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ON THE RELATION BETWEEN BOUNDEDNESS AND OSCILLATION OF SOLUTIONS OF MANY-DIMENSIONAL DIFFERENTIAL SYSTEMS WITH DEVIATING ARGUMENTS

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1. Boundedness and oscillation are, generally speaking, independent properties. Nevertheless, there exists a precise relation between them. Within the last ten years many papers have appeared that deal with the establishment of conditions of the relation between boundedness and oscillation of solutions of ordinary differential equations (see [2]) as well as differential equations with deviating arguments (see the references in [7]).

In this paper we give theorems on the relation between boundedness and oscillation of components of the solutions for many-dimensional systems with deviating arguments.

We note that until now few papers have been published dealing with the theory of oscillation and asymptotic behaviour of the solutions of many-dimensional systems with deviating arguments (see [1], [3], [4], [6], [9]).

2. We will consider a system of the form

$$(S_\lambda) \qquad \begin{aligned} y_i'(t) &= a_i(t) f_i(y_{i+1}(g_{i+1}(t))) , \quad i = 1, 2, \ldots, n-1 , \\ y_n'(t) &= (-1)^\lambda a_n(t) f_n(y_1(g_1(t))) , \quad t \geqq 0 , \quad \lambda \in \{1, 2\} , \end{aligned}$$

where $n \geqq 2$ and the following conditions hold:

$$(1) \qquad a_i \in C([0, \infty), [0, \infty)) , \quad i = 1, 2, \ldots, n ,$$

is not identically zero on any subinterval $[T, \infty) \subset [0, \infty)$,

$$(2) \qquad \int^\infty a_i(t) \, dt = \infty , \quad i = 1, 2, \ldots, n-1 ,$$

$$(3) \qquad g_i \in C([0, \infty), [0, \infty)) , \quad \lim_{t \to \infty} g_i(t) = \infty , \quad i = 1, 2, \ldots, n ,$$

$$(4) \qquad f_i \in C(R, R) , \quad u f_i(u) > 0 \quad \text{for} \quad u \neq 0 , \quad i = 1, 2, \ldots, n .$$

Denote by W the set of all solutions $y(t) = (y_1(t), \ldots, y_n(t))$ of (S_λ) which exist on some ray $[T_y, \infty) \subset [0, \infty)$ and satisfy $\sup \{ \sum_{i=1}^{n} |y_i(t)| : t \geqq T \} > 0$ for any $T \geqq T_y$.

Definition 1. A solution $y = (y_1, \ldots, y_n) \in W$ is called *oscillatory* if each of its components has arbitrarily large zeros.

Definition 2. A solution $y \in W$ is called *nonoscillatory* (weakly nonoscillatory) on $[T, \infty)$, $T \geqq 0$, if each of its components (at least one component) is eventually of a constant sign on $[T_1, \infty) \subset [T, \infty)$.

We will use the following notation:

$$(5) \qquad \gamma_i(t) = \sup \{x \geqq 0;\ g_i(x) \leqq t\} \quad \text{for} \quad t \geqq 0\,, \qquad i = 1, 2, \ldots, n\,;$$
$$\gamma(t) = \max \{\gamma_1(t), \ldots, \gamma_n(t)\}\,;$$

$$(6) \qquad h_1(t) = g_1(t)\,, \quad h_k(t) = g_k(h_{k-1}(t))\,, \quad t \in [0, \infty)\,, \quad k = 2, \ldots, n\,;$$

$$(7) \quad J_k(h_k(t), g_k(s); a_1, \ldots, a_k) = \int_{g_1(s)}^{h_1(t)} (a_1(x) \int_{g_2(x)}^{h_2(t)} (a_2(x_2) \ldots \int_{g_k(x_{k-1})}^{h_k(t)} a_k(x_k)\, dx_k \ldots) dx_2) dx\,,$$
$$k = 1, 2, \ldots, n\,;$$

$$(8) \qquad A_k(h_k(t), g_k(s);\ a_1 f_1, a_2 f_2, \ldots, a_{k-1} f_{k-1}, a_k) =$$
$$= \int_{g_1(s)}^{h_1(t)} a_1(x_1) f_1 \left(\int_{g_2(x_1)}^{h_2(t)} a_2(x_2) \ldots f_{k-1} \left(\int_{g_k(x_{k-1})}^{h_k(t)} a_k(x_k)\, dx_k \right) \ldots dx_2 \right) dx_1\,,$$
$$k = 1, 2, \ldots, n\,.$$

