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ON INTEGRATION IN BANACH SPACES, VIII (POLYMEASURES)

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INTRODUCTION

Our approach to integration enables us to extend the theory to an interesting theory of integration with respect to operator valued polymeasures separately countably additive in the strong operator topology, whose development we will start in Part IX. In this preparatory paper we deduce some basic properties of vector and operator valued polymeasures and their semivariations, needed later. However, note that the important problem on existence of control polymeasures is not solved in general, see Section 3. The theory of set functions of several variables, polymeasures, doubtless exploits results and ideas of the theory of measures, nonetheless it cannot be reduced to the latter.

To the author's best knowledge, except papers on bimeasures, see References, there are no published papers dealing with polymeasures of higher dimensions. Nevertheless, the notion of an operator valued polymeasure is known, see the end of p. 164 in [31]. Complex valued bimeasures and the related Morse-Transue integral have already found important applications in the structure theory of harmonizable stochastic processes, see [39], [3] and [28]. In Part XI which is under preparation we will clarify the notion of the strict Morse-Transue integral from [3]. Extensions of polymeasures are treated in [14].

1. POLYMEASURES

Let us first set down some basic notations. In the following, d (dimension) will be a fixed positive integer, X_i, $i = 1, \ldots, d$, and Y will be given Banach spaces over the same scalar field K of real or complex numbers. By $L^{(d)}(X_1, \ldots, X_d; Y)$ we denote the Banach space of all separately linear and continuous mappings $U: X_1 \times \ldots \times X_d \to Y$ with the norm $|U|_{L^{(d)}} = \sup\{|U(x_1, \ldots, x_d)|,\ x_i \in X_i,\ |x_i| \leqq 1,\ i = 1, \ldots, d\}$, see [38]. The elements of $L^{(d)}(X_1, \ldots, X_d; Y)$ are called *d-linear operators*. By the projective tensor product $X_1 \hat{\otimes} \ldots \hat{\otimes} X_d$ we mean the completion of the algebraic tensor product of the spaces X_i, $i = 1, \ldots, d$ under the greatest cross norm

$$\inf_z \sum_{r=1}^{N} |x_1^{(r)}| \cdot \ldots \cdot |x_d^{(r)}|, \quad z = \sum_{r=1}^{N} x_1^{(r)} \otimes \ldots \otimes x_d^{(r)}.$$

There is a "natural" isometric isomorphism between the space $L^{(d)}(X_1, \ldots, X_d; Y)$ and the space $L(X_1 \otimes^{\wedge} \ldots \otimes^{\wedge} X_d, Y)$ given by the equality $U(x_1, \ldots, x_d) = U(x_1 \otimes^{\wedge} \ldots \otimes^{\wedge} x_d)$. Clearly $L^{(d)}(K, \ldots, K; Y) = Y$, and $L^{(d)}(X_1, \ldots, X_d; Y) = L^{(d_1)}(X_1, \ldots, X_{d_1}; L^{(d-d_1)}(X_{d_1+1}, \ldots, X_d, Y))$ for any positive integer $d_1 < d$ in the sense of an isometric isomorphism.

In the following T_i, $i = 1, \ldots, d$ will be non empty sets. By $\mathscr{R}_i$, $\mathscr{P}_i$ and $\mathscr{S}_i$ we denote a ring, a δ-ring and a σ-ring of subsets of T_i, respectively. For $T \neq \emptyset$ and $\mathscr{A} \subset 2^T$, by $\varrho(\mathscr{A})$, $\delta(\mathscr{A})$ and $\sigma(\mathscr{A})$ we denote the smallest ring, δ-ring and σ-ring containing $\mathscr{A}$, respectively. We note that $\varrho(\mathscr{A})$ is countable whenever $\mathscr{A}$ is countable, see [25, Th. 5C]. In accordance with [9], by definition $\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_d = \delta(\mathscr{P}_1 \times \ldots \times \mathscr{P}_d)$. It is easy to verify that $\sigma(\mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_d) = \sigma(\mathscr{P}_1) \otimes \ldots \otimes \sigma(\mathscr{P}_d)$.

