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RELATIVE CONTINUITY OF THE FUNCTOR β

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0. INTRODUCTION AND PRELIMINARY RESULTS

0.1. The set of positive integers is denoted by N. The symbol ω_m denotes the set of all ordinals of the cardinality $\aleph_{m-1}$, $m \geqq 1$.

0.2. We use the notion of the inverse systems as in [2]. The inverse system is denoted by $\boldsymbol{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ and its limit by $X = \lim \boldsymbol{X}$.

0.3. We say that a covariant functor $F: \mathscr{A} \to \mathscr{B}$ is continuous [3; p. 258] if for every inverse system $\boldsymbol{X} = \{X_\alpha, f_{\alpha\beta}, A\}$, the object $F(\varprojlim \boldsymbol{X})$ is a limit of the inverse system $F\boldsymbol{X} = \{F(X_\alpha). F(f_{\alpha\beta}), A\}$.

It is well known that the Čech-Stone functor β is not continuous. It suffices to consider an inverse system with the empty limit space.

A covariant functor $F: \mathscr{A} \to \mathscr{B}$ is said to be *relatively continuous* with respect to $\boldsymbol{X} = \{X_\alpha, f_{\alpha\beta}, A\}$, or $\boldsymbol{X}$-continuous, if the object $F(\varprojlim \boldsymbol{X})$ is a limit of the inverse system $F\boldsymbol{X} = \{F(X_\alpha), F(f_{\alpha\beta}), A\}$.

The following two theorems were proved by Nagata [10].

0.4. Theorem. *Let $\boldsymbol{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence with a countably compact limit X. If X_n, $n \in N$, are normal firstcountable spaces, then $\beta X = \varprojlim \{\beta X_n, \beta f_{nm}, N\}$.*

0.5. Theorem. *Let $\boldsymbol{X} = \{X_n, f_{nm}, N\}$ be an inverse sequence of normal spaces X_n and open mappings f_{nm}; $n, m \in N$. If $X = \varprojlim \boldsymbol{X}$ is pseudocompact, then $\beta X = \varprojlim \{\beta X_n, \beta f_{nm}, N\}$.*

1. THEOREMS

We start with the following key lemma.

1.1. Lemma. *Let $\boldsymbol{X} = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system of completely regular spaces X_α. Then*: a) *$\beta\boldsymbol{X} = \{\beta X_\alpha, \beta f_{\alpha\beta}, A\}$ is an inverse system*; b) *if the projections $f_\alpha: \varprojlim \boldsymbol{X} \to X_\alpha$, $\alpha \in A$, are onto mappings, then $\varprojlim \beta\boldsymbol{X}$ is the compactification of*

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the space $\varprojlim X$. *Moreover, the spaces* $\varprojlim \beta X$ *and* $\beta(\varprojlim X)$ *are homeomorphic iff for every pair* F_1, F_2 *of completely separated subsets of* $\varprojlim X$ *there exists* $\alpha \in A$ *such that* $f_\alpha(F_1)$ *and* $f_\alpha(F_2)$ *are completely separated subsets of* X_α.

Proof. The proof of a) is trivial and can be omitted. For the first assertion of b) it suffices to prove that $\varprojlim \beta X$ is an extension of $\varprojlim X$ (since $\varprojlim \beta X$ is compact). For every nonempty open set $U \subseteq \varprojlim \beta X$ there is a nonempty open $U_\alpha \subseteq \beta X_\alpha$ such that $f_\alpha^{-1}(U_\alpha) \subseteq U$. The set $V_\alpha = U_\alpha \cap X_\alpha$ is nonempty since X_α is dense in βX_α. The surjectivity of f_α implies that $f_\alpha^{-1}(V_\alpha)$ is a nonempty subset of $\varprojlim X$. Clearly, $f_\alpha^{-1}(V_\alpha) \subseteq U$. This means that $\varprojlim X$ is dense in $\varprojlim \beta X$.

We prove now the second part of b).

