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THE *LMC*-COMPACTIFICATION OF A TOPOLOGIZED SEMIGROUP

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1. Introduction. The theory of right topological semigroups (i.e. semigroups S with a Hausdorff topology such that, for each $s \in S$, the function ϱ_s: $S \to S$ defined by $\varrho_s(t) = ts$ is continuous), especially compact right topological semigroups, has been extensively developed. See for example [2]. The existence of a maximal right topological compactification of a semigroup with topology (or a semitopological or a right topological or a left topological semigroup) would be of considerable interest. However, Example V.I.II of [2], essentially due to J. W. Baker, shows that such a maximal compactification cannot in general exist. (By a compactification of a semigroup S with topology, we mean a pair (ϕ, X) consisting of a compact Hausdorff semigroup X and a continuous homomorphism $\phi: S \to X$ with $\phi[S]$ dense in X. A compactification (ϕ, X) of S is maximal with respect to a given property if it possesses the property and satisfies the universal extension condition: whenever (λ, Y) is a compactification of S possessing the property, there is a continuous homomorphism $\eta: X \to Y$ such that $\eta \circ \phi = \lambda$. Note that ϕ is not required to be an embedding so the pair (ϕ, X) need not be a topological compactification.)

It was shown in Theorem III.4.5 of [2] that any Hausdorff semitopological semigroup (i.e. one which is both left and right topological) has a compactification (e, X) maximal with respect to the property that it is right topological and the requirement that $\lambda_{e(s)}$ be continuous for each $s \in S$. (Here $\lambda_x(y) = xy$.) We show here in Section 2 that the same conclusion applies to any semigroup S with a topology. No continuity assumptions need be made. One does not even need any separation axioms to apply.

In Section 3 we show that similar results hold with respect to the strong almost periodic, almost periodic, and weak almost periodic compactification of S.

Of course, since $e[S]$ will be semitopological, if S is not semitopological e cannot be an embedding. We determine in Section 4 when e is one-to-one and when, as a mapping to $e[S]$ it is open. We then present an example showing that one can have Hausdorff semigroups which are neither left nor right topological with e one-to-one

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and also such semigroups with e open. We also show that one can have completely regular Hausdorff semitopological semigroups where e is neither one-to-one nor open.

In Section 5, we provide explicit descriptions of the LMC-compactification for several different semigroups.

We conclude this section with the following preliminary results.

1.1. Lemma. *Let M and T be Hausdorff right topological semigroups and let S be a semigroup and a topological space. Let $\gamma: S \to M$ and $\phi: S \to T$ be continuous homomorphisms. Assume that $\gamma[S]$ is dense in M and that, for each $s \in S$, the functions $\lambda_{\gamma(s)}$ and $\lambda_{\phi(s)}$ are continuous (in M and T respectively). If $\eta: M \to T$ is continuous and $\eta \circ \gamma = \phi$, then η is a homomorphism.*

Proof. We must prove that $\eta(ab) = \eta(a)\,\eta(b)$ for all $a, b \in M$. Note first that if $a, b \in \gamma[S]$ (so that $a = \gamma(s)$ and $b = \gamma(t)$) we have $\eta(ab) = \eta(\gamma(s)\,\gamma(t)) = \eta(\gamma(st)) = \phi(st) = \phi(s)\,\phi(t) = \eta(\gamma(s))\,\eta(\gamma(t)) = \eta(a)\,\eta(b)$.

Now, given $a \in \gamma[S]$ with $a = \gamma(s)$, the continuous functions $\lambda_{\phi(s)} \circ \eta$ and $\eta \circ \lambda_{\gamma(s)}$ agree on the dense subspace $\gamma[S]$ of M. Thus given $b \in M$, $\eta(ab) = \eta \circ \lambda_{\gamma(s)}(b) = \lambda_{\phi(s)} \circ \eta(b) = \eta(a)\,\eta(b)$.

Finally, given $b \in M$, the continuous functions $\eta \circ \varrho_b$ and $\varrho_{\eta(b)} \circ \eta$ agree on $\gamma[S]$. Thus given $a \in M$, $\eta(ab) = \eta \circ \varrho_b(a) = \varrho_{\eta(b)} \circ \eta(a) = \eta(a)\,\eta(b)$. □

The proof of the following lemma is similar, and we omit it.

1.2. Lemma. *Let T be a Hausdorff topological space and let S be a semigroup which is also a topological space. Let $\cdot$ be a binary operation on T which is right continuous, let $\phi: S \to T$ be a continuous homomorphism, and assume $\phi[S]$ is dense in T and $\lambda_{\phi(s)}$ is continuous for each $s \in S$. Then $\cdot$ is associative.*

2. The LMC-compactification. Throughout this section, let S represent a semigroup which is also a topological space. (No separation axioms are assumed.) Let K represent either the real or complex numbers. (It is customary to take $K = C$ but for the theory the reals suffice.) Denote by $C(S)$ the set of continuous bounded functions from S to K.

If S is semitopological and Hausdorff and $(e, \delta S)$ is the compactification maximal with respect to the properties that δS is right topological and that $\lambda_{e(s)}$ is continuous for each $s \in S$, then the functions from S to K which extend continuously to δS are called the *LMC-functions*, and δS is called the *LMC-compactification of S*.

The class of LMC functions has been studied extensively; see for example [2], [7] and [8], the first of which also gives the LMC compactification theorem.

The LMC functions were characterized [6], for semitopological S, as those $f \in C(S)$ such that $\mathrm{cl}\,\{f \circ \varrho_s: s \in S\} \subseteq C(S)$, where the closure is taken in the product space K^S. We extend the definition of LMC to our arbitrary S with topology and observe that the notions coincide if S is semitopological.

2.1. Definition. The function f is defined to be in LMC if and only if
(a) $f \in C(S)$,

(b) $\{f \circ \lambda_s: s \in S\} \subseteq C(S)$,

(c) $\mathrm{cl}\,\{f \circ \varrho_s: s \in S\} \subseteq C(S)$, and

(d) for each $t \in S$, $\mathrm{cl}\,\{f \circ \lambda_t \circ \varrho_s: s \in S\} \subseteq C(S)$.

We can express this definition more succinctly by agreeing that $\lambda_1 = \varrho_1 = \iota$, the identity map. Then $f \in LMC$ if and only if for each $t \in S \cup \{1\}$, $\mathrm{cl}\,\{f \circ \lambda_t \circ \varrho_s: s \in S \cup \cup \{1\}\} \subseteq C(S)$.

2.2. Lemma. *Let T be a compact Hausdorff right topological semigroup and let ϕ be a continuous homomorphism from S to T such that $\lambda_{\phi(s)}$ is continuous for each $s \in S$. If $f \in C(T)$, then $f \circ \phi \in LMC$.*

Proof. Let $t \in S \cup \{1\}$ and let $g \in \mathrm{cl}\,\{(f \circ \phi) \circ \lambda_t \circ \varrho_s: s \in S \cup \{1\}\}$. Observe that if $g = f \circ \phi \circ \lambda_t$ then g is continuous; the boundedness of g follows from the boundedness of f. Thus we assume $g \in \mathrm{cl}\,\{(f \circ \phi) \circ \lambda_t \circ \varrho_s: s \in S\}$. It suffices to show that there is some $a \in T$ such that for each $x \in S$, $f \circ \lambda_{\phi(tx)}(a) = g(x)$. For then, if $t = 1$, $g = f \circ \varrho_a \circ \phi$ and if $t \in S$, $g = f \circ \lambda_{\phi(t)} \circ \varrho_a \circ \phi$. In either case, g is the composition of continuous functions.

Pick a net $\langle s_\alpha \rangle_{\alpha \in I}$ in S such that $\langle f \circ \phi \circ \lambda_t \circ \varrho_{s_\alpha} \rangle_{\alpha \in I}$ converges to g. By taking a subnet if necessary we may assume $\langle \phi(s_\alpha) \rangle_{\alpha \in I}$ converges to some $a \in T$. Then for each $x \in S$ and $\alpha \in I$, $f \circ \lambda_{\phi(tx)}(\phi(s_\alpha)) = \pi_x(f \circ \phi \circ \lambda_t \circ \varrho_{s_\alpha})$, so $f \circ \lambda_{\phi(tx)}(a) = g(x)$ as required. □

We are now ready to define the function e and to define δS as a topological space. We use the product space $X_{f \in LMC}\, \mathrm{cl}_K f[S]$. Those familiar with the terminology customarily used in analysis may wish to observe that the function e takes S to the dual space $C(S)^*$ of $C(S)$ and that the relative topology on $C(S)^*$ is the weak*-topology.

2.3. Definition. (a) *Define $e: S \to X_{f \in LMC}\, \mathrm{cl}_K f[S]$ by $e(s)(f) = f(s)$.*

(b) $\delta S = \mathrm{cl}\, e[S]$.

We observe immediately that δS is compact (by the Tychonoff Theorem) and Hausdorff and that $e[S]$ is dense in δS. We set out now to define the multiplication on δS.

2.4. Lemma. *Let $f \in LMC$ and let $x \in S$. Then $f \circ \lambda_x \in LMC$.*

Proof. Let $t \in S \cup \{1\}$. Then $\{(f \circ \lambda_x) \circ \lambda_t \circ \varrho_s: s \in S \cup \{1\}\} = \{f \circ \lambda_{xt} \circ \varrho_s: s \in S \cup \{1\}\}$. □

2.5. Definition. *For $f \in LMC$ and $v \in \delta S$, define $h_{v,f}: S \to K$ by $h_{v,f}(s) = v(f \circ \lambda_s)$.*
Observe that by Lemma 2.4, v is defined at $f \circ \lambda_s$.