We say that a set function defined on $\mathscr{R}_1 \times \ldots \times \mathscr{R}_d$ has locally a property P if its restriction to $(A_1 \cap \mathscr{R}_1) \times \ldots \times (A_d \cap \mathscr{R}_d)$ has the property P for each $(A_1, \ldots, A_d) \in \mathscr{R}_1 \times \ldots \times \mathscr{R}_d$.

Definition 1. We say that a set function $\gamma: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to Y$ is a *vector d-polymeasure* if it is separately countably additive, i.e., if:

1) $\gamma(\cdot, A_2, \ldots, A_d): \mathscr{R}_1 \to Y$ is a countably additive vector measure for each $(A_2, \ldots, A_d) \in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$,

. . . .

d) $\gamma(A_1, \ldots, A_{d-1}, \cdot): \mathscr{R}_d \to Y$ is a countably additive vector measure for each $(A_1, \ldots, A_{d-1}) \in \mathscr{R}_1 \times \ldots \times \mathscr{R}_{d-1}$.

If the countable addtivities in 1) – d) are uniform, then we say that γ is a *uniform vector d-polymeasure.*

If $Y = K$, then γ is called a *scalar d-polymeasure.*

We say that $\Gamma: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ is an operator valued d-polymeasure separately countably additive in the strong operator topology (in the uniform operator topology) if $\Gamma(\ldots)(x_1, \ldots, x_d): \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to Y$ is a vector d-polymeasure for each $(x_1, \ldots, x_d) \in X_1 \times \ldots \times X_d$ (if $\Gamma: \mathscr{R}_1 \times \ldots \times \mathscr{R}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ is a vector d-polymeasure).

2-polymeasures are called *bimeasures* and for them we sometimes write β and B instead of γ and Γ.

Let us introduce some types of polymeasures. If $\Gamma: \mathscr{P}_1 \otimes \ldots \otimes \mathscr{P}_d \to L(X_1 \otimes^{\wedge} \ldots \otimes^{\wedge} X_d, Y)$ is an operator valued measure countably additive in the strong operator topology (in the uniform operator topology), then its restriction to $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$ is an $L^{(d)}(X_1, \ldots, X_d; Y) = L(X_1 \otimes^{\wedge} \ldots \otimes^{\wedge} X_d, Y)$ valued d-polymeasure of the same type. On the other hand, most vector valued polymeasures are not restrictions of vector valued measures on product σ-rings. (Note, however, that each vector valued d-polymeasure, being finitely additive on a semiring