The "if" part. From the assumption $f_\alpha(F_1) \cap f_\alpha(F_2) = \emptyset$ it follows that $f_\alpha(F_1)^{\beta X_\alpha} \cap \cap f_\alpha(F_2)^{\beta X_\alpha} = \emptyset$ [4; p. 226]. It follows that $\bar{F}_1^{X'} \cap \bar{F}_2^{X'} = \emptyset$, $X' = \varprojlim \beta X$ since $\mathrm{cl}\, F_i^{X'} \subseteq (\beta f_\alpha)^{-1}\ \mathrm{cl}\,(f_\alpha(F_i))^{\beta X_\alpha}$, $i = 1, 2$. From [4: p. 226] it follows that X' is homeomorphic to $\beta(\varprojlim X)$.

The "only if" part. Now, from $\mathrm{cl}\, F_1^{X'} \cap \mathrm{cl}\, F_2^{X'} = \emptyset$ it follows that there exists $\alpha \in A$ such that $\mathrm{cl}\,(f_\alpha(F_1)^{\beta X_\alpha} \cap \mathrm{cl}(f_\alpha(F_2))^{\beta X_\alpha} = \emptyset$ since βX is an inverse system of compact spaces βX_α. From the normality of βX_α it follows that $\mathrm{cl}(f_\alpha(F_1)^{\beta X_\alpha}$ and $\mathrm{cl}(f(F_2)^{\beta X_\alpha}$ are completely separated. This means that $f_\alpha(F_1)$ and $f_\alpha(F_2)$ are completely separated. The proof is complete.

1.2. Remark. *If* X_α, $\alpha \in A$, *are normal spaces, then* $\beta(\varprojlim X)$ *and* $\varprojlim \beta X$ *are homeomorphic iff for every pair* F_1, F_2 *of completely separated subsets of* $\lim X$ *there exists* $\alpha \in A$ *such that* $\mathrm{cl}(f_\alpha(F_1))^{X_\alpha'} \cap \mathrm{cl}(f_\alpha(F_2))^{X_\alpha'} = \emptyset$.

If $\varprojlim X$ is normal, then we can assume that F_1 and F_2 are closed subsets of $\varprojlim X$.

Applying Lemma 1.1 and Remark 1.2 we prove the following theorems.

1.3. Theorem. *Let* $X = \{X_n, f_{nm}, N\}$ *be an inverse sequence such that the mappings* f_{nm} *are onto and the spaces* X_n *are normal. If* $X = \varprojlim X$ *is a countably compact space, then* $\beta X = \varprojlim \beta X$.

Proof. Let us prove that X is normal. Consider two disjoint closed subsets F_1, F_2 of X. If we assume that $Y_n = \mathrm{cl}(f_n(F_1))^{X_n} \cap \mathrm{cl}(f_n(F_2))^{X_n}$ is nonempty for every $n \in N$, then we obtain a contradiction: $\emptyset \neq \bigcap\{f_n^{-1}(Y_n): n \in N\} \subseteq F_1 \cap F_2 = \emptyset$ since the space X is countably compact. It follows that there exists $n_0 \in N$ such that $\mathrm{cl}(f_{n_0}(F_1))^{X_{n_0}} \cap \cap\ \mathrm{cl}(f_{n_0}(F_2))^{X_{n_0}} = \emptyset$. Since X_{n_0} is normal, it follows that there exist disjoint open sets $U \supseteq \mathrm{cl}(f_{n_0}(F_1))^{X_{n_0}}$ and $V \supseteq \mathrm{cl}(f_{n_0}(F_2))^{X_{n_0}}$. Clearly, $f_{n_0}^{-1}(U) \supseteq F_1$ and $f_{n_0}^{-1}(V) \supseteq \supseteq F_2$. The normality of X is proved.

Let $X' = \varprojlim \beta X$. For two disjoint closed subsets $F_1, F_2 \subseteq X$ we have $n_0 \in N$ such that $\mathrm{cl}(f_{n_0}(F_1))^{X_{n_0}} \cap \mathrm{cl}(f_{n_0}(F_2))^{X_{n_0}} = \emptyset$. Lemma 1.1 and Remark 1.2 imply that $\beta X = \varprojlim \beta X$. The proof is complete.

1.4. Corollary. *Let* $X = \{X_n, f_{nm}, N\}$ *be an inverse sequence of normal countably compact spaces* X_n. *If* f_{nm}, $n, m \in N$, *are closed onto mappings, then* $\beta(\varprojlim X) = = \varprojlim \beta X$.

Proof. In [8] it is proved that $\varprojlim X$ is a normal countably compact space. Now, apply Theorem 1.3.