2.6. Lemma. Let $f \in LMC$ and $v \in \delta S$. Then $h_{v,f} \in LMC$.

Proof. Let $t \in S \cup \{1\}$. We show that for each $x \in S \cup \{1\}$, $h_{v,f} \circ \lambda_t \circ \varrho_x \in \in \mathrm{cl}\,\{f \circ \lambda_t \circ \varrho_s: s \in S \cup \{1\}\}$. One then has immediately that $\mathrm{cl}\,\{h_{v,f} \circ \lambda_t \circ \varrho_x: x \in S \cup \cup \{1\}\} \subseteq \mathrm{cl}\,\{f \circ \lambda_t \circ \varrho_s: s \in S \cup \{1\}\} \subseteq C(S)$.

To this end, let $x \in S \cup \{1\}$, let F be a finite subset of S, and for each $y \in F$, let U_y be a neighbourhood of $h_{v,f} \circ \lambda_t \circ \varrho_x(y)$. Then $U = \bigcap_{y \in F} \pi_y^{-1}[U_y]$ is a basic neighborhood of $h_{v,f} \circ \lambda_t \circ \varrho_x$. Let $V = \bigcap_{y \in F} \pi_{f \circ \lambda_{tyx}}^{-1}[U_y]$. (Observe that the projections $\pi_y: K^S \to K$ and $\pi_{f \circ \lambda_{tyx}}: X_{g \in LMC}\, \mathrm{cl}_K\, g[S] \to K$.) Now given $y \in F$, $h_{v,f} \circ \lambda_t \circ \varrho_x(y) = h_{v,f}(tyx) = v(f \circ \lambda_{tyx})$ so that V is a neighbourhood of v. Pick $s \in S$ such that $e(s) \in V$. Then, given $y \in F$, $f \circ \lambda_t \circ \varrho_{xs}(y) = f(tyxs) = f \circ \lambda_{tyx}(s) = e(s)\,(f \circ \lambda_{tyx}) \in U_y$. Thus $f \circ \lambda_t \circ \varrho_{xs} \in U$ as required. □

2.7. Definition *For $\mu, v \in \delta S$, define $\mu v \in K^{LMC}$ by $\mu v(f) = \mu(h_{v,f})$.*

2.8. Lemma. *With the operation just defined, δS is a right topological semigroup and for each $s \in S$, $\lambda_{e(s)}$ is continuous.*

Proof. We show first that for $\mu, v \in \delta S$, $\mu v \in \delta S$. Let F be a finite subset of LMC and for each $f \in F$, let U_f be a neighbourhood of $\mu\, v(f)$, so that $U = \bigcap_{f \in F} \pi_f^{-1}[U_f]$ is a basic neighborhood of μv. Now given $f \in F$, $\mu(h_{v,f}) = \mu\, v(f) \in U_f$ so $\bigcap_{f \in F} \pi_{h_{v,f}}^{-1}[U_f]$ is a neighborhood of μ. Pick $s \in S$ such that $e(s) \in \bigcap_{f \in F} \pi_{h_{v,f}}^{-1}[U_f]$. Then given $f \in F$, $v(f \circ \lambda_s) = h_{v,f}(s) = e(s)\,(h_{v,f}) \in U_f$ so that $\bigcap_{f \in F} \pi_{f \circ \lambda_s}^{-1}[U_f]$ is a neighbourhood of v. Pick $t \in S$ such that $e(t) \in \bigcap_{f \in F} \pi_{f \circ \lambda_s}^{-1}[U_f]$. Then given $f \in F$, $e(st)\,(f) = f(st) = f \circ \lambda_s(t) = e(t)\,(f \circ \lambda_s) \in U_f$ so $e(st) \in U$ as required.

To see that the operation is associative, let $\mu, v, \eta \in \delta S$. Let $f \in LMC$. Then $(\mu v)\, \eta(f) = \mu\, v(h_{\eta,f}) = \mu(h_{v,h_{\eta,f}})$ and $\mu(v_\eta)\,(f) = \mu(h_{v\eta,f})$ so it suffices to show $h_{v,h_{\eta,f}} = h_{v\eta,f}$. Let $s \in S$. Then $h_{v,h_{\eta,f}}(s) = v(h_{\eta,f} \circ \lambda_s)$ and $h_{v\eta,f}(s) = v\eta(f \circ \lambda_s) = v(h_{\eta,f\circ\lambda_s})$ so it suffices to show $h_{\eta,f} \circ \lambda_s = h_{\eta,f\circ\lambda_s}$. Let $t \in S$. Then $h_{\eta,f} \circ \lambda_s(t) = h_{\eta,f}(st) = \eta(f \circ \lambda_{st}) = \eta(f \circ \lambda_s \circ \lambda_t) = h_{\eta,f\circ\lambda_s}(t)$.

To see that δS is right topological, let $v \in \delta S$, let $f \in LMC$ and let U be open in K. (So $\pi_f^{-1}[U] \cap \delta S$ is a subbasic open set in δS.) Then $\varrho_v^{-1}[\pi_f^{-1}[U] \cap \delta S] = \pi_{h_{v,f}}^{-1}[U] \cap \delta S$.

Finally let $s \in S$. To see that $\lambda_{e(s)}$ is continuous let $f \in LMC$ and let U be open in K. Then $\lambda_{e(s)}^{-1}[\pi_f^{-1}[U] \cap \delta S] = \pi_{f\circ\lambda_s}^{-1}[U] \cap \delta S$. □

2.9. Lemma. *The function $e: S \to \delta S$ is a continuous homomorphism.*

Proof. Let $f \in LMC$ and let U be open in K. Then $e^{-1}[\pi_f^{-1}[U] \cap \delta S] = f^{-1}[U]$ so e is continuous.

To see that e is a homomorphism, let $s, t \in S$ and let $f \in LMC$. Then $e(s)\, e(t)\, (f) = e(s)\,(h_{e(t),f}) = h_{e(t),f}(s) = e(t)\,(f \circ \lambda_s) = f(st) = e(st)\,(f)$. □

The following theorem says that $(e, \delta S)$ is the LMC-compactification of S. We say "the" because, if (ϕ, T) is any other such compactification, then δS and T are isomorphic and homeomorphic via a map η with $\eta \circ e = \phi$.

2.10. Theorem. *Given a semigroup S with a topology, δS is a compact Hausdorff right topological semigroup, $e: S \to \delta S$ is a continuous homomorphism, $e[S]$ is dense in δS, and $\lambda_{e(s)}$ is continuous for each $s \in S$. Further, if T is a compact Hausdorff right topological semigroup, $\phi: S \to T$ is a continuous homomorphism,* *and $\lambda_{\phi(s)}$ is continuous for each $s \in S$, then there is continuous homomorphism $\eta: \delta S \to T$ such that $\eta \circ e = \phi$.*

Proof. Everything has been established except the existence of η. By Lemma 1.1 it suffices to show there exists a continuous $\eta: \delta S \to T$ such that $\eta \circ e = \phi$. For this it in turn suffices to show that given any nets $\langle s_\alpha \rangle_{\alpha \in I}$ and $\langle t_\gamma \rangle_{\gamma \in J}$, any $\mu \in \delta S$ and any $a, b \in T$, if $e(s_\alpha) \to \mu$, $e(t_\gamma) \to \mu$, $\phi(s_\alpha) \to a$, and $\phi(t_\gamma) \to b$, then $a = b$. (For then if $e(s) = e(t)$, taking $s_\alpha = s$ and $t_\gamma = t$ one sees that one can define η on $e[S]$ by $\eta(e(s)) = \phi(s)$. One then extends η continuously to $\mu \in \delta S$ by picking a net $\langle s_\alpha \rangle_{\alpha \in I}$ such that $e(s_\alpha) \to \mu$ and defines $\eta(\mu) = \lim_{\alpha \in I} \phi(s_\alpha)$.)

Suppose we have such $\langle s_\alpha \rangle_{\alpha \in I}$, $\langle t_\gamma \rangle_{\gamma \in J}$, $\mu \in \delta S$ and $a, b \in T$ but that $a \neq b$. Pick $f \in C(T)$ such that $f(a) \neq f(b)$ and let $\varepsilon = |f(a) - f(b)|$. By Lemma 2.2, $f \circ \phi \in LMC$. Let $U = \{x \in K: |x - \mu(f \circ \phi)| < \varepsilon/4\}$, $V = \{x \in K: |x - f(a)| < \varepsilon/4\}$, and $W = \{x \in K: |x - f(b)| < \varepsilon/4\}$. Then $\pi_{f \circ \phi}^{-1}[U]$ is a neighborhood of μ so pick $\alpha_0 \in I$ and $\gamma_0 \in J$ such that $e(s_\alpha)$ and $e(t_\gamma)$ are in $\pi_{f \circ \phi}^{-1}[U]$ whenever $\alpha \geqq \alpha_0$ and $\gamma \geqq \gamma_0$. Since $f^{-1}[V]$ and $f^{-1}[W]$ are neighborhoods of a and b respectively pick $\alpha \geqq \alpha_0$ and $\gamma \geqq \gamma_0$ such that $\phi(s_\alpha) \in f^{-1}[V]$ and $\phi(t_\gamma) \in f^{-1}[W]$. Then $|f(a) - f(b)| \leqq$
$\leqq |f(a) - f(\phi(s_\alpha))| + |f(\phi(s_\alpha)) - \mu(f \circ \phi)| + |\mu(f \circ \phi) - f(\phi(t_\gamma))| +$
$+ |f(\phi(t_\gamma)) - f(b)| < \varepsilon$, a contradiction. □

2.11. Theorem. *Let $f: S \to K$. Then f extends continuously to δS (i.e. there exists $g \in C(\delta S)$ with $g \circ e = f$) if and only if $f \in LMC$.*

Proof. For the sufficiency, define $g: \delta S \to K$ by $g(v) = v(f)$. Given $s \in S$ $g(e(s)) = e(s)(f) = f(s)$. Given U open in K, $g^{-1}[U] = \pi_f^{-1}[U] \cap \delta S$.