$\mathscr{R}_1 \times \ldots \times \mathscr{R}_d$, has a unique finitely additive extension to the ring $\varrho(\mathscr{R}_1 \times \ldots \times \mathscr{R}_d)$.) Probably the first real valued bimeasure which is not a restriction of a measure on a product σ-algebra was constructed de facto by G. Fichtenholz in [23]. Namely, he constructed a Borel measurable function $f: [a, b] \times [c, d] \to R$ which is not Lebesgue integrable on the rectangle $[a, b] \times [c, d]$ but for which the iterated Lebesgue integrals $\int_{A_2}(\int_{A_1} f(x, y)\,\mathrm{d}x)\,\mathrm{d}y$ and $\int_{A_1}(\int_{A_2} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$ exist and coincide for each Borel sets $A_1 \subset [a, b]$ and $A_2 \subset [c, d]$. Here the bimeasure β, $\beta(A_1, A_2) = \int_{A_2}(\int_{A_1} f(x, y)\,\mathrm{d}x)\,\mathrm{d}y$ is of type $\beta(A_1, A_2) = \int_{A_2}\mu_1(y, A_1)\,\mathrm{d}\mu_2(y)$, which we call the *indirect product* of measures $\mu_1(y, \cdot)$, $y \in Y$, with the measure μ_2; see [13], where such indirect products of operator valued measures were considered. Examples of bimeasures which are indirect products of vector valued measures by non negative measures are given in [30]. In particular, we note that the Dvoretzky-Rogers theorem, see [4], implies that there are bimeasures $\beta: 2^N \times 2^N \to R$ which are not restrictions of measures on $2^{N \times N} = 2^N \otimes 2^N$, see Example 2 in [30]. Other two types of bimeasures were de facto introduced in the pioneering paper [41] of K. Ylinen. Namely, in [41] K. Ylinen proved a Riesz type representation of a wide class of bilinear operators on $C_0(T_1) \times C_0(T_2)$, T_1 and T_2 being locally compact Hausdorff, as certain integrals with respect to separately regular vector Borel bimeasures. Hence such bilinear operators, which, moreover, cannot be extended to weakly compact linear operators on $C_0(T_1 \times T_2)$, are represented by vector separately regular Borel bimeasures which are not restrictions of regular vector Borel measures in the product $T_1 \times T_2$. The second type of bimeasures introduced by K. Ylinen is the following one: Let $\mathrm{ca}(\mathscr{S}_2)$ denote the Banach space of countably additive scalar measures on $\mathscr{S}_2$ with the variation (= semivariation) norm. Then clearly any vector measure $\mu_1: \mathscr{S}_1 \to \mathrm{ca}(\mathscr{S}_2)$ induces by the equality $\beta(A_1, A_2) = \mu_1(A_1)(A_2)$, $A_1 \in \mathscr{S}_1$, $A_2 \in \mathscr{S}_2$, a scalar bimeasure $\beta: \mathscr{S}_1 \times \mathscr{S}_2 \to K$. From the remarkable Theorem 4.4 in [41] we know that each scalar bimeasure is of this type. More precisely, there is an isometric isomorphism between scalar bimeasures $\beta: \mathscr{S}_1 \times \mathscr{S}_2 \to K$, the vector measures $\mu_1: \mathscr{S}_1 \to \mathrm{ca}(\mathscr{S}_2)$, and the vector measures $\mu_2: \mathscr{S}_2 \to \mathrm{ca}(\mathscr{S}_1)$, the norms being the semivariations (for polymeasures the semivariation is defined in Definition 3 below). (Actually K. Ylinen proved this result when $\mathscr{S}_1$ and $\mathscr{S}_2$ are σ-algebras, but small modifications of the proof yield the result for σ-rings). From here we easily obtain an isometric isomorphism between scalar d-polymeasures $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to K$, and vector valued $d - 1$ polymeasures $\gamma': \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d-1} \to \mathrm{ca}(\mathscr{S}_d)$, see the paragraph after Theorem 3 below. Another consequence of this result is the following fact:

(Y) *Each scalar bimeasure $\beta: \mathscr{S}_1 \times \mathscr{S}_2 \to K$ is uniform.*

The author is indebted to Hans Weber from Potenza for the following nice

Example of a Hilbert space valued bimeasure on Cartesian product of σ-algebras which is not a uniform bimeasure; Let $T_1 = N = \{1, 2, \ldots\}$, $\mathscr{P}_1 = 2^N$, $T_2 = [0, 1]$, $\mathscr{P}_2 = \mathscr{B}([0, 1])$ = the Borel measurable subsets of $[0, 1]$, $Y = L^2([0, 1])$, and let e_n,

$n = 1, 2, \ldots$ be an orthonormal base for Y. For $A \in \mathscr{P}_1$ and $B \in \mathscr{P}_2$ put $\beta(A, B) = P'_A(\chi_B)$, where P'_A is the projection operator of $L^2([0, 1])$ into its subspace $\overline{\mathrm{sp}}(\{e_i\}_{i \in A})$.