Lemma 1. *Let the conditions* (1)–(4) *hold and let* $y = (y_1, \ldots, y_n) \in W$ *be a nonoscillatory solution of* (S_λ) *on the interval* $[0, \infty)$. *Then there exist a* $t_0 \geqq 0$ *and an integer* $l \in \{1, 2, \ldots, n\}$ *with* $n + \lambda + l$ *odd or* $l = n$ *such that for* $t \geqq t_0$,

$$(9_l) \qquad y_i(t)\, y_1(t) > 0\,, \quad i = 1, 2, \ldots, l\,,$$

$$(10_l) \qquad (-1)^{l+i}\, y_i(t)\, y_1(t) > 0\,, \quad i = l, l+1, \ldots, n\,.$$

Proof. If $\lambda = 1$, then Lemma 1 coincides with Lemma 1 [3]. For $\lambda = 2$, the proof of Lemma 1 is done similarly as that of [3, Lemma 1].

It is easy to prove the following statement.

Lemma 2. *Let the conditions of Lemma* 1 *hold.*

a) *Then there exists a* $\bar{t}_0 \geqq 0$ *such that for* $t \geqq \bar{t}_0$,

$$(11) \qquad y_i'(t)\, y_1(t) > 0\,, \quad i = 1, 2, \ldots, l-1 \quad \text{if} \quad l > 1\,,$$
$$(-1)^{l+i+1}\, y_i'(t)\, y_1(t) > 0\,, \quad i = l, l+1, \ldots, n \quad (n + \lambda + l \text{ is odd})\,.$$

b) *In addition, let* $\lim_{t \to \infty} |y_l(t)| = L_l$, $0 \leqq L_l \leqq \infty$. *Then*

$$(12) \qquad l > 1\,, \quad L_l > 0 \ \Rightarrow \lim_{t \to \infty} |y_i(t)| = \infty\,, \quad i = 1, 2, \ldots, l-1\,,$$

$$(13) \qquad l < n\,, \quad L_l < \infty \Rightarrow \lim_{t \to \infty} |y_i(t)| = 0\,, \quad i = l+1, \ldots, n\,.$$

Lemma 3 (Lemma 1 [6]). *Let the conditions* (1)–(4) *hold. Let* $y = (y_1, \ldots, y_n) \in W$ *be such that* $y_k(t) \neq 0$ *in* $[t_0, \infty)$ *for some* $k \in \{1, 2, \ldots, n\}$.

Then there exists a $T \geqq t_0$ *such that each component* y_i *of* y *is in* $[T, \infty)$ *different from zero, monotone and the limit* $\lim_{t \to \infty} y_i(t) = L_i$ *exists (finite or infinite).*

Theorem 1. *Let the conditions* (1)–(4) *hold. In addition, suppose that*

(14) $g_i(t)$ $(i = 1, 2, \ldots, n)$ *is nondecreasing and* $h_n(t) \leqq t$ *for* $t \geqq t_0$;

(15) $g_1'(t) > 0$ *for* $t \geqq t_0$;

(16) $f_{n-1}(u), f_n(u)$ *are nondecreasing*;

(17) $\inf_{0 \leqq |u| < \varepsilon} f_i(u)/u > 0$ $(i = 1, 2, \ldots, n-1)$ *for some* $\varepsilon > 0$;

(18) $\limsup_{t \to \infty} \int_{h_n(t)}^{t} a_n(s)\, J_{n-1}(h_{n-1}(t), g_{n-1}(s);\ a_1, \ldots, a_{n-1})\, ds > 0.$

Then, for $n + \lambda$ *even, every solution* $y = (y_1, \ldots, y_n) \in W$ *of* (S_λ) *with a bounded component* y_1 *is either oscillatory, or* y_i $(i = 1, 2, \ldots, n)$ *monotonically tend to zero as* $t \to \infty$.

