

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Author's address: 814 73 Bratislava, Obrancov mieru 49, Czechoslovakia (Matematický ústav SAV).