We now deduce some basic properties of polymeasures. The Orlicz-Pettis theorem, see any of the books [4], [5], [22] and [26], implies that a set function $\gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to Y$ which is separately weakly countably additive is a vector d-polymeasure. Moreover if Y does not contain an isomorphic copy of c_0, for example if Y is weakly sequentially complete, see [2], then it is enough to suppose that $y^*\gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to K$ is a scalar d-polymeasure for each $y^* \in Y^*$.

Similarly the Vitali-Hahn-Saks-Nikodým theorem, see [5], immediately yields its generalization to polymeasures:

(VHSN)-**Theorem for polymeasures.** *Let $\Gamma_n: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$, $n = 1, 2, \ldots$ be an operator valued d-polymeasures separately countably additive in the strong operator topology and let $\lim_{n\to\infty} \Gamma_n(A_1, \ldots, A_d)(x_1, \ldots, x_d) \in Y$ exist for each $(A_1, \ldots, A_d) \in \mathscr{P}_1 \times \ldots \times \mathscr{P}_d$ and each $(x_1, \ldots, x_d) \in X_1 \times \ldots \times X_d$. Put $\Gamma(A_1, \ldots, A_d)(x_1, \ldots, x_d) = \lim_{n\to\infty} \Gamma_n(A_1, \ldots, A_d)(x_1, \ldots, x_d)$. Then $\Gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$, Γ is separately countably additive in the strong operator topology, and the vector d-polymeasures $\Gamma_n(\ldots)(x_1, \ldots, x_d): \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to Y$, $n = 1, 2, \ldots$ are separately uniformly countably additive for each $(x_1, \ldots, x_d) \in X_1 \times \ldots \times X_d$.*

We now prove also the following (N – Nikodým) result:

(N) – **Uniform boundedness theorem for polymeasures.** *Let $\Gamma_{\mathfrak{b}}: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$, $\mathfrak{b} \in \mathfrak{J}$ = an index set, be operator valued d-polymeasures separately countably additive in the strong operator topology and let $\sup_{\mathfrak{b}\in\mathfrak{J}} |\Gamma_{\mathfrak{b}}(A_1, \ldots, A_d)(x_1, \ldots, x_d)| < +\infty$ for each $(A_1, \ldots, A_d) \in \mathscr{S}_1 \times \ldots \times \mathscr{S}_d$ and each $(x_1, \ldots, x_d) \in X_1 \times \ldots \times X_d$. Then*

$$\sup_{\mathfrak{b}\in\mathfrak{J}} \sup_{(A_1,\ldots,A_d)\in\mathscr{S}_1\times\ldots\times\mathscr{S}_d} |\Gamma_{\mathfrak{b}}(A_1, \ldots, A_d)|_{L^{(d)}} < +\infty .$$

Proof. We proceed by induction for d. For $d = 1$ the assertion follows from the Nikodým uniform boundedness theorem for vector measures, see [5] or [22], and from the uniform boundedness principle for operators. Suppose now the assertion of the theorem holds for $d - 1$. Define $\Gamma_{\mathfrak{b}}(A_1, \ldots, A_{d-1}, \cdot)(x_1, \ldots, x_{d-1}, \cdot): \mathscr{S}_d \to L(X_d, Y)$ by the equality $\Gamma_{\mathfrak{b}}(A_1, \ldots, A_{d-1}, \cdot)(x_1, \ldots, x_{d-1}, \cdot)(A_d)(x_d) = \Gamma_{\mathfrak{b}}(A_1, \ldots, A_d)(x_1, \ldots, x_d)$, $(A_1, \ldots, A_{d-1}) \in \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d-1}$, and $(x_1, \ldots, x_{d-1}) \in X_1 \times \ldots \times X_{d-1}$. Then by the induction hypothesis,

$$\sup_{\mathfrak{b}\in\mathfrak{J}} \sup_{(A_1,\ldots,A_{d-1})\in\mathscr{S}_1\times\ldots\times\mathscr{S}_{d-1}} |\Gamma_{\mathfrak{b}}(A_1, \ldots, A_{d-1}, \cdot)(x_1, \ldots, x_{d-1}, \cdot)(A_d)(x_d)| =$$

