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AN EXTENSION THEOREM FOR CONTINUOUS FUNCTIONS

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1. INTRODUCTION

In this paper we present some results concerning the extension to a residual subset of a continuous function defined on a dense subset of a topological space. It is well-known that if the range space is completely metrizable, any such function has a continuous extension to a G_δ subspace of the domain.

We will consider the non-metrizable case. Zdeněk Frolík showed in [1] that if the range space is a m-space for $m = \aleph_0$ the extension problem has a solution.

We give a generalization of this result. For example, if the range space is a Čech-complete space with a G_δ diagonal the extension problem has a solution also. There exists a Čech-complete space with a G_δ diagonal which is not a m-space for $m = \aleph_0$ ([3]).

Further Zdeněk Frolík proved in [1] the following theorem: Let Y be a m-space. Let A be a dense subset of a space X. Let f be a continuous mapping from A to Y. Then there exist a $G(m)$-subset S of X containing A and a continuous mapping F from S to Y such that f is a restriction of F.

We give a generalization of this result for m^*-spaces and also an extension theorem for upper semicontinuous set-valued mappings.

The terminology and notation of J. Kelley will be used throughout. Moreover we shall use following notions and notations. A system is a synonym for indexed family. If m is a cardinal number, then an m-system is a system whose index set is of cardinal m. A family $\mathscr{U}$ of sets has the finite intersection property if the intersection of every finite subfamily is not empty. A centered family is a family of sets having the finite intersection property.

The closure of a subset M of a space X will be denoted by $\mathrm{cl}M$. If $\mathscr{U}$ is a family of subsets of X then the family of closures of all sets of $\mathscr{U}$ will be denoted by $\overline{\mathscr{U}}$. An open (closed) family of a space X is a family consisting of open (closed) subsets of X. Analoguous conventions will be used for systems.

The intersection of a family $\mathscr{U}$ of sets will be denoted by $\bigcap\mathscr{U}$, the union by $\bigcup\mathscr{U}$.

If Y is a set, 2^Y denotes the collection of subsets of Y and $F: X \to 2^Y$ a set-valued mapping from X to Y.

In what follows X and Y are topological spaces. All (topological) spaces will be supposed to be Hausdorff.

A set-valued mapping $F: X \to 2^Y$ is upper semicontinuous at $x \in X$ if for every open set V in Y such that $F(x) \subset V$ there exists an open set U in X such that $x \in U$ and $F(z) \subset V$ for every $z \in U$.

$F: X \to 2^Y$ is lower semicontinuous at $x \in X$ if for every open set V in Y such that $F(x) \cap V \neq \emptyset$ there exists an open set U in X such that $x \in U$ and $F(z) \cap V \neq \emptyset$ for every $z \in U$.

F is upper semicontinuous (lower semicontinuous) if F is upper semicontinuous (lower semicontinuous) at every $x \in X$.

Let A be a subset of X and $F: A \to 2^Y$ be a mapping (set-valued or single-valued). A set-valued mapping $F^*: A \to 2^Y$ is an extension of F if $F^*(x) = F(x)$ for every x in A.

N denotes the natural numbers.

2. AN EXTENSION THEOREM FOR UPPER SEMICONTINUOUS SET-VALUED MAPPINGS

Definition 1. (See [1]) A subset G of a space X is said to be a *$G(m)$-subset* of X, if it is the intersection of some open m-system in X.

Definition 2. (See [1]) A system $\{\mathcal{B}_i: i \in I\}$ of open coverings of a space X is said to be *complete* if the following condition is satisfied: If $\mathcal{U}$ is an open centered family in X such that $\mathcal{U} \cap \mathcal{B}_i \neq \emptyset$ for each $i \in I$ then $\bigcap \overline{\mathcal{U}} \neq \emptyset$.

We shall need this proposition

Proposition 1. (See [1]) *Let $\{\mathcal{B}_i: i \in I\}$ be a complete system of open coverings of a regular space X. Suppose that $\mathcal{M}$ is a centered family of subsets of X such that for each i in I there exists a $M \in \mathcal{M}$ and a finite subfamily $\mathcal{U}_i$ of $\mathcal{B}_i$ which covers M. Then $\bigcap \overline{\mathcal{M}} \neq \emptyset$.*

Remark 1. It follows from Theorem 2.8 in [1] that if Y is a Čech-complete space then Y possesses a complete countable system of open coverings of Y.