The necessity is an immediate consequence of Lemma 2.2. □

3. Other compactifications. It seems clear that the other compactifications produced in [2] for semitopological semigroups also exist for an arbitrary semigroup with topology. To define the relevant class of functions on S, one simply adds the requirement of [2] to the requirement that $f \in LMC$ and then goes through the steps of Section 2.

In certain cases one can get the desired conclusion more quickly. For example, let us consider the almost periodic compactification $(a, \alpha S)$ of S, characterized in [2] as the compactification maximal subject to being a topological semigroup.

3.1. Theorem. *Let S be a semigroup with topology. There is a compact Hausdorff topological semigroup αS and a continuous homomorphism $a: S \to \alpha S$ such that $a[S]$ is dense in αS and whenever T is a compact Hausdorff topological semigroup and $\phi: S \to T$ is a continuous homomorphism with $\phi[S]$ dense in T, there is a continuous homomorphism $\eta: \alpha S \to T$ such that $\eta \circ a = \phi$.*

Proof. Since $e[S]$ is a semitopological semigroup we have by [2, Theorem III.9.4] an almost periodic compactification $(a^*, \alpha(e[S]))$ of $e[S]$. We let $\alpha S =$

$= \alpha(e[S])$ and $a = a^* \circ e$. Then a is a continuous homomorphism and $a[S]$ is dense in αS.

Let T be a compact topological semigroup and let $\phi: S \to T$ be a continuous homomorphism with $\phi[S]$ dense in T. Then for each $s \in S$, $\lambda_{\phi(s)}$ is continuous so pick by Theorem 2.10 a continuous homomorphism $\gamma: \delta S \to T$ such that $\gamma \circ e = \phi$.

Since $\alpha(e[S])$ is the almost periodic compactification of $e[S]$ and $\gamma[e[S]] = \phi[S]$ which is dense in T, we may pick a continuous homomorphism $\eta: \alpha(e[S]) \to T$ such that $\eta \circ a^* = \gamma|_{e[S]}$. Then $\eta: \alpha S \to T$ and $\eta \circ a = \eta \circ a^* \circ e = \gamma \circ e = \phi$ as required. □

Nearly verbatim proofs establish that we can obtain the strong almost periodic and weak almost periodic compactification for any semigroup S with topology.

Denote by $(w, \omega S)$ the weak almost periodic compactification of S (maximal with respect to ωS being semitopological). We obtain in Theorem 3.3 an amusing characterization of ωS.

3.2. Lemma. *If S is compact, then $\delta S = \omega S$.*

Proof. Since S is compact and $e[S]$ is dense in δS, $e[S] = \delta S$. Thus δS is semitopological and hence $\delta S = \omega S$. □

3.3. Theorem. *Let S be any semigroup with topology. Then $\delta(\delta S) = \omega S$.*

Proof. By Lemma 3.2, we have $\delta(\delta S) = \omega(\delta S)$ so we show $\omega(\delta S) = \omega S$. Observe $\omega(\delta S)$ is semitopological and $w \circ e[S]$ is dense in $\omega(\delta S)$. Let T be a compact semitopological semigroup and let $\phi: S \to T$ be a continuous homomorphism with $\phi[S]$ dense in T. Pick a continuous homomorphism $\gamma: \delta S \to T$ with $\gamma \circ e = \phi$. Pick $\eta: \omega(\delta S) \to T$ with $\eta \circ w = \gamma$. Then $\eta \circ (w \circ e) = \phi$ as required. □

For the next theorem, we remind the reader that the strong almost periodic compactification $(m, \mu S)$ of S is the compactification maximal with respect to μS being a topological group.

3.4. Theorem. *If S is a group and a compact space, then δS is the strong almost periodic compactification of S (and αS as well).*

Proof. By Lemma 3.2, δS is semitopological. As the homomorphic image of a group δS is a group. Therefore, by Ellis' Theorem [3], δS is a topological group. □

4. One-to-one and open. It is not clear that one loses much by extending the LMC-compactification to apply to an arbitrary semigroup with topology. On the one hand, since $e[S]$ is a semitopological semigroup, e cannot be an embedding unless S is semitopological. On the other hand, as we shall see, e may fail to be one-to-one and open (as a map to $e[S]$) even when S is a completely regular semitopological semigroup. (Complete regularity is important since the absence of this property also trivially forces e to not be an embedding.) It may also be one-to-one and it may be open when S is neither left nor right topological.

4.1. Definition. (a) For $f \in C(S)$, $\operatorname{coz}(f) = \{x \in S: f(x) \neq 0\}$.

(b) For $x, y \in S$, $x \approx y$ if and only if $f(x) = f(y)$ for all $f \in LMC$.

4.2. Theorem. *Let S be a semigroup and a topological space. Then e is*

(a) *one-to-one if and only if LMC separates the points of S;*

(b) *open as a map to $e[S]$ if and only if whenever $x \in S$ and U is a neighborhood of x there exists $f \in LMC$ such that $x \in \operatorname{coz}(f)$ and $\operatorname{coz}(f) \subseteq \{y \in S$: there exists $z \in U$ with $z \approx y\}$;*

(c) *an embedding if and only if LMC separates the points of S and $\{\operatorname{coz}(f): f \in LMC\}$ is a basis for the topology of S.*

Proof. Statement (a) is trivial and (c) is a trivial consequence of (a) and (b). We establish (b).

For the sufficiency, let U be open in S and let $a \in e[U]$. Pick $x \in U$ such that $e(x) = a$. Pick $f \in LMC$ such that $x \in \operatorname{coz}(f)$ and $\operatorname{coz}(f) \subseteq \{y \in S$: there exists $z \in U$ such that $z \approx y\}$. Then $a \in \pi_f^{-1}[K \setminus \{0\}] \cap e[S] \subseteq e[U]$.

For the necessity let $x \in S$ and let U be a neighborhood of x. Then $e[U]$ is a neighborhood of $e(x)$ in $e[S]$ so pick V open in δS such that $e(x) \in V \cap e[S] \subseteq e[U]$. Since δS is compact Hausdorff, it is completely regular, so pick $g \in C(\delta S)$ such that $g(e(x)) = 1$ and $g[\delta S \setminus V] = \{0\}$. Let $f = g \circ e$. By Lemma 2.2, $f \in LMC$. Immediately, $x \in \operatorname{coz}(f)$. Let $y \in \operatorname{coz}(f)$. Then $e(y) \in V \cap e[S]$ so $e(y) \in e[U]$. Pick $z \in U$ such that $e(y) = e(z)$. Then $z \approx y$. □

4.3. Example. A completely regular Hausdorff semigroup S which is neither left nor right topological but for which $e: S \to e[S]$ is open.

Let S be the set of positive integers under addition. Define $\phi: S \to (0, 1)$ as follows. Given $n \in S$, let $a = [\log_2(n)]$ and let $\phi(n) = (2(n - 2^a) + 1)/2^{a+1}$. (Thus ϕ enumerates the dyadic rationals in $(0, 1)$ in their natural order: 1/2, 1/4, 3/4, 1/8, 3/8, 5/8, 7/8, 1/16, ...). Define a topology T on S by $T = \{\phi^{-1}[U]$: U is open in the usual topology on $(0, 1)\}$. Trivially T is Hausdorff and completely regular. Observe that each non-empty member of T contains arbitrarily long blocks of S. That is, if U is open in $(0, 1)$ and $m \in S$ then there exists $n \in S$ with $\{n, n + 1, n + 2, \ldots, n + m\} \subseteq \phi^{-1}[U]$.

It now suffices, in order to see that e is open, to show that LMC is the set of constant functions. (For then $e[S]$ is a singleton.) To this end, let $f \in C(S)$ such that f is not constant, pick $x, y \in S$ such that $f(x) \neq f(y)$, and let $\varepsilon = |f(x) - f(y)|$. Let $U = \{z \in S: |f(z) - f(y)| < \varepsilon/2\}$. Then U is open and $x \notin \operatorname{cl} U$. Pick a neighborhood V of x such that $V \cap U = \phi$. Since U is infinite, we can pick $m \in S$ such that $x + m \in U$. We claim that $f \circ \varrho_m$ is not continuous. Suppose that it is. Then $|f \circ \varrho_m(x) - f(y)| < \varepsilon/2$ so pick a neighbourhood W of x such that for all $z \in W$, $|f \circ \varrho_m(z) - f(y)| < \varepsilon/2$. Pick $n \in S$ such that $\{n, n + 1, n + 2, \ldots, n + m\} \subseteq V \cap W$. Since $n \in W$, $|f(n + m) - f(y)| < \varepsilon/2$ so that $n + m \in U$ and hence $U \cap V \neq \emptyset$, a contradiction.