1.5. Remark. Corollary 1.4 implies Theorem 0.4 since a continuous mapping $f: X \to Y$ is closed if X is countably compact, Y is a regular first-countable space, and for every inverse system there exists another one which has onto projections f'_α.

We say that a mapping $f: X \to Y$ is fully closed [6] if for every point $y \in Y$ and every finite cover $\{U_1, \ldots, U_s\}$ of $f^{-1}(y)$ by open sets $U_i \subseteq X$, $i = 1, \ldots, s$, the set $\{y\} \cup \cup (f^\# U_1 \cup \ldots f^\# U_s\}$ is an open set in Y. The set $f^\# U_i$ is defined by $f^\# U_i = \{y: f^{-1}(y) \subseteq \subseteq U_i\}$.

1.6. Theorem. *Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse system such that $f_{\alpha\beta}$ are perfect fully closed onto mappings. If the spaces X_α are normal countably compact spaces, then $X = \lim X$ is normal and $\beta X = \varprojlim \beta X$.*

Proof. The projections $f_\alpha: X \to X_\alpha$, $\alpha \in A$, are perfect fully closed onto mappings [6; Lemma 3]. It is readily seen that X is countably compact. On the other hand, for every pair F_1, F_2 of disjoint closed subsets of X, the sets $Y_\alpha = f_\alpha(F_1) \cap f_\alpha(F_2)$, $\alpha \in A$, are discrete [6; Lemma 1(b)]. This means that Y_α, $\alpha \in A$, are finite sets since X_α, $\alpha \in A$, are countably compact. It follows that $Y = \{Y_\alpha, f_{\alpha\beta}/Y_\beta, A\}$ has a non-empty limit Y, if all Y_α are nonempty. The contradiction $\emptyset \neq Y \subseteq F_1 \cap F_2 = \emptyset$ implies that there exists $\alpha \in A$ such that $Y_\alpha = f_\alpha(F_1) \cap f_\alpha(F_2) = \emptyset$. This fact yields that X is normal. Moreover, the conditions of Lemma 1.1 and Remark 1.2 are satisfied. The proof is complete.

For a sequentially compact (strongly countably compact, D-compact) spaces [8; p. 158] we prove

1.7. Theorem. *If $X = \{X_n, f_{nm}, N\}$ is an inverse system of normal sequentially compact (strongly countably compact, D-compact) spaces X_n, then $X = \lim X$ is a normal space and $\beta X = \varprojlim \beta X$.*

Proof. X is a countably compact space [4; p. 268] ([8; p. 164]). Applying Theorem 1.3 we complete the proof.

We say that an inverse system $X = \{X_\alpha, f_{\alpha\beta}, A\}$ is a factorizable, or f-system [12], if for every real-valued function $f: \lim X \to R$ there exist $\alpha \in A$ and a real-valued function $g_\alpha: X_\alpha \to X$ such that $f = g_\alpha f_\alpha$, where $f_\alpha: \lim X \to X_\alpha$ is the projection.

1.8. Remark. If X is a limit space of an inverse f-system $X = \{X_\alpha, f_{\alpha\beta}, A\}$ of pseudocompact spaces X_α, then X is pseudocompact.

1.9. Theorem. *If X is a limit space of an inverse f-system with surjective projections $f_\alpha: X \to X_\alpha$, $\alpha \in A$, then $\beta X = \varprojlim \beta X$.*

Proof. Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be an inverse f-system and $f: X \to I$ a continuous real-valued function. There exist $\alpha \in A$ and $g_\alpha: X_\alpha \to I$ such that $f = g_\alpha f_\alpha$. Let

f'_α: $\underleftarrow{\lim}\ \beta X \to \beta X_\alpha$ be the projection. The function $f' = (\beta g_\alpha) \cdot f'_\alpha$ is an extension of f. By virtue of [4; 3.6.3 Corollary] it follows that $\underleftarrow{\lim}\ \beta X$ is equivalent to the compactification βX.

1.10. Corollary. *Let $X = \{X_\alpha, f_{\alpha\beta}, A\}$ be a σ-directed inverse system such that the projections f_α: $\underleftarrow{\lim}\ X \to X_\alpha$, $\alpha \in A$, are onto mappings. If* $\lim X$ *is a Lindelöf space, then $\beta(\underleftarrow{\lim}\ X) = \underleftarrow{\lim}\ \beta X$.*

Proof. X is an f-system [12; pp. 28]. Theorem 1.9 completes the proof.