Proof. Suppose the contrary. Let the system (S_λ) for $n + \lambda$ even have a weakly nonoscillatory solution $y = (y_1, \ldots, y_n)$ with a bounded component y_1. Then by Lemma 3, y is nonoscillatory. Without loss of generality we may suppose that $y_1(g_1(t)) > 0$ for $t \geqq t_0$. Then the n-th equation of (S_λ) implies that $(-1)^\lambda y_n'(t) \geqq 0$ for $t \geqq t_0$, and is not identically zero on any subinterval $[t_1, \infty) \subset [t_0, \infty)$. Then by Lemma 1 and Lemma 2 there exist a $t_2 \geqq t_0$ and an integer $l \in \{1, 2, \ldots, n\}$ with $n + \lambda + l$ odd or $l = n$ such that (9)–(11) hold for $t \geqq t_2$. If y_1 is bounded, then in view of (9), (11), Lemma 3, (12) and (2) we get that l must be only one, i.e. $l = 1$. With regard to Lemma 2 we obtain $\lim_{t \to \infty} y_1(t) = b \geqq 0$, $\lim_{t \to \infty} y_i(t) = 0$, $i = 2, 3, \ldots, n$.

Let $b > 0$. Integrating the first equation of (S_λ) from $g_1(s)$ to $g_1(t)$ $(t \geqq s \geqq t_3 = \gamma(t_2))$, we get

$$(19) \qquad y_1(g_1(t)) - y_1(g_1(s)) = \int_{g_1(s)}^{g_1(t)} a_1(x_1) f_1(y_2(g_2(x_1)))\, dx_1\,, \quad s \geqq t_3\,.$$

We denote

$$(20_i) \qquad M_{i-1} = \inf_{0 \leqq |u| \leqq y_i(g_i(t_3))} \frac{f_{i-1}(u)}{u}\,, \quad i = 2, \ldots, n\,, \quad t \geqq t_3\,.$$

In view of (10), (4) and (20_1), from (19) we have

$$(21) \qquad y_1(g_1(t)) - y_1(g_1(s)) \leqq M_1 \int_{g_1(s)}^{g_1(t)} a_1(x_1)\, y_2(g_2(x_1))\, dx_1\,, \quad s \geqq t_3\,.$$

Integrating the second equation of (S_λ) from $g_2(x_1)$ to $h_2(t)$, then using (10), (20_2), we get

$$-y_2(g_2(x_1)) \geqq M_2 \int_{g_2(x_1)}^{h_2(t)} a_2(x_2)\, y_3(g_3(x_2))\, dx_2\,, \quad x_1 \geqq t_4 = \gamma(t_3)\,.$$

Taking into account this inequality, we obtain from (19)

$$(22) \quad y_1(g_1(t)) - y_1(g_1(s)) \leqq -M_1 M_2 \int_{g_1(s)}^{h_1(t)} \left(a_1(x_1) \int_{g_2(x_1)}^{h_2(t)} a_2(x_2)\, y_3(g_3(x_2))\, dx_2\right) dx_1 =$$
$$= -M_1 M_2\, J_2(h_2(t), g_2(s);\ a_1, a_2\, y_3(g_3))\,.$$

Integrating the third equation of (S_λ) from $g_3(x_2)$ to $h_3(t)$, then using (7), (22), we have

$$y_1(g_1(t)) - y_1(g_1(s)) \leqq M_1 M_2 M_3\, J_3(h_3(t), g_3(s);\ a_1, a_2, a_3\, y_4(g_4))\,, \quad s \geqq t_3\,.$$

Integrating the fourth equation of (S_λ) (if $n > 4$) and then proceeding analogously $n - 4$-times, we get

(23) $$y_1(g_1(t)) - y_1(g_1(s)) \leqq (-1)^n M\, J_{n-1}(h_{n-1}(t), g_{n-1}(s);$$
$$a_1, \ldots, a_{n-2}, a_{n-1} f_{n-1}\, y_n(g_n))\,, \quad s \geqq t_3\,, \quad M = M_1 M_2 \ldots M_{n-2}\,.$$