Note that we have also established that S is not right topological (nor left topological since S is commutative). Indeed, since there do exist non constant continuous functions, for example ϕ, one has the function ϱ_m produced above cannot be continuous. □

4.4. Example. A completely regular Hausdorff semigroup S which is neither right nor left topological but for which e is one-to-one.

Let βN be the Stone-Čech compactification of the discrete set of positive integers, let $p \in \beta N \setminus N$ and let $S = N \cup \{p\}$ where S has the relative topology. Let the operation on S be ordinary addition on N with, for $n \in N$ $n + p = p + n = p + p = p$.

Since S is commutative, in order to show that S is neither right nor left topological, it suffices to show that λ_1 is not continuous. (Note 1 is not an identity.)

Let A be the set of even members of N. By [4, 6S] either $A \cup \{p\}$ or $(A + 1) \cup \{p\}$ is a neighborhood of p. Since $\lambda_1^{-1}[A \cup \{p\}] = (A - 1) \cup \{p\}$ and $\lambda_1^{-1}[(A + 1) \cup \cup \{p\}] = A \cup \{p\}$, λ_1 is not continuous at p.

To see that e is one-to-one it suffices, by Theorem 3.2, to show that for each $n \in N$, the characteristic function $\chi_{\{n\}}$ is in LMC. Trivially each $\chi_{\{n\}}$ is continuous. Let $n \in N$. Observe that if $t \in S \cup \{0\}$ and $s \in S \cup \{0\}$, then $\chi_{\{n\}} \circ \lambda_t \circ \varrho_s = \chi_{\{n\}} \circ \varrho_{t+s}$ so that given $t \in S \cup \{0\}$, $\{\chi_{\{n\}} \circ \lambda_t \circ \varrho_s : s \in S \cup \{0\}\} \subseteq \{\chi_{\{n\}} \circ \varrho_s : s \in S \cup \{0\}\}$. But $\{\chi_{\{n\}} \circ \varrho_s : s \in S \cup \{0\}\} = \{\chi_{\{m\}} : m \leqq n\} \cup \{\bar{0}\} = \mathrm{cl}(\{\chi_{\{m\}} : m \leqq n\} \cup \{\bar{0}\})$ where $\bar{0}$ is the function constantly 0. □

Example V.2.3(b) of [2] is an example of a completely regular Hausdorff semitopological semigroup such that $e\colon S \to e[S]$ is not open. (See Section 5 for a detailed analysis of this example.)

Example 92 of [11] (due to Hewitt in [5]) is an example of a regular Hausdorff space X with $C(X)$ consisting solely of the constant functions. If one then defines a trivial multiplication on X (for example $xy = y$ for all x and y) one makes X a semitopological semigroup. Then $e[X]$ is a singleton.

What we are after is an example of a completely regular Hausdorff semitopological semigroup for which e is not one-to-one and is not open as a map to $e[S]$.

We remark that it would now suffice to obtain such S with e not one-to-one. Indeed if $e_1\colon S_1 \to e_1[S_1]$ is not open and $e_2\colon S_2 \to \delta S_2$ is not one-to-one, then $e\colon S_1 \times (S_2 \cup \{1\}) \to \delta(S_1 \times (S_2 \cup \{1\}))$ is not one-to-one and not open as a map to $e[S_1 \times (S_2 \cup \{1\})]$. (Here 1 is adjoined as an isolated identity – whether or not S_2 originally had an identity.) We omit the verification of the above assertion since it turns out that we don't need it. That is, the example we construct with e not one-to-one also fails to have $e\colon T \to e[T]$ open.

4.5. Definition. Let T be the free semigroup on the set of distinct letters $\{a, b\} \cup \cup \{x_1, x_2, \ldots\} \cup \{s_1, s_2, \ldots\}$.

The idea of the construction is simple enough. We define a topology on T so that $x_n \to b$, $bs_n \to b$ and for each k, $x_k s_n \to a$, while we keep the operations continuous from the left and right. Unfortunately, the details of the construction are somewhat complicated, and we will require several lemmas.

4.6. Definition. (a) Define a relation R on T by $w_1 R w_2$ if and only if there exist $u, v \in T \cup \{\emptyset\}$ such that

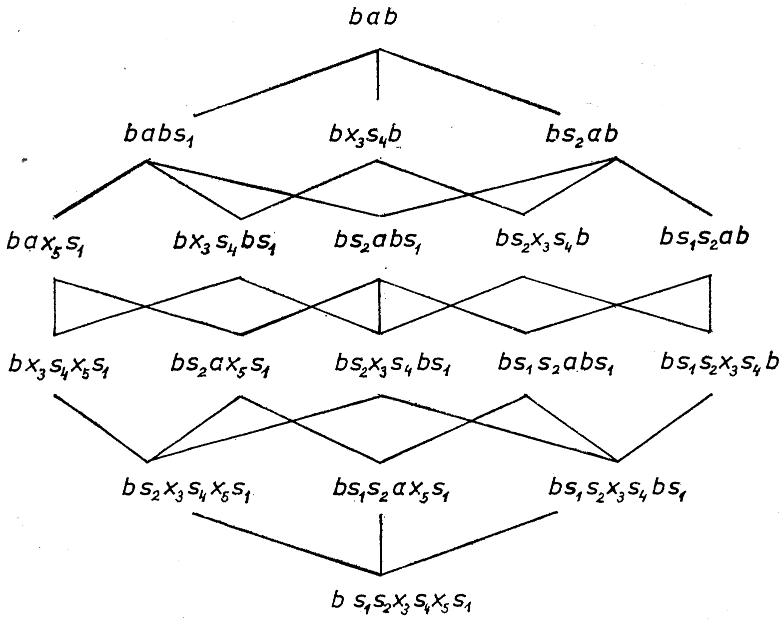
(i) $w_2 = ubv$ and $w_1 = ux_k v$ for some $k \in N$ and the leftmost letter of v, if any, is not a member of $\{s_n : n > k\}$; or

(ii) $w_2 = ubv$ and $w_1 = ubs_n v$ for some $n \in N$; or

(iii) $w_2 = uav$ and $w_1 = ux_k s_n v$ for some $k, n \in N$ with $n > k$.

(b) Let $<$ be the transitive closure of R.

We illustrate this order by drawing the lattice of all words greater than or equal to the word $bs_1 s_2 x_3 s_4 x_5 s_1$.



We omit the routine proof of the following lemma.

4.7. Lemma. (a) *Let $w_2, u_1, \ldots, u_l$ be members of T and let $w_1 = u_1 u_2, \ldots, u_l$. Assume that for each $i \in \{1, 2, \ldots, l-1\}$ none of the following cases hold:*

(i) $u_i = t_1 x_k$ *and* $u_{i+1} = s_n t_2$ *for some* $k, n \in N$;

(ii) $u_i = t_1 x_k s_n t_2$ *and* $u_{i+1} = s_m t_3$ *where* $k \geqq n$ *and all letters of* t_2, *if any, are in* $\{s_r : r \in N\}$;

(iii) $u_i = t_1 b t_2$ *and* $u_{i+1} = s_m t_3$ *and all letters of* t_2, *if any, are in* $\{s_r : r \in N\}$.

If $w_1 \leqq w_2$, *then there exist* $v_1, v_2, \ldots, v_l$ *in* T *such that* $w_2 = v_1 v_2 \ldots v_l$ *and each* $u_i \leqq v_i$.

(b) *Let* $u_1, u_2, \ldots, u_l$ *and* $v_1, v_2, \ldots, v_l$ *be members of* T *such that for each* i, $u_i \leqq v_i$. *Assume that for each* $i \in \{1, 2, \ldots, l-1\}$ *we do not have some* $k < n$ *with* $u_i = t_1 x_k$, $v_i = t_2 b$, *and* $u_{i+1} = s_n t_3$. *Then* $u_1 u_2 \ldots u_l \leqq v_1 v_2 \ldots v_l$.

Observe that the restrictions in part (a) are needed by considering $w_1 = x_2s_3$, $w_2 = a$; $w_1 = x_2s_1$, $w_2 = b$; $w_1 = x_2s_1s_2$, $w_2 = b$; and $w_1 = bs_1s_2$, $w_2 = b$. To see that the restriction in part (b) is needed let $u_1 = x_3$, $u_2 = s_4$, $v_1 = b$, $v_2 = s_4$.

4.8. Lemma. *Let w_1, w_2, w_3, and w_4 be members of T.*

(a) *If $w_1 \leqq w_2$ and $w_2 \leqq w_1$, then $w_1 = w_2$.*

(b) *$\{q \in T: w_1 \leqq q\}$ is finite.*

(c) *If $w_2 \nleqq w_1$, then $\{q \in T: qRw_2$ and $q \leqq w_1\}$ is finite.*

(d) *If w_1Rw_2 and $w_2 < w_4$ and $w_1 \leqq w_3 < w_4$, then there exists $w_5 \in T$ such that w_3Rw_5 and $w_2 \leqq w_5 \leqq w_4$.*

Proof. Statements (a) and (b) are trivial. To establish (c), let $A = \{q \in T: qRw_2$ and $q \leqq w_1\}$. Since there are finitely many choices of u and v for which $w_2 = ubv$ or $w_2 = uav$, it suffices to show for a given choice of u and v that

(i) if $w_2 = ubv$, then $\{q \in A: q = ux_kv$ for some $k \in N\}$ is finite;

(ii) if $w_2 = ubv$, then $\{q \in A: q = ubs_nv$ for some $n \in N\}$ is finite; and

(iii) if $w_2 = uav$, then $\{q \in A: q = ux_ks_nv$ for some $k, n \in N$ with $k < n\}$ is finite.