$$= \sup_{\mathfrak{b}\in\mathfrak{J}} \sup_{(A_1,\ldots,A_{d-1})\in\mathscr{S}_1\times\ldots\times\mathscr{S}_{d-1}} |\Gamma_{\mathfrak{b}}(\ldots, A_d)(\ldots, x_d)(A_1, \ldots, A_{d-1})(x_1, \ldots, x_{d-1})| < +\infty$$

for each $(x_1, \ldots, x_d) \in X_1 \times \ldots \times X_d$ and each $A_d \in \mathscr{S}_d$. Hence by Nikodým's uniform boundedness theorem for vector measures we have

$$\sup_{\mathfrak{d}\in\mathfrak{I}} \sup_{(A_1,\ldots,A_d)\in\mathscr{S}_1\times\ldots\times\mathscr{S}_d} \left|\Gamma_{\mathfrak{d}}(A_1,\ldots,A_d)(x_1,\ldots,x_d)\right| < +\infty$$

$$\text{for each} \quad (x_1,\ldots,x_d) \in X_1 \times \ldots \times X_d \,.$$

It remains to apply the uniform boundedness principle for operators to obtain the assertion of the theorem for d. Hence the theorem is valid for any positive integer d.

As an immediate consequence we have that any operator valued d-polymeasure $\Gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ separately countably additive in the strong operator topology is a bounded set function.

We will need

Lemma 1. *Let a family of vector measures $\nu_{\mathfrak{d}}: \mathscr{S} \to Y$, $\mathfrak{d} \in \mathfrak{I}$, be uniformly countably additive, let $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$, and let $A_n \to A$, i.e., let* $\liminf_n A_n = \limsup_n A_n = A$. *Then* $\lim_{n\to\infty} \nu_{\mathfrak{d}}(A_n) = \nu_{\mathfrak{d}}(A)$ *uniformly with respect to* $\mathfrak{d} \in \mathfrak{I}$.

Proof. For $\mathfrak{d} \in \mathfrak{I}$ and $E \in \mathscr{S}$ put $\bar{\nu}_{\mathfrak{d}}(A) = \sup_{F\subset E, F\in\mathscr{S}} \left|\nu_{\mathfrak{d}}(F)\right|$. Then $\bar{\nu}_{\mathfrak{d}}: \mathscr{S} \to [0, +\infty]$ is monotone, subadditive and bounded by (N). Further, the countable additivity of $\nu_{\mathfrak{d}}$ implies that $\bar{\nu}_{\mathfrak{d}}$ has Fatou property: $E_n \in \mathscr{S}$, $n = 1, 2, \ldots$ and $E_n \nearrow E$ implies $\bar{\nu}_{\mathfrak{d}}(E_n) \nearrow \bar{\nu}_{\mathfrak{d}}(E)$; and that $\bar{\nu}_{\mathfrak{d}}$ is exhaustive: if $E_n \in \mathscr{S}$, $n = 1, 2, \ldots$ are pairwise disjoint, then $\bar{\nu}_{\mathfrak{d}}(E_n) \to 0$. These two properties imply that $\bar{\nu}_{\mathfrak{d}}$ is continuous: $E_n \in \mathscr{S}$, $n = 1, 2, \ldots$ and $E_n \searrow \emptyset$ implies $\bar{\nu}_{\mathfrak{d}}(E_n) \to 0$. Hence $\bar{\nu}_{\mathfrak{d}}$ is a subadditive submeasure in the sense of Definition 1 in [16].