1.11. Corollary. *Let X be a limit space $X = \{X_\alpha, f_{\alpha\beta}, A\}$ of an m-directed inverse system, where the Souslin number $c(X_\alpha) \leqq m$. If the projections f_α:* $\lim X \to X_\alpha$*, $\alpha \in A$, are onto mappings, then $\beta(\lim X) = \underleftarrow{\lim}\ \beta X$.*

Proof. X is an f-system [12; pp. 28, Predloženije 1.8]. Now, apply Theorem 1.9.

1.12. Corollary. *Let $X = \{X_\alpha, f_{\alpha\beta}, \Omega\}$ be a well-ordered inverse system of hereditarily Lindelöf spaces X_α such hat* $\mathrm{cf}(\Omega) > \omega_1$*. If the projections f_α: $\underleftarrow{\lim}\ X \to X_\alpha$, $\alpha \in A$, are onto mappings, then $\beta(\underleftarrow{\lim}\ X) = \underleftarrow{\lim}\ \beta X$.*

Proof. By virtue of [13] it follows that $\lim X$ is a hereditarily Lindelöf space. Corollary 1.10 implies that $\beta(\underleftarrow{\lim}\ X) = \underleftarrow{\lim}\ \beta X$. The proof is complete.

We close this paper with the following theorem.

1.13. Theorem. *Let $X = \{X_\alpha, f_{\alpha\beta}, \Omega\}$ be an inverse system, where the spaces X_α are normal and such that the hereditary Lindelöf number* $\mathrm{hl}(X_\alpha) < \aleph_m$ *and* $\mathrm{cf}(\Omega) > \aleph_{m+1}$*. If the projections f_α: $\underleftarrow{\lim}\ X \to X_\alpha$, $\alpha \in A$, are onto mappings, then $\beta(\underleftarrow{\lim}\ X) = \underleftarrow{\lim}\ \beta X$.*

Proof. From [9] and [13] it follows that for every pair F_1, F_2 of disjoint closed subsets of $\lim X$ there exists $\alpha \in \Omega$ such that $F_i = f_\alpha^{-1}\,\mathrm{cl}(f_\alpha(F))$, $i = 1, 2$. This means that $\underleftarrow{\lim}\ X$ is a normal space and that the conditions of Lemma 1.1 and Remark 1.2 are satisfied. The proof is complete.

References

[1] *Arhangelskij A. V.* and *Ponomarev V. I.:* Osnovy obščej topologii v zadačah i upražnenijah, Nauka, Moskva, 1974.

[2] *Bucur I.* and *Deleanu A.:* Vvedenije v teoriju kategorij i funktorov, Mir, Moskva, 1972.

[3] *Eilenberg S.* and *Steenrod N.:* Foundations of algebraic topology, Princeton, 1952.

[4] *Engelking R.:* General Topology, PWN, Warszawa, 1977.

[5] *Engelking R.:* Remarks on real-compact spaces, Fund. Math. *55* (1964), 303—308.

[6] *Fedorčuk V. V.:* Metod razvertyvaemyh spektrov i vpolne zamknutyh otobraženij v obščej topologii, UMN *35* (1980), 112—121.

[7] *Gillman L.* and *Jerison M.:* Rings of continuous functions, D. van Nostrand co., New York, 1960.

[8] *Lončar I.:* Inverse limits which generalize compact spaces, Glasnik matematički *17* (*37*) (1982), 155—173.

[9] *Lonačar I.:* Lindelöfov broj i inverzni limesi, Zbornik radova Fakulteta organizacije i informatike Varaždin *6* (1982), 155—165.
[10] *Nagota Jun-iti:* Some aspects of extension theory in general topology, Contribs. Extens. Theory. Topol. Struct. Proc. Sympos. Berlin 1967, Berlin 1969, 157—161.
[11] *Semadeni Z.:* Banach spaces of continuous functions I, PWN, Warszawa, 1971.
[12] *Ščepin E. V.:* Funktory i nesčetnye stepeni kompaktov, UMN *36* (1981), 3—62.
[13] *Tkačenko M. G.:* Cepy i kardinaly, DAN SSSR *239* (1978), 546—549.

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