We establish (i), the other cases being similar. To do this, we show that if $q = ux_kv$ then there exist u', v' with $u \leqq u'$ and $v \leqq v'$ such that $w_1 = u'x_kv'$. Since w_1 has only finitely many occurences of x's this will suffice. We may write $v = t_1t_2$ where t_1 is a possibly empty word from $\{s_n: n \in N\}$ and the leftmost letter of t_2 if any, is not in $\{s_n: n \in N\}$. Then $w_2 = ubt_1t_2$ and $q = ux_kt_1t_2$. Pick by Lemma 4.7(a) u', t_3, and t_4 in $T \cup \{\emptyset\}$ such that $w_1 = u't_3t_4$, $u \leqq u'$, $x_kt_1 \leqq t_3$, and $t_2 \leqq t_4$. Now if $x_kt_1 \neq t_3$, then t_3 is b followed by a tail of t_1 so that $bt_1 \leqq t_3$ and hence by Lemma 4.7(b), $w_2 \leqq w_1$, a contradiction. Thus $t_3 = x_kt_1$ so letting $v' = t_1t_4$, we have $w_1 = u'x_kv'$ as required.

To see (d), assume w_1Rw_2, $w_2 < w_4$, and $w_1 \leqq w_3 < w_4$. We shall assume we have $u, v \in T \cup \{\emptyset\}$ and $k < n$ in N such that $w_2 = uav$ and $w_1 = ux_ks_nv$, the other cases being similar. Since $w_1 \leqq w_3$, pick by Lemma 4.7(a) t_1, t_2, and t_3 such that $w_3 = t_1t_2t_3$, $u \leqq t_1$, $x_ks_n \leqq t_2$, and $v \leqq t_3$. Since $w_3 < w_4$, pick t_4, t_5, and t_6 such that $t_1 \leqq t_4$, $t_2 \leqq t_5$, and $t_3 \leqq t_6$ and $w_4 = t_4t_5t_6$. Then $t_2 = x_ks_n$ or $t_2 = a$. If $t_2 = x_ks_n$, let $w_5 = t_1at_3$. Then w_3Rw_5 and $w_2 \leqq w_5 \leqq w_4$ as required. Thus we assume $t_2 = a$. Then $w_2 \leqq w_3$. Since $w_3 < w_4$, pick w_5 such that w_3Rw_5 and $w_5 \leqq w_4$.

In fact the set in Lemma 4.8(c) can have at most one member, but this is not important to us. We now proceed to describe the topology on T.

4.9. Definition. (a) Let $\mathcal{U} = \{U \subseteq T$: for each $w \in U$, $\{v \in T: vRw$ and $v \notin U\}$ is finite$\}$.

(b) For $U \subseteq T$ and $w_1 \in T$, let $N(w_1, U) = \{w_2 \in T: w_2 \leqq w_1$ and $\{v \in T: w_2 \leqq v \leqq w_1\} \subseteq U\}$.

4.10. Lemma. *If $U \in \mathcal{U}$ and $w_1 \in U$, then $N(w_1, U) \in \mathcal{U}$.*

Proof. Let $w_2 \in N(w_1, U)$. Let $A = \{w \in T: wRw_2$ and $w \notin N(w_1, U)\}$. We need to show that A is finite. Suppose instead that A is infinite. For each $w \in A$, pick $v(w)$

such that $w \leqq v(w) \leqq w_1$ and $v(w) \notin U$. Since $w_1 \in U$ we have in fact that $w \leqq \leqq v(w) < w_1$. Since $w_2 \in N(w_1, U)$, we cannot have $w_2 \leqq v(w)$ for any $w \in A$. Let $B = \{v(w): w \in A\}$. By Lemma 4.8(c), for each $v \in B$, $\{w \in A: v = v(w)\}$ is finite. Thus B must be infinite. For each $v \in B$, pick by Lemma 4.8(d), $u(v) \in T$ such that $vR\,u(v)$ and $w_2 \leqq u(v) \leqq w_1$. By Lemma 4.8(b), $\{u \in T: w_2 \leqq u \leqq w_1\}$ is finite, pick $u \in T$ such that $\{v \in B: u = u(v)\}$ is infinite. But $w_2 \leqq u \leqq w_1$ so $u \in U$. Since $u \in U$, $\{v \in T: vRu \text{ and } v \notin U\}$ is finite. Since $B \cap U = \emptyset$, this is a contradiction. □

4.11. Lemma. *$\mathcal{U}$ is a completely regular Hausdorff topology on T.*

Proof. $\mathcal{U}$ is trivially a topology on T. To see that $\mathcal{U}$ is Hausdorff, let w_1 and w_2 be distinct members of T. By Lemma 4.8(a), we assume without loss of generality that $w_2 \nleqq w_1$. Let $U = \{w \in T: w \leqq w_1\}$ and let $V = \{w \in T: w \leqq w_2 \text{ and } w \nleqq w_1\}$. Then $w_1 \in U$, $w_2 \in V$ and trivially U is open. To see that V is open, let $w \in V$ and note that $\{u \in T: uRw \text{ and } u \notin V\} = \{u \in T: uRw \text{ and } u \leqq w_1\}$. By Lemma 4.8(c), this latter set is finite.

By Lemma 4.10, $\{N(w_1, U): U \in \mathcal{U} \text{ and } w_1 \in U\}$ is a basis for $\mathcal{U}$. To see that $\mathcal{U}$ is a completely regular topology, it suffices to show that each $N(w_1, U)$ is closed. (For then $\chi_{N(w_1,U)}$ is continuous.) Indeed, let $U \in \mathcal{U}$ and $w_1 \in U$. Let $w_2 \in T \setminus N(w_1, U)$. If $w_2 \nleqq w_1$, let $V = \{w \in T: w \leqq w_2 \text{ and } w \nleqq w_1\}$. As above V is open and $V \cap \cap N(w_1, U) = \emptyset$. Thus we assume $w_2 \leqq w_1$. Pick $v \in T$ such that $w_2 \leqq v \leqq w_1$ and $v \notin U$. Let $V = \{u \in T: u \leqq v\}$. Then $V \in \mathcal{U}$, $w_2 \in V$, and $V \cap N(w_1, U) = \emptyset$. □

4.12. Lemma. *With the topology $\mathcal{U}$, T is a semitopological semigroup.*

Proof. Let $z \in T$ and let U be open. We first let $V = \lambda_z^{-1}[U]$ and show that V is open. Let $w_1 \in V$ and let $B = \{w \in T: wRw_1 \text{ and } w \notin V\}$. Let $C = \{w \in T: wRzw_1 \text{ and } w \notin U\}$. Since $zw_1 \in U$, C is finite. Then $\lambda_z[B] \subseteq C$ and hence, since λ_z is one-to-one, B is finite.

To see that T is right topological, let $V = \varrho_z^{-1}[U]$ and let $w_1 \in V$. The proof here is identical to the left case unless we have $z', w' \in T \cup \{\emptyset\}$ and some $n \in N$ such that $z = s_n z'$ and $w_1 = w'b$ so we assume this case holds. Let as before $B = \{w \in T: wRw_1 \text{ and } w \notin V\}$ and let $C = \{w \in T: wRw_1 z \text{ and } w \notin U\}$. Then $\varrho_z[B \setminus \{w'x_k: k < n\}] \subseteq C$ so $B \setminus \{w'x_k: k < n\}$ is finite so B is finite. □

4.13. Lemma. *Let $f \in LMC(T)$. Then $f(a) = f(b)$. Consequently $e(a) = e(b)$.*

Proof. Suppose $f(a) \neq f(b)$ and let $\varepsilon = |f(b) - f(a)|$. Since f is bounded pick a compact subset A of K such that $f[T] \subseteq A$. Then $\langle f \circ \varrho_{s_n} \rangle_{n=1}^{\infty}$ is a sequence in the compact product A^T so pick an accumulation point g of this sequence. Since $f \in LMC$, $g \in C(T)$.

Let $U = \{w \in T: |f(w) - f(b)| < \varepsilon/5\}$, $V = \{w \in T: |f(w) - f(a)| < \varepsilon/5\}$; and $W = \{w \in T: |g(w) - g(b)| < \varepsilon/5\}$. Then U, V, and W are open. Since $b \in W$ and for each $k \in N$, $x_k Rb$ we may pick k such that $x_k \in W$. Since $a \in V$ and for $n > k$, $x_k s_n Ra$, we may pick $m \in N$ such that $x_k s_n \in V$ whenever $n \geqq m$. Since $b \in U$ and for each $n \in N$, $bs_n Rb$, we may pick $m' \in N$ such that $bs_n \in U$ whenever $n \geqq m'$.