If we use the monotonicity of f_{n-1}, y_n and g_n, from (23) we obtain

(24) $$y_1(g_1(t)) - y_1(g_1(s)) \leqq (-1)^n M f_{n-1}(y_n(h_n(t)))\, J_{n-1}(h_{n-1}(t), g_{n-1}(s);$$
$$a_1, \ldots, a_{n-2}, a_{n-1})\,, \quad s \geqq t_3\,.$$

Consider the function $F(s, t)$ defined by

$$F(s, t) = (-1)^n[y_n(h_n(t)) - y_n(s)] \int_s^t \frac{y_1'(g_1(x))\, g_1'(x)}{f_n(y_1(g_1(x)))}\, dx\,, \quad t \geqq s \geqq t_3\,.$$

Obviously we have $F(t, t) = 0 = F(h_n(t), t)$ for $t \geqq t_3$. Calculating the partial derivative of $F(s, t)$ with respect to s, using (S_λ), (10_1), (11), (15) and the fact that f_n is nondecreasing, we obtain

$$F_s'(s, t) \geqq a_n(s)\,[y_1(g_1(s)) - y_1(g_1(t))] +$$
$$+ (-1)^{n+1}\, y_n(h_n(t))\, \frac{y_1'(g_1(s))\, g_1'(s)}{f_n(y_1(g_1(s)))}\,.$$

Integrating the last inequality from $h_n(t)$ to t, using (24), Lemma 2, we have

(25) $$\frac{f_{n-1}(y_n(h_n(t)))}{y_n(h_n(t))} M \int_{h_n(t)}^t a_n(s)\, J_{n-1}(h_{n-1}(t), g_{n-1}(s);$$
$$a_1, \ldots, a_{n-1})\, ds + \int_{y_1(g_1(h_n(t)))}^{y_1(g_1(t))} \frac{du}{f_n(u)} \leqq 0\,.$$

Because $\lim_{t\to\infty} y_1(t) = b > 0$, we get

(26) $$\lim_{t\to\infty} \int_{y_1(g_1(h_n(t)))}^{y_1(g_1(t))} \frac{du}{f_n(u)} = 0\,.$$

From (25), (26) and (17) we conclude that

(27) $$\limsup_{t\to\infty} \int_{h_n(t)}^t a_n(s)\, J_{n-1}(h_{n-1}(t), g_{n-1}(s);\; a_1, \ldots, a_{n-1})\, ds \leqq 0\,,$$

which contradicts (18). Therefore $b = 0$ and by (13) $\lim_{t\to\infty} y_i(t) = 0$ for $i = 1, 2, \ldots, n$. Theorem 1 is proved.

Theorem 2. *Let the conditions of Theorem 1 hold. In addition, suppose that*

(28) $$\lim_{x\to 0+} \int_x^1 \frac{du}{f_n(u)} = d_1 < \infty\,, \quad \lim_{x\to 0-} \int_x^{-1} \frac{du}{f_n(u)} = d_2 < \infty\,.$$

Then for $n + \lambda$ even all solutions $y \in W$ of (S_λ) with a bounded component y_1 are oscillatory.

Proof. Suppose the contrary. Let the system (S_λ) for $n + \lambda$ even have a weakly nonoscillatory solution $y \in W$ with a bounded component y_1. Then by Lemma 3, y is nonoscillatory. Without loss of generality we suppose that $y_1(g_1(t)) > 0$ for $t \geqq t_0$. Since the conditions of Theorem 1 hold, in view of this theorem we have $\lim_{t\to\infty} y_1(t) = 0$. From (28) we get (26). Then from (25), (26) we obtain (27), which contradicts (18). Theorem 2 is proved.

The system (S_λ), where $g_i(t) \equiv t$, $a_i(t) \equiv 1$, $f_i(u) \equiv u$ for $i = 1, 2, \ldots, n-1$, $g_n(t) = g(t)$, $a_n(t) = a(t)$, $f_n(u) = f(u)$, $n + \lambda$ even, is equivalent to the n-th order scalar differential equation

$$\text{(E)} \qquad y^{(n)}(t) + (-1)^{n+1}\, a(t)\, f(y(g(t))) = 0\,.$$

Theorem 1, 2 are generalizations of [5, Theorem 2] for (E) and also of [9, Theorem 3, 4].