Theorem 1. *Let X, Y be topological spaces, A be a dense subset of X, Y regular. Suppose that Y possesses a complete m-system of open coverings. Let $F: A \to 2^Y$ be an upper semicontinuous compact-valued mapping. There exist a $G(m)$-subset S of X containing A and an upper semicontinuous compact-valued extension F^* of F defined on S.*

Proof. Let $\{\mathcal{U}_i: i \in I\}$ be a complete m-system of open coverings of the space Y. For each i in I denote by $\mathcal{M}_i$ the family of all open subsets W of X such that there exists a finite subfamily $\mathcal{V}_i \subset \mathcal{U}_i$ with property cl $F(W \cap A) \subset \bigcup V_i$. For each i in I denote by A_i the union of the family $\mathcal{M}_i$. Consider the space $S = \bigcap\{A_i: i \in I\}$. It is obvious that S is a $G(m)$-subset of X. We show $A \subset S$. Let $x \in A$ and $i \in I$.

Consider the family $\mathscr{G}_i$ of all open sets V in Y such that $V \subset \overline{V} \subset U$ for some $U \in \mathscr{U}_i$. Since $F(x)$ is compact, there exists a finite subfamily $\mathscr{H}_i \subset \mathscr{G}_i$ such that $F(x) \subset \bigcup \mathscr{H}_i$. Then there exists a finite subfamily $\mathscr{H}'_i \subset \mathscr{U}_i$ such that $\bigcup \overline{\mathscr{H}}_i \subset \bigcup \mathscr{H}'_i$. The upper semicontinuity of F at x implies there exists an open neighbourhood G of x such that $F(G \cap A) \subset \bigcup \mathscr{H}_i$. Then cl $F(G \cap A) \subset \bigcup \mathscr{H}'_i$, that means $x \in A_i$.

We shall now construct the set-valued mapping F^*. Let $x \in S$ and let $\mathscr{B}(x)$ denote an open neighbourhood base at the point x.

First we show that $\bigcap\{\text{cl } F(H \cap A): H \in \mathscr{B}(x)\} \neq \emptyset$. The system $\{\text{cl } F(H \cap A): H \in \mathscr{B}(x)\}$ is centered and satisfies the conditions of Proposition 1. By this proposition $\bigcap\{\text{cl } F(H \cap A): H \in \mathscr{B}(x)\} \neq \emptyset$. Denote this intersection by $F^*(x)$.

For every $x \in A$ we have $F^*(x) = F(x)$. The inclusion $F(x) \subset F^*(x)$ is obvious. Suppose there exists $y \in F^*(x) \setminus F(x)$. Regularity of Y implies there exist open sets G_1, G_2 such that $F(x) \subset G_1$, $y \in G_2$ and $G_1 \cap G_2 = \emptyset$. Let G'_1 be an open subset of G_1 such that $F(x) \subset G'_1 \subset \text{cl}G'_1 \subset G_1$. The upper semicontinuity of F at x implies there exists $H \in \mathscr{B}(x)$ such that $F(H \cap A) \subset G'_1$. Then cl $F(H \cap A) \subset G_1$. That is a contradiction since $y \in \text{cl } F(H \cap A)$ but $y \notin G_1$.

$F^*(x)$ is compact for every $x \in S$. Let $x \in S \setminus A$. Let $\mathscr{K}$ be a centered family of subsets of $F^*(x)$ such that for every $F \in \mathscr{K}$ F is a closed set in $F^*(x)$. Since $F^*(x)$ is closed in Y, $\mathscr{K}$ is centered family of closed sets in Y, which satisfies the conditions of Proposition 1. Then $\bigcap \mathscr{K} \neq \emptyset$. That means $F^*(x)$ is compact.