Let $B = \pi_{x_k}^{-1}[\{z \in K: |z - g(x_k)| < \varepsilon/5\}] \cap \pi_b^{-1}[\{z \in K: |z - g(b)| < \varepsilon/5\}]$. Then B is a neighbourhood of g so pick $n > \max\{m, m'\}$ such that $f \circ \varrho_{s_n} \in B$. Then $|f(b) - f(bs_n)| < \varepsilon/5$ since $bs_n \in U$, $|f(bs_n) - g(b)| < \varepsilon/5$ since $f \circ \varrho_{s_n} \in B$, $|g(b) - g(x_k)| < \varepsilon/5$ since $x_k \in W$, $|g(x_k) - f(x_k s_n)| < \varepsilon/5$ since $f \circ \varrho_{s_n} \in B$, and $|f(x_k s_n) - f(a)| < \varepsilon/5$ since $x_k s_n \in V$. Thus $|f(b) - f(a)| < \varepsilon$, a contradiction. □

4.14. Lemma. *For each $k \in N$, $\chi_{\{x_k\}} \in LMC(T)$.*

Proof. Since $\{x_k\}$ is open and closed, $\chi_{\{x_k\}}$ is continuous. Given $u, v \in T \cup \{\emptyset\}$ with $\{u, v\} \neq \{\emptyset\}$, $\chi_{\{x_k\}} \circ \lambda_u \circ \varrho_v = \bar{0}$. Thus, given $u \in T \cup \{\emptyset\}$, $\{\chi_{\{x_k\}} \circ \lambda_u \circ \varrho_v: v \in T \cup \{\emptyset\}\} \subseteq \{\chi_{\{x_k\}}, \bar{0}\} \subseteq C(T)$. □

4.15. Theorem. *T is a completely regular Hausdorff semitopological semigroup for which $e: T \to e[T]$ is neither one-to-one nor open.*

Proof. By Lemmas 4.11 and 4.12 T is a completely regular Hausdorff semitopological semigroup. By Theorem 4.2 and Lemma 4.13, e is not one-to-one. To see that e is not open, let $U = \{a\} \cup \{x_k s_n: k, n \in N \text{ and } k < n\}$. Then U is open in T and by Lemma 4.13 $e(b) \in e[U]$. Suppose $e[U]$ is open in $e[T]$ and pick, by the continuity of e, a neighbourhood V of b such that $e[V] \subseteq e[U]$. Pick $k \in N$ such that $x_k \in V$. Pick $z \in U$ such that $e(x_k) = e(z)$. But $\chi_{\{x_k\}}(x_k) = 1$, $\chi_{\{x_k\}}(z) = 0$, and, by Lemma 4.14, $\chi_{\{x_k\}} \in LMC$, a contradiction. □

5. Some examples of δS. We present here three examples where we have identified δS is a reasonably concrete fashion. The first two examples are right topological groups which are based on the circle group which we denote by T. The ideas for these two examples are derived from [9].

We let T^T have the product topology with coordinate -wise operations.

5.1. Theorem. *Let $S = T^T \times T$ where S has the product topology and where, for (h_1, w_1) and $(h_2, w_2) \in S$, $(h_1, w_1) . (h_2, w_2) = ((h_1 \circ \lambda_{w_2}) . h_2,\ w_1 . w_2)$. Then $T = \delta S$.*

Proof. Let $\pi_2(h, w) = w$. When we say „$T = \delta S$" we mean that (π_2, T) satisfies the conditions of Theorem 2.10. To see this let M be a compact Hausdorff right topological semigroup and let $\phi: S \to M$ be a continuous homomorphism such that $\lambda_{\phi(x)}$ is continuous for each $x \in X$. Define $\eta: T \to M$ by $\eta(w) = \phi(\bar{1}, w)$ where $\bar{1}$ is the function constantly. Then η is a continuous homomorphism. To complete the proof, it suffices to show that for $(h, w) \in S$, $\phi(h, w) = \phi(\bar{1}, w)$. (For then $\eta \circ \pi_2 = \phi$.) For this it in turn suffices to show that given $h \in T^T$ $\phi(h, 1) = \phi(\bar{1}, 1)$, since $(h, w) = (\bar{1}, w) . (h, 1)$.

Suppose instead that $\phi(h, 1) \neq \phi(\bar{1}, 1)$ and pick disjoint neighborhoods U_1 and U_2 of $\phi(h, 1)$ and $\phi(\bar{1}, 1)$ respectively. Since $\phi(\bar{1}, 1) . \phi(h, 1) \in U_1$, pick a neighborhood U_3 of $\phi(\bar{1}, 1)$ such that $U_3 . \phi(h, 1) \subseteq U_1$. Pick neighborhoods V of $\bar{1}$ and L_1 of 1 such that $\phi[V \times L_1] \subseteq U_2 \cap U_3$. Pick finite $F \subseteq T$ and for each $x \in F$, a neighborhood P_x of 1 such that $\bigcap_{x \in F} \pi_x^{-1}[P_x] \subseteq V$.

Inductively, choose a sequence $\langle w_n \rangle_{n=1}^{\infty}$ such that $|w_n - 1| < 1/n$, $w_1 F \cap F = \emptyset$ and for $n > 1$, $w_n F \cap (F \cup \bigcup_{t=1}^{n-1} w_t F) = \emptyset$. Define $h' \in T^T$ by $h'(w_n x) = h(x)^{-1}$ for each $x \in F$ and each $n \in N$ and $h'(x) = 1$ otherwise. (Note that, by the choice of the w_n's, h' is well defined.)

Now, given $x \in F$, $h'(x) = 1$ so $h' \in V$ and thus $\phi(h', 1) \in U_3$ so that $\phi(h', 1) . . \phi(h, 1) \in U_1$. Pick a neighborhood U_4 of $\phi(h, 1)$ such that $\phi(h', 1) . U_4 \subset U_1$. Pick neighborhoods Q of h and L_2 of 1 such that $\phi[Q \times L_2] \subset U_4$. Pick n such that $w_n \in L_1 \cap L_2$.

Now $(h, w_n) \in Q \times L_2$ so $\phi(h', 1) . \phi(h, w_n) \in U_1$. But $(h', 1) . (h, w_n) = ((h' \circ \lambda_{w_n}) . . h, w_n)$. Given $x \in F$, $(h' \circ \lambda_{w_n})(x) = h'(w_n x) = h(x)^{-1}$ so $((h' \circ \lambda_{w_n}) . h)(x) = 1$. Thus $(h' \circ \lambda w_n) . h \in V$. Thus $\phi((h', 1) . (h, w_n)) \in \phi[V \times L_1] \subset U_2$, a contradiction. □

In Theorem 5.1, the topological center of S ($\Lambda(S) = \{x \in S: \lambda_x$ is continuous$\}$) is dense. To be precise, $(h, w) \in \Lambda(S)$ if and only if h is continuous. By way of contrast, in Theorem 5.2, $\Lambda(S)$ will consist of exactly two points (namely $(1, 1)$ and $(-1, 1)$).

The topological space in Theorem 5.2 is familiar. See for exampe [2, p. 172].

5.2. Theorem. *Let $S = T \times \{-1, 1\}$ where, for (w_1, x_1) and (w_2, x_2) in S, $(w_1, x_1) . (w_2, x_2) = (w_1^{x_2} w_2, x_1 x_2)$. For $\varepsilon > 0$ and $(w, x) \in S$, let $N((w, x), \varepsilon) = = \{(we^{i\delta x}, x): 0 \leqq \delta < \varepsilon\} \cup \{(we^{i\delta x}, -x): 0 < \delta < \varepsilon\}$ and take $\{N((w, x), \varepsilon): \varepsilon > 0\}$ as a basis for the neighborhoods of (w, x). Then $\delta S = \{1\}$.*

Proof. As in Theorem 5.1, we let M be a compact Hausdorff right topological semigroup and let $\phi: S \to M$ be a continuous homomorphism with $\lambda_{\phi(x)}$ continuous for each $x \in S$. We show that ϕ must be constant. For this it suffices to show that $\phi(1, 1) = \phi(1, -1)$. (For then $\phi(w, 1) = \phi(1, 1) . \phi(w, 1) = \phi(1, -1) . \phi(w, 1) = = \phi(w, -1)$. From this $\phi(w^2, 1) = \phi(w, 1) . \phi(w, 1) = \phi(w, 1) . \phi(w, -1) = = \phi(1, -1)$. Since every element of T is a square, this suffices.)

Suppose instead $\phi(1, 1) \neq \phi(1, -1)$ and pick disjoint neighborhoods U_1 and U_2 of $\phi(1, 1)$ and $\phi(1, -1)$ respectively. Pick a neighborhood V_1 of $(1, 1)$ such that $\phi[V_1] \subseteq U_1$ and pick $\varepsilon > 0$ such that $N((1, 1), \varepsilon) \subseteq V_1$. Pick a neighborhood U_3 of $\phi(1, 1)$ such that $U_3 . \phi(1, -1) \subseteq U_2$. Pick a neighborhood V_2 of $(1, 1)$ with $\phi[V_2] \subseteq U_3$. Pick δ, $0 < \delta < \varepsilon$, with $(e^{i\delta}, 1) \in V_2$.

Then $\phi(e^{i\delta}, 1) . \phi(1, -1) \in U_2$ so pick a neighborhood U_4 of $\phi(1, -1)$ with $\phi(e^{i\delta}, 1) . U_4 \subseteq U_2$. Pick a neighborhood W of $(1, -1)$ with $\phi[W] \subseteq U_4$. Pick τ, $0 < \tau < \delta$, such that $(e^{-i\tau}, 1) \in W$. Then $\phi(e^{i\delta}, 1) . \phi(e^{-i\tau}, 1) \in U_2$. But $(e^{i\delta}, 1) . . (e^{-i\tau}, 1) = (e^{i(\delta-\tau)}, 1)$ and $0 < \delta - \tau < \delta < \varepsilon$ so $(e^{i(\delta-\tau)}, 1) \in V_1$. Thus $\phi((e^{i\delta}, 1) . . (e^{-i\tau}, 1)) \in U_1$, a contradiction. □

The remainder of this section is devoted to a characterization of a familiar semi-topological semigroup as a quotient of δR, where R is the real numbers under addition with the usual topology. As in [2, Example V.2.3(b)], we let $S = R \cup \{\theta\}$ where topologically θ is a point at $+\infty$ and algebraically $\theta + x = x + \theta = \theta$ for all $x \in S$.