Theorem 3. *Let the conditions* (1)–(4), (14)–(16) *hold. In addition, let*

$$\text{(29)} \qquad \inf_{0 \leqq |u| < \varepsilon} \frac{f_i(u)}{u} > 0\,, \quad i = 1, 2, \ldots, n \quad \textit{for some} \quad \varepsilon > 0\,,$$

$$\text{(30)} \qquad \limsup_{t\to\infty} \int_{h_n(t)}^{t} a_n(s)\, J_{n-1}(h_{n-1}(t), g_{n-1}(s); a_1, \ldots, a_{n-1})\, ds >$$

$$> \prod_{i=1}^{n} \left(\limsup_{u\to 0} \frac{u}{f_i(u)} \right).$$

Then the conclusion of Theorem 2 holds.

Proof. Let the system (S_λ) have a nonoscillatory solution $y \in W$ with a bounded component y_1. Without loss of generality we suppose that $y_1(g_1(t)) > 0$ for $t \geqq t_0$. As in the proof of Theorem 1 we obtain (9)–(11), where $l = 1$. By Lemma 2,

$$\text{(31)} \qquad \lim_{t\to\infty} y_1(t) = b \geqq 0\,, \quad \lim_{t\to\infty} y_i(t) = 0\,, \quad i = 2, \ldots, n\,.$$

Let $b > 0$. Proceeding in the same way as in the proof of Theorem 1, we get (23). From (23), with regard to (7) and $y_1(g_1(t)) > 0$ for $t \geqq t_0$, we have

$$\text{(32)} \qquad y_1(g_1(s)) \geqq (-1)^{n+1} M\, J_{n-1}(h_{n-1}(t), g_{n-1}(s); a_1, \ldots, a_{n-2}, a_{n-1}\, y_n(g_n))\,,$$
$$s \geqq t_3\,.$$

Because (18) follows from (30), in view of Theorem 1 we get $\lim_{t\to\infty} y_1(t) = 0$.

Integrating the last equation of (S_λ) from $h_n(t)$ to t and using

$$\text{(33)} \qquad M_n = \inf_{0 \leqq |u| \leqq |y_1(g_1(t_3))|} \frac{f_n(u)}{u}\,,$$

Lemma 1 ($n + \lambda + 1$ is odd), we have

$$\text{(34)} \qquad 0 < (-1)^{n+1}\, y_n(t) \leqq (-1)^{n+1}\, y_n(h_n(t)) - M_n \int_{h_n(t)}^{t} a_n(s)\, y_1(g_1(s))\, ds\,.$$

From (32), (34), by virtue of the monotonicity of y_n, g_n, we get

$$0 < (-1)^{n+1} y_n(h_n(t)) \{1 - M . M_n \int_{h_n(t)}^t a_n(s) J_{n-1}(h_{n-1}(t), g_{n-1}(s); a_1, \dots, a_{n-1}) ds\}.$$

Taking into account the inequality $(-1)^{n+1} y_n(h_n(t)) \geqq 0$ and (29) for $i = n$, we obtain

$$\limsup_{t\to\infty} \int_{h_n(t)}^t a_n(s) J_{n-1}(h_{n-1}(t), g_{n-1}(s); a_1, \dots, a_{n-1}) ds \leqq$$
$$\leqq \prod_{i=1}^n \left(\limsup_{u\to 0} \frac{u}{f_i(u)}\right),$$

which contradicts (30). Theorem 3 is proved.

Theorem 3 extends Theorem 2.7 [8] and Theorem 5 [9].