F^* is upper semicontinuous. Let $x \in A$. Let U be an open set in Y such that $F^*(x) \subset U$. $F^*(x) = F(x)$ is a compact set and so there exists an open set G in Y such that $F(x) \subset G \subset \text{cl}G \subset U$. The upper semicontinuity of F at x implies there exists an open neighbourhood W of x such that $F(W \cap A) \subset G$. Let $z \in W \cap (S \setminus A)$. Since W is an open neighbourhood of z $F^*(z) \subset \text{cl } F(W \cap A)$ and cl $F(W \cap A) \subset \text{cl}G \subset U$. Upper semicontinuity of F at $x \in A$ is proved.

Now let $x \in S \setminus A$. Let U be an open set in Y such that $F^*(x) \subset U$. There exists $V \in \mathscr{B}(x)$ such that cl $F(V \cap A) \subset U$. Suppose contrary. Then $(\text{cl } F(H \cap A)) \cap (Y \setminus U) \neq \emptyset$ for every $H \in \mathscr{B}(x)$. The system $\{(\text{cl }(H \cap A)) \cap (Y \setminus U): H \in \mathscr{B}(x)\}$ is centered and satisfies the conditions of Proposition 1. By this proposition $\bigcap\{(\text{cl } F(H \cap A)) \cap (Y \setminus U): H \in \mathscr{B}(x)\} \neq \emptyset$. That is a contradiction since $\bigcap\{(\text{cl } F(H \cap A)) \cap (Y \setminus U): H \in \mathscr{B}(x)\} \subset F^*(x) \subset U$.

Corollary 1. *Let X, Y be topological spaces, A be a dense subset of X. Let Y be a Čech-complete space. Let $F: A \to 2^Y$ be an upper semicontinuous compact-valued mapping. There exist a G_δ subset S of X containing A and an upper semicontinuous compact-valued extension F^* of F defined on S.*

Proof. By Remark 1 Y possesses a complete m-system of open coverings for $m = \aleph_0$.

3. AN EXTENSION THEOREM FOR CONTINUOUS FUNCTIONS

Definition 3. (See [1]) A space Y is said to be a *m-space* if there exists a complete m-system $\{\mathscr{B}_i: i \in I\}$ of open coverings of Y such that for each y in Y the family $\{\mathrm{St}(y, \mathscr{B}_i): i \in I\}$ is a local base at y. (If $\mathscr{U}$ is a family of subsets, then $\mathrm{St}(y, \mathscr{U}) = \bigcup\{A: A \in \mathscr{U},\ y \in A\}$.

Definition 4. A space Y is said to be a *m*-space* if there exists a complete m-system $\{B_i: i \in I\}$ of open coverings of Y such that for each $y \in Y \bigcap\{\mathrm{St}(y, \mathscr{B}_i): i \in I\} = \{y\}$.

Remark 2. It is obvious that a m-space is a m^*-space. (Let Y be a m-space. Let $y \in Y$. We show that $\bigcap\{\mathrm{St}(y, \mathscr{B}_i): i \in I\} = \{y\}$. Suppose there exist $v \neq y$ such that $v \in \bigcap\{\mathrm{St}(y, \mathscr{B}_i): i \in I\}$. There exist open sets U_1, U_2 in Y such that $v \in U_1$, $y \in U_2$ and $U_1 \cap U_2 = \emptyset$. There exists $i \in I$ such that $\mathrm{St}(y, \overline{\mathscr{B}}_i) \subset U_2$. That means $v \notin \mathrm{St}(y, \mathscr{B}_i)$ and thus $v \notin \bigcap\{\mathrm{St}(y, \mathscr{B}_i): i \in I\}$.)

Remark 3. If Y is a Čech-complete space with a G_δ diagonal or Y is a Čech-complete Moore space [2] then Y is a m^*-space for $m = \aleph_0$. It is obvious that for $m = \aleph_0$ Y is a m^*-space if and only if Y possesses a complete countable system of open coverings and Y has a G_δ diagonal. (By result of Ceder [4] Y has a G_δ diagonal if and only if Y has a sequence of open covers $\{\mathscr{G}_i\}$ such that for every $y \in Y$ $\bigcap \mathrm{St}(y, \mathscr{G}_i) = \{y\}$.)