As is well known, $LMC(R)$ is the set of bounded uniformly continuous functions on R. (See for example [2, Theorem III.14.6].) Consequently, by Theorem 4.2(c), $e: R \to \delta R$ is an embedding. Therefore we are justified in pretending that $R \subseteq \delta R$ and we will do so. We begin by characterizing (in a negative fashion) the members of $LMC(S)$.

5.3. Lemma. *Let $f: S \to K$. Then $f \notin LMC(S)$ if and only if either $f|_R \notin LMC(R)$ or there exist sequences $\langle x_n \rangle_{n=1}^{\infty}$, $\langle y_n \rangle_{n=1}^{\infty}$, and $\langle a_n \rangle_{n=1}^{\infty}$ in R such that*

(a) $\lim_{n \to \infty} y_n = +\infty$,

(b) *for each $k \in N$, $\lim_{n \to \infty} f(y_k + x_n) = a_k$, and*

(c) $\lim_{k \to \infty} a_k$ *exists and* $\lim_{k \to \infty} a_k \neq f(\theta)$.

Proof. Observe that, since S is commutative and ϱ_0 is the identity, $f \in LMC(S)$ if and only if $\mathrm{cl}\,\{f \circ \varrho_s\colon s \in S\} \subseteq C(S)$.

For the necessity, pick $g \in \mathrm{cl}\,\{f \circ \varrho_s\colon s \in S\} \setminus C(S)$. Observe that $f \circ \varrho_\theta$ is constantly equal to $f(\theta)$. Thus $g \in \mathrm{cl}\,\{f \circ \varrho_s\colon s \in R\}$ and hence $g|_R \in \mathrm{cl}\,\{f|_R \circ \varrho_s, s \in R\}$. If $g|_R \notin C(R)$, then $f|_R \notin LMC(R)$. We thus assume that $g|_R \in C(R)$ so that g is bounded and g is not continuous at θ.

Trivially $g(\theta) = f(\theta)$, since $f \circ \varrho_s(\theta) = f(\theta)$ for each $s \in R$. Pick a neighborhood V of $g(\theta)$ in K such that $g^{-1}[V]$ is not a neighborhood of θ. For each $n \in N$ pick $z_n > n$ such that $g(z_n) \notin V$. Then $\langle g(z_n) \rangle_{n=1}^{\infty}$ is a bounded sequence in K so pick a subsequence $\langle y_n \rangle_{n=1}^{\infty}$ of $\langle z_n \rangle_{n=1}^{\infty}$ such that $\langle g(y_n) \rangle_{n=1}^{\infty}$ converges. Let for each n, $a_n = g(y_n)$. Thus statements (a) and (c) hold.

For each $n \in N$, let $U_n = \bigcap_{k=1}^{n} \pi_{y_k}^{-1}[\{z \in K\colon |z - g(y_k)| < 1/n\}]$ and, since U_n is a neighborhood of g, pick $x_n \in R$ such that $f \circ \varrho_{x_n} \in U_n$. Then given $n > k$ we have $|f \circ \varrho_{x_n}(y_k) - g(y_k)| < 1/n$ so $\lim_{n \to \infty} f(y_k + x_n) = g(y_k)$ as required.

For the sufficiency observe that trivially if $f|_R \notin LMC(R)$, then $f \notin LMC(S)$. We thus assume we have sequences $\langle x_n \rangle_{n=1}^{\infty}$, $\langle y_n \rangle_{n=1}^{\infty}$, and $\langle a_n \rangle_{n=1}^{\infty}$ in R satisfying (a), (b), and (c). Let g be any cluster point (in K^S) of $\langle f \circ \varrho_{x_n} \rangle_{n=1}^{\infty}$. Again $g(\theta) = f(\theta)$. We show that $\lim_{k \to \infty} g(y_k) = \lim_{k \to \infty} a_k$, establishing that g is not continuous. To this end, let $\varepsilon > 0$ be given and let $b = \lim_{k \to \infty} a_k$. Pick l such that for $k > l$, $|a_k - b| < \varepsilon/3$. We claim that for $k > l$, $|g(y_k) - b| < \varepsilon$, so let $k > l$. Let $U = \pi_{y_k}^{-1}[\{z \in K\colon |z - g(y_k)| < \varepsilon/3\}]$. Pick m such that, for $n > m$, $|f(y_k + x_n) - a_k| < \varepsilon/3$. Pick $n > m$ such that $f \circ \varrho_{x_n} \in U$. Then $|f(y_k + x_n) - g(y_k)| < \varepsilon/3$, $|f(y_k + x_n) - a_k| < \varepsilon/3$, and $|a_k - b| < \varepsilon/3$ so $|g(y_k) - b| < \varepsilon$ as required. □

We denote by R^+ and R^- the sets $\{x \in R\colon x > 0\}$ and $\{x \in R\colon x < 0\}$, respectively.

5.4. Lemma. *Let U be open in δR. There exist $q \in \delta R$ and $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R$ such that $r + q \in U$ if and only if there exist V open in δR and sequences $\langle y_n \rangle_{n=1}^{\infty}$ and $\langle x_n \rangle_{n=1}^{\infty}$ in R such that*

(a) $\mathrm{cl}_{\delta R} V \subseteq U$,

(b) $\lim\limits_{n \to \infty} y_n = +\infty$, *and*

(c) $\{y_k + x_n \colon k, n \in N \text{ and } k < n\} \subseteq V$.

Proof. For the necessity, pick $q \in \delta R$ and $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R$ such that $r + q \in U$ and pick a neighborhood V of $r + q$ with $\mathrm{cl}_{\delta R} V \subseteq U$. Then V is a neighborhood of $\varrho_q(r)$ so pick a neighborhood W of r such that $\varrho_q[W] \subseteq V$. Now $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R$ so pick a sequence $\langle y_n \rangle_{n=1}^{\infty}$ in $W \cap R^+$ with $\lim\limits_{n \to \infty} y_n = +\infty$. Now, given $n \in N$, $y_n + q \in V$ and λ_{y_n} is continuous so pick a neighborhood A_n of q such that $\lambda_{y_n}[A_n] \subseteq V$. Given $n \in N$, $\bigcap_{k=1}^{n} A_k$ is a neighborhood of q so pick $x_n \in R \cap \bigcap_{k=1}^{n} A_k$. Then $\{y_k + x_n \colon k, n \in N \text{ and } k < n\} \subseteq V$ as required.

For the sufficiency, let V, $\langle y_n \rangle_{n=1}^{\infty}$, and $\langle x_n \rangle_{n=1}^{\infty}$ satisfy statements (a), (b) and (c). Let r be a cluster point of $\langle y_n \rangle_{n=1}^{\infty}$ in δR. Since $\lim\limits_{n \to \infty} y_n = +\infty$, $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R$. Let q be a cluster point of $\langle x_n \rangle_{n=1}^{\infty}$ in δR. Suppose that $r + q \notin U$. Then $r + q \notin \mathrm{cl}_{\delta R} V$ so pick a neighborhood W_1 of $r + q$ such that $W_1 \cap V = \emptyset$. Since W_1 is a neighborhood of $\varrho_q(r)$, pick a neighborhood W_2 of r such that $\varrho_q[W_2] \subseteq W_1$. Pick k such that $y_k \in W_2$. Then $y_k + q \in W_1$ so pick a neighborhood W_3 of q such that $\lambda_{y_k}[W_3] \subseteq W_1$. Pick $n > k$ such that $x_n \in W_3$. Then $y_k + x_n \in W_1 \cap V$, a contradiction. □

Now given $e \colon S \to \delta S$, $e|_R$ is a continuous homomorphism from R to δS and $\lambda_{e(s)}$ is continuous for each $s \in R$. Thus, by Theorem 2.10 we have a continuous homomorphism η so that the following diagram commutes. (Recall that we are assuming that $R \subseteq \delta R$.)

$$\begin{array}{ccc} \delta R & \xrightarrow{\ \eta\ } & \delta S \\ \uparrow{\scriptstyle \iota_1} & & {\scriptstyle \iota_2}\uparrow{\scriptstyle e} \\ R & \xrightarrow{\ L_2\ } & S \end{array}$$

Since R is dense in S, η is onto δS. Consequently δS is a quotient of δR via η. Theorem 5.6 shows that there is only one equivalence class which is not a singleton and identifies precisely what the members of that equivalence class are.