Theorem 4. *Let the conditions* (1)–(4), (14) *hold. In addition, let*

(35) f_i *be nondecreasing and* $f_i(-u) = -f_i(u)$ *for* $u \in R$, $i = 1, 2, \dots, n$;

(36) $f_i(uv) \geqq K_i f_i(u), f_i(v)$, $uv > 0$, $0 < K_i = \text{const.}$ *for* $i = 1, 2, \dots, n$;

(37) $$\limsup_{t\to\infty} \int_{h_n(t)}^t a_n(s) f_n(A_{n-1}(h_{n-1}(t), g_{n-1}(s); a_1 f_1, \dots, a_{n-2} f_{n-2}, a_{n-1})) ds \geqq$$
$$\geqq \limsup_{u\to 0} \frac{u}{K_n f_n(K_1 f_1(\dots K_{n-2} f_{n-2}(f_{n-1}(u)) \dots))}.$$

Then the conclusion of Theorem 2 *holds.*

Proof. Let the system (S_λ) for $n + \lambda$ even have a nonoscillatory solution $y \in W$ with a bounded component y_1. Without loss of generality we suppose that $y_1(g_1(t)) > 0$ for $t \geqq t_0$. Proceeding in the same way as in the proof of Theorem 1, we get (9)–(11), where $l = 1$. By virtue of Lemma 2 we have (31) and

(38) $$(-1)^{i+1} y_i(t) > 0, \quad (-1)^i y_i'(t) > 0, \quad i = 1, 2, \dots, n, \quad t \geqq t_1.$$

Let $\lim_{t\to\infty} y_1(t) = b > 0$. Integrating the first equation of (S_λ) from $g_1(s)$ to $g_1(t)$, we get (19). In view of $y_1(g_1(t)) > 0$ for $t \geqq t_0$, (19) implies

(39) $$y_1(g_1(s)) \geqq - \int_{g_1(s)}^{g_1(t)} a_1(x_1) f_1(y_2(g_2(x_1))) dx_1, \quad s \geqq t_3.$$

Integrating the second equation of (S_λ) from $g_2(x_1)$ to $h_2(t)$ and using $y_2(g_2(t)) < 0$ for $t \geqq t_3$, we obtain

$$-y_2(g_2(x_1)) \geqq \int_{g_2(x_1)}^{h_2(t)} a_2(x_2) f_2(y_3(g_3(x_2))) dx_2.$$

From (39), by virtue of the last inequality, (38) and (35) we get

(40) $$y_1(g_1(s)) \geqq \int_{g_1(s)}^{g_1(t)} a_1(x_1) f_1(\int_{g_2(x_1)}^{h_2(t)} a_2(x_2) f_2(y_3(g_3(x_2))) dx_2) dx_1, \quad s \geqq t_3.$$

Integrating the third equation of (S_λ) and then proceeding analogously $\overline{n-3}$-times, we get

(41) $$y_1(g_1(s)) \geqq (-1)^{n+1} \int_{g_1(s)}^{g_1(t)} a_1(x_1) f_1(\int_{g_2(x_1)}^{h_2(t)} a_2(x_2) \dots$$

$$\ldots f_{n-2}\Big(\int_{g_{n-1}(x_{n-2})}^{h_{n-1}(t)} a_{n-1}(x_{n-1})\, f_{n-1}(y_n(g_n(x_{n-1})))\,\mathrm{d}x_{n-1}\Big)\ldots \mathrm{d}x_2\Big)\,\mathrm{d}x_1\,,$$
$$s \geqq t_3\,, \quad (n + \lambda \text{ is even})\,.$$

From (41), with regard to the monotonicity of f_{n-1}, y_n, g_n, (6) and (36), we obtain

(42) $$y_1(g_1(s)) \geqq (-1)^{n+1} K_1 f_1(\ldots K_{n-2} f_{n-2}(f_{n-1}(y_n(h_n(t))) \ldots) \times$$
$$\times\, A(h_{n-1}(t), g_{n-1}(x_{n-2});\; a_1 f_1, \ldots, a_{n-2} f_{n-2}, a_{n-1})\,.$$

Integrating the last equation of (S_λ) from $h_n(t)$ to t, we have

(43) $$0 < (-1)^{n+1}\, y_n(t) = (-1)^{n+1}\, y_n(h_n(t)) - \int_{h_n(t)}^{t} a_n(s)\, f_n(y_1(g_1(s)))\,\mathrm{d}s\,.$$