Theorem 2. *Let X, Y be topological spaces, A be a dense subset of X. Let Y be a regular m^*-space. Let f be a continuous mapping from A to Y. Then there exist a $G(m)$-subset S of X containing A and a continuous mapping F from S to Y such that f is a restriction of F.*

Proof. Let $\{\mathscr{U}_i: i \in I\}$ be a complete m-system of open coverings of Y such that for each y in $Y \bigcap\{\mathrm{St}(y, \mathscr{U}_i): i \in I\} = \{y\}$. For each i in I denote by $\mathscr{M}_i$ the family of all open subsets W of X such that there exists a set $U_i \in \mathscr{U}_i$ with property $\mathrm{cl} f(W \cap A) \subset U_i$. For each i in I denote by A_i the union of the family $\mathscr{M}_i$. Consider the space $S = \bigcap\{A_i: i \in I\}$. It is obvious that S is a $G(m)$-subset of X. We show $A \subset S$. Let $x \in A$ and $i \in I$. There exists a set $U_i \in \mathscr{U}_i$ such that $f(x) \in U_i$. Since Y is regular there exists an open set V_i in Y such that $f(x) \in V_i \subset \mathrm{cl} V_i \subset U_i$.

The continuity of f at x implies there exists an open neighbourhood G of x such that $f(G \cap A) \subset V_i$. Then $\mathrm{cl} f(G \cap A) \subset U_i$, that means $x \in A_i$. Now we shall construct the set-valued mapping $F: S \to 2^Y$ analogical as in the proof of Theorem 1. Also $F(x) = \bigcap\{(\mathrm{cl} f(V \cap A)): V \in \mathscr{B}(x)\}$, where $\mathscr{B}(x)$ denotes an open neighbourhood base at the point x.

Then F is upper semicontinuous and $F(x) = \{f(x)\}$ for every $x \in A$. (The proof of this fact is analogical as in the proof of Theorem 1.) F is single-valued at every $x \in S$. Let $x \in S \setminus A$. Suppose there exist z, v such that $z \neq v$, $z \in F(x)$, $v \in F(x)$. Then for every i in I there exists $U_i \in \mathscr{U}_i$ such that $\mathrm{cl} f(A \cap G) \subset U_i$ for some $G \in \mathscr{B}(x)$, that means $z \in U_i$, $v \in U_i$. But $\{z\} = \bigcap\{\mathrm{St}(z, \mathscr{U}_i): i \in I\} \supset \bigcap\{U_i: i \in I\} \ni v$ and that is a contradiction.

Since $F: X \to 2^Y$ is single-valued and upper semicontinuous, F is continuous function.

Corollary 2. (See [1]) *Let Y be a m-space. Let A be a dense subset of a space X. Let f be a continuous mapping from A to Y. Then there exist a $G(m)$-subset S of X containing A and a continuous mapping F from S to Y such that f is a restriction of F.*

Corollary 3. (See [2]) *Let Y be a Čech-complete Moore space and let $f: A \to Y$ be a continuous mapping, with A dense in X. Then there exist a G_δ subset S of X containing A and a continuous mapping F from S to Y such that f is a restriction of F.*

Corollary 4. *Let Y be a Čech-complete space with a G_δ diagonal and let $f: A \to Y$ be a continuous mapping, with A dense in X. Then there exist a G_δ subset S of X containing A and a continuous mapping F from S to Y such that f is a restriction of F.*

Suppose A is dense subset of X and a function $f: A \to Y$ is continuous. If there exists a set-valued mapping $F: X \to 2^Y$ such that F is an extension of f (that means $F(x) = \{f(x)\}$ for every x in A) and F is lower semicontinuous, then F is single-valued.

Let $x \in X \setminus A$. Suppose there exist y, v such that $y \neq v$ and $y \in F(x)$, $v \in F(x)$. There exist open sets U_1, U_2 in Y such that $y \in U_1$, $v \in U_2$ and $U_1 \cap U_2 = \emptyset$. Lower semicontinuity of F at x implies there exist open sets V_1, V_2 in X such that $x \in V_1$, $x \in V_2$ and $F(z) \cap U_1 \neq \emptyset$ for every $z \in V_1$ and $F(z) \cap U_2 \neq \emptyset$ for every $z \in V_2$. Put $V = V_1 \cap V_2$. A is dense in X, that means $V \cap A \neq \emptyset$. Let $t \in V \cap A$. Then $f(t) \in U_1, f(t) \in U_2$ and that is a contradiction.