For $E \in \mathscr{S}$ put $\mu(E) = \sup_{\mathfrak{d}\in\mathfrak{I}} \bar{\nu}_{\mathfrak{d}}(E)$. Then the properties of $\bar{\nu}_{\mathfrak{d}}$ imply that $\mu: \mathscr{S} \to [0, +\infty]$ is monotone, subadditive and has the Fatou property. Further, the uniform countable additivity of the family $\nu_{\mathfrak{d}}$, $\mathfrak{d} \in \mathfrak{I}$, implies that μ is also exhaustive. Hence μ is continuous. Since $E_n \to \emptyset$ if and only if $\bigcup_{k=n}^{\infty} E_k \searrow \emptyset$, by monotonocity and continuity of μ we conclude that $E_n \to \emptyset$ implies $\mu(E_n) \to 0$.

Let $\mathfrak{d} \in \mathfrak{I}$ and $E, F \in \mathscr{S}$. Then by additivity of $\nu_{\mathfrak{d}}$ and monotonocity of $\bar{\nu}_{\mathfrak{d}}$ we have the inequalities

$$\left|\nu_{\mathfrak{d}}(E) - \nu_{\mathfrak{d}}(F)\right| = \left|\nu_{\mathfrak{d}}(E - F) - \nu_{\mathfrak{d}}(F - E)\right| \leqq \left|\nu_{\mathfrak{d}}(E - F)\right| +$$
$$+ \left|\nu_{\mathfrak{d}}(F - E)\right| \leqq \bar{\nu}_{\mathfrak{d}}(E - F) + \bar{\nu}_{\mathfrak{d}}(F - E) \leqq 2\bar{\nu}_{\mathfrak{d}}(E \,\Delta\, F) \leqq 2\mu(E \,\Delta\, F)\,,$$

for each $\mathfrak{d} \in \mathfrak{I}$.

If now $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$ and $A_n \to A$, then $A_n \,\Delta\, A \to \emptyset$. Hence $\mu(A_n \,\Delta\, A) \to 0$ by continuity of μ, and thus the above inequalities imply the assertion of the lemma.

Theorem 1. *Let $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ be a vector d-polymeasure, let $A_{i,n} \in \mathscr{S}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$, and let $A_{i,n} \to A_i$ for each $i = 1, \ldots, d$. Then*

$$\lim_{n_1,\ldots,n_d\to\infty} \gamma(A_{1,n_1}, \ldots, A_{d,n_d}) = \gamma(A_1, \ldots, A_d)\,.$$

Proof. We prove the theorem by induction for d. For $d = 1$ the result is well known, see for example Theorem 3.5 in [24], and also follows from Lemma 1. Suppose now the assertion of the theorem holds for $d - 1$, $d \geqq 2$. In the given setting put $\nu_{n_1,\dots,n_{d-1}}(E_d) = \gamma(A_{1,n_1}, \dots, A_{d-1,n_{d-1}}, E_d)$, $E_d \in \mathscr{S}_d$, $n_1, \dots, n_{d-1} = 1, 2, \dots$. Then, by the separate countable additivity of γ, for any multiindex $(n_1, \dots, n_{d-1})$ the set function $\nu_{n_1,\dots,n_{d-1}}: \mathscr{S}_d \to Y$ is a countably additive vector measure. We now show that the vector measures $\nu_{n_1,\dots,n_{d-1}}$, $n_1, \dots, n_{d-1} = 1, 2, \dots$ are uniformly countably additive on $\mathscr{S}_d$, and this by Lemma 1 will imply the assertion of the theorem for d. Suppose the vector measures $\nu_{n_1,\dots,n_{d-1}}$, $n_1, \dots, n_{d-1} = 1, 2, \dots$ are not uniformly countably additive on $\mathscr{S}_d$. Then they are clearly not uniformly exhaustive in $\mathscr{S}_d$. Hence there is an $\varepsilon > 0$, a sequence of pairwise disjoint sets $E_{d,k} \in \mathscr{S}_d$, $k = 1, 2, \dots$, and a sequence of multiindexes $m_k = (n_{1,k}, \dots, n_{d-1,k})$, $k = 1, 2, \dots$ such that $|\nu_{m_k}(E_{d,k})| > \varepsilon$ for each $k = 1, 2, \dots$. Take first a subsequence $\{k_1\} \subset \{k\}$ such that $\{n_{1,k_1}\}$ is either a constant sequence, or $n_{1,k_1} \to +\infty$. Next take a subsequence $\{k_{1,2}\} \subset \{k_1\}$ such that $\{n_{2,k_{1,2}}\}$ is either a constant sequence, or $n_{2,k_{1,2}} \to +\infty$. Continuing in this way, finally take a subsequence $\{k_{1,\dots,d-1}\} \subset \{k_{1,\dots,d-2}\}$ such that it is either a constant sequence, or tends to infinity. Then by the induction hypothesis $\lim\limits_{k_{1,\dots,d-1}\to\infty} \nu_{m_{k_{1,\dots,d-1}}}(E_d) \in Y$ exists for each $E_d \in \mathscr{S}_d$. But then by (VHSN)-Theorem the vector measures $\nu_{m_{k_{1,\dots,d-1}}}: \mathscr{S}_d \to Y$, $k_{1,\dots,d-1} = 1, 2, \dots$ are uniformly countably additive, equivalently, uniformly exhaustive, a contradiction. Thus the theorem is valid for any positive integer d.