If we substitute (42) in (43) and use (36), we get

$$0 \leqq (-1)^{n+1}\, y_n(h_n(t)) \Bigg[1 - \frac{K_n f_n(K_1 f_1(\ldots K_{n-2} f_{n-2}(f_{n-1}(y_n(h_n(t)))) \ldots))}{y_n(h_n(t))} \times$$
$$\times \int_{h_n(t)}^{t} a_n(s)\, f_n(A_{n-1}(h_{n-1}(t), g_{n-1}(s);\; a_1 f_1, \ldots, a_{n-2} f_{n-2}, a_{n-1})\,\mathrm{d}s\Bigg]\,.$$

The last inequality, in view of $(-1)^{n+1}\, y_n(h_n(t)) > 0$ for $t \geqq t_3$, implies

$$\limsup_{t\to\infty} \int_{h_n(t)}^{t} a_n(s)\, f_n(A_{n-1}(h_{n-1}(t), g_{n-1}(s);\; a_1 f_1, \ldots, a_{n-2} f_{n-2}, a_{n-1}))\,\mathrm{d}s \leqq$$
$$\leqq \limsup_{u\to 0} \frac{u}{K_n f_n(K_1 f_1(\ldots K_{n-2} f_{n-2}(f_{n-1}(u)) \ldots))}$$

and this contradicts (37). Theorem 4 is proved.

Theorem 4 generalizes Theorem 2.7 [8].

Now we consider the system (S_λ) where $f_i(u) = u^{\alpha_i}$, $i = 1, 2, \ldots, n$, i.e.

$(\bar{S}_\lambda)$ $$y_i'(t) = a_i(t)\,(y_{i+1}(g_{i+1}(t)))^{\alpha_i}\,, \quad i = 1, 2, \ldots, n-1\,,$$
$$y_n'(t) = (-1)^{\lambda}\, a_n(t)\,(y_n(g_1(t)))^{\alpha_n}\,,$$

where $0 < \alpha_i$ is the ratio of odd numbers, $i = 1, 2, \ldots, n$.

From Theorem 3 we get

Corollary 1. *Let the conditions* (1)–(3), (14), (15) *hold. In addition, let* $0 < \alpha_i \leqq 1$; $i = 1, 2, \ldots, n$,

$$\limsup_{t\to\infty} \int_{h_n(t)}^{t} a_n(s)\, J_{n-1}(h_{n-1}(t), g_{n-1}(s);\; a_1, \ldots, a_{n-1})\,\mathrm{d}s >$$
$$> \begin{cases} 0 & \text{if} \quad \alpha_1 \alpha_2 \ldots \alpha_n < 1\,, \\ 1 & \text{if} \quad \alpha_1 \alpha_2 \ldots \alpha_n = 1\,. \end{cases}$$

Then for $n + \lambda$ even all solutions $y \in W$ of $(\bar{S}_\lambda)$ with a bounded component y_1 are oscillatory.

From Theorem 4 we get

Corollary 2. *Let the conditions* (1)–(3), (14) *hold. In addition, let* $\alpha_1 . \alpha_2 \dots \alpha_n \leqq 1$,

$$\limsup_{t \to \infty} \int_{h_n(t)}^{t} a_n(s) \left(\int_{g_1(s)}^{h_1(t)} a_1(x_1) \left(\int_{g_2(x_1)}^{h_2(t)} a_2(x_2) \dots \right.\right.$$

$$\dots \left(\int_{g_{n-1}(x_{n-2})}^{h_{n-1}(t)} a_{n-1}(x_{n-1}) \, dx_{n-1} \right)^{\alpha_{n-2}} \dots dx_2 \Big)^{\alpha_1} dx_1 \Big)^{\alpha_n} ds >$$

$$> \begin{cases} 0 & \text{if} \quad \alpha_1 . \alpha_2 \dots \alpha_n < 1 \,, \\ 1 & \text{if} \quad \alpha_1 . \alpha_2 \dots \alpha_n = 1 \,. \end{cases}$$

Then the conclusion of Corollary 1 holds.

From Corollary 2, for $\alpha_1 . \alpha_2 \dots \alpha_n = 1$, we get Theorem 6 [9].

Theorems given above are specific in the sense that they do not hold for the corresponding differential systems without deviating arguments.

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