5.5. Lemma. *Let η be the continuous homomorphism from δR to δS such that $\eta(s) = e(s)$ for all $s \in R$. For $p, t \in \delta R$, agree that $p \approx t$ if and only if $\eta(p) = \eta(t)$. Given $p, t \in \delta R$, $p \not\approx t$ if and only if there if there exist $f \in C(\delta R)$ and $g \in LMC(S)$ such that $f(p) \neq f(t)$ and $f|_R = g|_R$.*

Proof. For the necessity, let $p, t \in \delta R$ such that $p \not\approx t$. Then $\eta(p) \neq \eta(t)$ so pick $h \in C(\delta S)$ such that $h(\eta(p)) \neq h(\eta(t))$. Let $g = h \circ e$. By Theorem 2.11, $g \in LMC(S)$. By Lemma 5.3, $g|_R \in LMC(R)$ so, again by Theorem 2.11, there exists $f \in C(\delta R)$ such that $f|_R = g|_R$. Now f and $h \circ \eta$ agree on the dense subset R of δR so $f = h \circ \eta$ and hence $f(p) \neq f(t)$ as required.

For the sufficiency assume we have $p, t \in \delta R$, $f \in C(\delta R)$, and $g \in LMC(S)$ such that $f(p) \neq f(t)$ and $f|_R = g|_R$. Pick by Theorem 2.11, $h \in C(\delta S)$ such that $h \circ e = g$. Then as above $h \circ \eta$ and f agree on R so $h \circ \eta = f$. Thus $h(\eta(p)) = f(p) \neq f(t) = h(\eta(t))$ and hence $\eta(p) \neq \eta(t)$ as required. □

5.6. Theorem. *Let η and $\approx$ be as in Lemma 5.5. The $\approx$-equivalence classes of δR are the singletons and $\mathrm{cl}\delta_R\{r + q: q \in \delta R$ and $r \in \mathrm{cl}\delta_R(R^+) \setminus R\}$.*

Proof. Let $A = \{r + q: q \in \delta R$ and $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R\}$. We first let $p, t \in \mathrm{cl}_{\delta R}A$ and show that $p \approx t$. Suppose instead that $p \not\approx t$ and pick, by Lemma 5.5, $f \in C(\delta R)$ and $g \in LMC(S)$ such that $f(p) \neq f(t)$ and $f|_R = g|_R$. We assume without loss of generality that $f(p) \neq g(\theta)$ and let $\varepsilon = |f(p) - g(\theta)|$. Pick a neighborhood U of p such that $f[U] \subseteq \{z \in K: |z - f(p)| < \varepsilon/2\}$. Since $p \in \mathrm{cl}_{\delta R}A$, pick by Lemma 5.4 V open in δR and sequences $\langle y_n\rangle_{n=1}^{\infty}$ and $\langle x_n\rangle_{n=1}^{\infty}$ such that $\mathrm{cl}_{\delta R}V \subseteq U$, $\lim_{n\to\infty} y_n = +\infty$, and $\{y_k + x_n: k, n \in N$ and $k < n\} \subseteq V$. By thinning $\langle x_n\rangle_{n=1}^{\infty}$ we may presume that for each $k \in N$, $\lim_{n\to\infty} f(y_k + x_n)$ exists. Let $a_k = \lim_{n\to\infty} f(y_k + x_n)$ and observe that $|a_k - f(p)| \leqq \varepsilon/2$. Thinning the sequence $\langle y_k\rangle_{k=1}^{\infty}$ so that $\lim_{n\to\infty} a_k$ exists, we have that $|\lim_{k\to\infty} a_k - f(p)| \leqq \varepsilon/2$ so that $\lim_{k\to\infty} a_k \neq g(\theta)$. Since $g|_R = f|_R$ we have that each $a_k = \lim_{k\to\infty} g(y_k + x_n)$. Thus, by Lemma 5.3, $g \notin LMC(S)$, a contradiction.

To complete the proof we let $p, t \in \delta R$, assume $p \neq t$ and $p \notin \mathrm{cl}_{\delta R}A$ and show that $p \not\approx t$. Since $p \neq t$ and $p \notin \mathrm{cl}_{\delta R}A$, pick a neighborhood U of p such that $t \notin U$ and $U \cap A = \emptyset$. Pick $f \in C(\delta R)$ such that $f(p) = 1$ and $f[\delta R \setminus U] = \{0\}$. Define $g: S \to K$ by $g|_R = f|_R$ and $g(\theta) = 0$. Since $f(p) = 1 \neq 0 = f(t)$, it suffices by Lemma 5.5 to show that $g \in LMC(S)$.

Suppose instead $g \notin LMC(S)$. Now $f|_R \in LMC(R)$ by Theorem 2.11, so $g|_R \in LMC(R)$. Thus by Lemma 5.3, we have sequences $\langle x_n\rangle_{n=1}^{\infty}$, $\langle y_n\rangle_{n=1}^{\infty}$, and $\langle a_n\rangle_{n=1}^{\infty}$ in R such that $\lim_{n\to\infty} y_n = +\infty$, for each $k \in N$ $\lim_{n\to\infty} g(y_k + x_n) = a_k$, $\lim_{n\to\infty} a_k$ exists, and $\lim_{k\to\infty} a_k \neq g(\theta)$. Let $b = \lim_{k\to\infty} a_k$. Then $b \neq 0$. By eliminating early terms, if necessary, we may presume each $|a_k| > |b|/2$. Likewise by thinning $\langle x_n\rangle_{n=1}^{\infty}$, we may presume that for $n > k$, we have $g(y_k + x_n) > |b|/2$. Let $V = \{q \in \delta R: |f(q)| > |b|/2\}$. Then V is open in δR and $\mathrm{cl}_{\delta R}V \subseteq U$, since $f[\delta R \setminus U] = \{0\}$. Further since $f|_R = g|_R$, we have $\{y_k + x_n: k, n \in N$ and $k < n\} \subseteq V$. But then, by Lemma 5.4, $U \cap A \neq \emptyset$, a contradiction. □

Theorem 5.6 tells us that δS is obtained from δR by collapsing $\mathrm{cl}_{\delta R}\{r + q: q \in \delta R$ and $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R\}$ to the point θ. This is similar to what occurs when R and S are considered as topological spaces. Then one obtains βS (the Stone Čech compactification of S) by collapsing $\mathrm{cl}_{\beta R}(R^+)$ to the point θ. The major difference is that $\mathrm{cl}_{\delta R}\{r + q: q \in \delta R$ and $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R\}$ includes points of $\mathrm{cl}_{\delta R}(R^-)$, as we shall see in Theorem 5.7. This result allows us to see in a graphic fashion why $e: S \to e[S]$ is not open. Given a neighborhood U of $e(\theta)$, $\eta^{-1}[U]$ is a neighborhood of points of $\mathrm{cl}_{\delta R}(R^-)$ and hence includes points of R^-. Thus each neighborhood of $e(\theta)$ includes points of $e[R^-]$ so that $e[R^+ \cup \{\theta\}]$ is not open in $e[S]$.

5.7. Theorem. *Let* $A = \{r + q: q \in \delta R$ *and* $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R\}$. *Then* $\mathrm{cl}_{\delta R}(R^+) \setminus R \subseteq A$, $R \cap \mathrm{cl}_{\delta R} A = \emptyset$, *and* $\mathrm{cl}_{\delta R}(R^-) \cap A \neq \emptyset$.

Proof. For the first assertion observe that each $r \in \delta R$ satisfies $r + 0 = r$. For the second assertion, let $s \in R$. Let $U = \{x \in R: s - 1 < x < s + 1\}$. Since R is locally compact, U is open in δR. We claim $U \cap A = \emptyset$. Suppose instead $U \cap A \neq \emptyset$ and pick V, $\langle y_n \rangle_{n=1}^{\infty}$ and $\langle x_n \rangle_{n=1}^{\infty}$ as guaranteed by Lemma 5.4. Since $\{y_1 + x_n: n \in N$ and $n > 1\} \subseteq V \subseteq U$ we have $\langle x_n \rangle_{n=1}^{\infty}$ is bounded. But then, since $\{y_k + x_n: n, k \in N$ and $n > k\} \subseteq V$, we must have $\langle y_k \rangle_{k=1}^{\infty}$ is bounded so that $\lim_{k \to \infty} y_k \neq +\infty$, a contradiction.

Finally, we show that $\mathrm{cl}_{\delta R}(R^-) \setminus R$ is a left ideal of δR. (Thus if $r \in \mathrm{cl}_{\delta R}(R^+) \setminus R$ and $q \in \mathrm{cl}\delta_R(R^-) \setminus R$, then $r + q \in A \cap \mathrm{cl}_{\delta R} R^-$.) To this end, let $q \in \mathrm{cl}_{\delta R}(R^-) \setminus R$ and let $p \in \delta R$. Let U be a neighborhood of $p + q$ and pick a neighborhood V of p such that $\varrho_q[V] \subseteq U$. Pick $x \in V \cap R$, so $x + q \in U$. Pick a neighborhood W of q such that $\lambda_x[W] \subseteq U$. Since $q \in \mathrm{cl}_{\delta R}(R^-) \setminus R$, pick $y \in W$ such that $y < -x$. Then $x + y \in U \cap R^-$ as required. □

The situation is the same if one starts with the group Z of integers under addition and lets $T = Z \cup \{\theta\}$ with θ topologically and algebraically at $+\infty$. In this case in fact $\delta Z = \beta Z$ [1, Theorem 2.4] so that δT is the quotient of βZ obtained by collapsing $\mathrm{cl}_{\beta Z}\{r + q: q \in \beta Z$ and $r \in \mathrm{cl}_{\beta Z}(-N) \setminus Z\}$ to a point. The proofs are similar to the ones we have done and in fact somewhat simpler. We omit them. (The reader should be cautioned however, that we write $r + q$ for what was called $q + r$ in [1].)

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