Definition 2. For a separately additive set function $\Gamma: \mathscr{R}_1 \times \dots \times \mathscr{R}_d \to L^{(d)}(X_1, \dots, X_d\ Y)$ *we define its supremation* $\bar{\Gamma}: \mathscr{R}_{1,\sigma} \times \dots \times \mathscr{R}_{d,\sigma} \to [0, +\infty]$ by the equality

$$\bar{\Gamma}(A_1, \dots, A_d) = \sup\{|\Gamma(B_1, \dots, B_d)|_{L^{(d)}},\ B_i \in A_i \cap \mathscr{R}_i,\ i = 1, \dots, d\}\,.$$

We will use the same notation if A_i is replaced by T_i for some $i \in \{1, \dots, d\}$.

We say that a d-tuple $(A_1, \dots, A_d) \in \mathscr{R}_{1,\sigma} \times \dots \times \mathscr{R}_{d,\sigma}$ is Γ-null if $\bar{\Gamma}(A_1, \dots, A_d) = 0$.

We say that a separately additive set function $\Gamma_1: \mathscr{R}_1 \times \dots \times \mathscr{R}_d \to L^{(d)}(X_1', \dots, X_d'; Y')$ is absolutely continuous with respect to Γ if $\Gamma_1(A_1, \dots, A_d) = 0$ for each Γ-null d-tuple $(A_1, \dots, A_d) \in \mathscr{R}_{1,\sigma} \times \dots \times \mathscr{R}_{d,\sigma}$. In that case we write $\Gamma_1 \ll \Gamma$.

Clearly, $\bar{\Gamma}(A_1, \dots, A_d) = \sup\limits_{|x_1|\leqq 1,\dots,|x_d|\leqq 1} \bar{\Gamma}(A_1, \dots, A_d)(x_1, \dots, x_d)$.

Theorem 2. *Let* $\gamma: \mathscr{S}_1 \times \dots \times \mathscr{S}_d \to Y$ *be a vector d-polymeasure. Then*:

1. $\bar{\gamma}(T_1, \dots, T_d) < +\infty$, *and* $\bar{\gamma}: \mathscr{S}_1 \times \dots \times \mathscr{S}_d \to [0, +\infty)$ *is separately monotone and separately subadditive*,

2. $\bar{\gamma}$ *has the following Fatou property*: $A_{i,n} \in \mathscr{S}_i$, $i = 1, \dots, d$, $n = 1, 2, \dots$ *and* $A_{i,n} \nearrow A_i$ *implies* $\bar{\gamma}(A_{1,n}, \dots, A_{d,n}) \nearrow \bar{\gamma}(A_1, \dots, A_d)$. *Hence by* 1) $\bar{\gamma}$ *is separately countably subaddtitive*,

3. $\bar{\gamma}$ *is continuous at* $(\emptyset, \ldots, \emptyset)$ *in the following sense:* $A_{i,n} \in \mathcal{S}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ *and* $A_{i,n} \to \emptyset$ *for each* $i = 1, \ldots, d$ *imply* $\lim_{n_1, \ldots, n_d \to \infty} \bar{\gamma}(A_{1,n_1}, \ldots, A_{d,n_d}) = 0$.