Under some conditions, every upper semicontinuous set-valued mapping is lower semicontinuous at points of a residual set. For example: Let $F: X \to 2^Y$ be an upper semicontinuous set-valued mapping with compact values. Then

[7] Fort: If Y is a metrizable space, F is lower semicontinuous at points of a residual set.

[5] Miškin: If Y is a σ-space, F is lower semicontinuous at points of a residual set.

[8] Kenderov: If there exists a metrizable topology ϱ on Y such that ϱ is weaker than τ, where τ is an origin topology on Y, then F is τ-lower semicontinuous at points of a residual set.

Theorem 3. *Let X and Y be topological spaces (Y regular). Let Y be such that each upper semicontinuous set-valued mapping defined on any topological space Z with compact values in Y is lower semicontinuous at points of a residual subset of Z. And suppose Y possesses a complete countable system of open coverings. Let $f: A \to Y$ be a continuous mapping with A dense in X. Then f has a continuous extension to a residual subset of X containing A.*

Proof. Let $\{\mathcal{G}_i : i \in N\}$ be a complete sequence of open coverings of Y. For each $i \in N$ denote by $\mathcal{M}_i$ the family of all open subsets W of X such that there exists a set $G_i \in \mathcal{G}_i$ with property $\operatorname{cl} f(W \cap A) \subset G_i$. For each $i \in N$ denote by A_i the union of the family $\mathcal{M}_i$. Consider the space $S = \bigcap\{A_i : i \in N\}$. Then S is a G_δ subset of X and $A \subset S$. We construct the set-valued mapping $F: S \to 2^Y$ analogical as in the proof of Theorem 1. F is upper semicontinuous with compact values and $F(x) = \{f(x)\}$ for every $x \in A$.

By assumption $F: S \to 2^Y$ is lower semicontinuous at points of a residual subset $L \subset S$. Then $A \subset L$, since $F(x) = \{f(x)\}$ for $x \in A$. By above argument F is single-valued at every $x \in L$, that means F is continuous function from L to Y.

L is residual set in X. ($X \setminus L = (X \setminus S) \cup (S \setminus L)$ and $X \setminus S$ is the set of the first category in X, $S \setminus L$ is the set of the first category in S. Then $S \setminus L = \bigcup\{E_n : n \in N\}$, where E_n is nowhere dense in S for every $n \in N$. Suppose there exists $n \in N$ such that E_n is not nowhere dense in X. There exists a non-empty open set V in X such that $V \subset \operatorname{cl} E_n$ ($\operatorname{cl} E_n$ is the closure of E_n in X). Then $V \cap S \neq \emptyset$ and that is a contradiction.)

References

[1] *Zdeněk Frolík:* Generalizations of the G_δ-property of complete metric spaces, Czechoslovak Mathematical Journal, *10* (85) 1960.

[2] *Sandro Levi:* Set-valued mappings and an extension theorem for continuous functions, to appear.

[3] *D. Burke:* A nondevelopable locally compact Hausdorff space with G_δ diagonal, Gen. Topology Appl. *2* (1972) 287—291.

[4] *J. Ceder:* Some generalizations of metric spaces, Pac. J. Math. *11* (1961) 105—125.

[5] *V. Miškin:* Upper and lower semi-continuous set-valued maps into $\mathfrak{S}$-spaces, in: J. Novák, ed., General Topology and its relations to modern analysis and algebra V (Helderman Verlag, Berlin, 1983) 486—487.

[6] *C. Bessaga, A. Pelczynski:* Infinite dimensional topology, Warszava 1975.

[7] *M. K. Fort:* Points of continuity of semi-continuous functions, Public. Math. Debrecen, *2* (1951) 100—102.

[8] *P. Kenderov:* Semi-continuity of set-valued monotone mappings, Fundamenta Mathematicae L XXXVIII. 1, 1975, 61—69.

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