Proof. 1) follows from (N)-Theorem and the definitions, while 2) is an easy consequence of Theorem 1.

3) For each $i = 1, \ldots, d$ let $B_{i,n} \in \mathcal{S}_i$, $n = 1, 2, \ldots$ be pairwise disjoint. Then $\gamma(B_{1,n_1}, \ldots, B_{1,n_d}) \to 0$ as $n_1, \ldots, n_d \to \infty$ by Theorem 1. This property of γ is called *exhaustivity*. But then evidently $\bar{\gamma}$ is also exhaustive. Now it is easy to see that exhaustivity and the Fatou property of $\bar{\gamma}$ imply its continuity from above at $(\emptyset, \ldots, \emptyset)$, i.e., if for each $i = 1, 2, \ldots, d$, $B_{i,n} \in \mathcal{S}_i$, $n = 1, 2, \ldots$ and $B_{i,n} \searrow \emptyset$, then $\bar{\gamma}(B_{1,n_1}, \ldots, B_{d,n_d}) \to 0$ as $n_1, \ldots, n_d \to \infty$. Hence the continuity of $\bar{\gamma}$ at $(\emptyset, \ldots, \emptyset)$ immediately follows from the fact that $A_{i,n_i} \to \emptyset$ if and only if $\limsup_{k_i} A_{i,k_i} \searrow \emptyset$, by virtue of the separate monotonocity of $\bar{\gamma}$.

Corollary 1. *Let* $\Gamma\colon \mathcal{S}_1 \times \ldots \times \mathcal{S}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ be an operator valued d-polymeasure separately countably additive in the strong operator topology. Then $\bar{\Gamma}(T_1, \ldots, T_d) < +\infty$, $\bar{\Gamma}$ has the Fatou property, and $\bar{\Gamma}$ is separately monotone and separately countably subadditive.

The next corollary is also immediate.

Corollary 2. *Let* $T_i = N = \{1, 2, \ldots\}$, *and let* $\mathcal{S}_i = 2^N$ *for each* $i = 1, \ldots, d$. *Then any vector* d*-polymeasure* $\gamma\colon \mathcal{S}_1 \times \ldots \times \mathcal{S}_d \to Y$ *is uniform.*

Denote by $S(\mathcal{P}_i, X_i)$, $i = 1, 2, \ldots, d$, the normed linear space of $\mathcal{P}_i$-simple functions $f_i\colon T_i \to X_i$ with the sup norm $\|f_i\|_{T_i} = \sup_{t_i \in T_i} |f(t_i)|$. If $X_i = K$, then we write simply $S(\mathcal{P}_i)$.

Let $\Gamma\colon \mathcal{P}_1 \times \ldots \times \mathcal{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ be an operator valued d-polymeasure separately countably additive in the strong operator topology and let $f_i \in S(\mathcal{P}_i, X_i)$, $i = 1, \ldots, d$, be of the form $f_i = \sum_{j_i=1}^{r_i} x_{i,j_i} \chi_{A_{i,j_i}}$ with $x_{i,j_i} \in X_i$, and with pairwise disjoint $A_{i,j_i} \in \mathcal{P}_i$. Then the integral of the d-tuple of functions $(f_1, \ldots, f_d) \in S(\mathcal{P}_1, X_1) \times \ldots \times S(\mathcal{P}_d, X_d)$ over the d-tuple of sets $(A_1, \ldots, A_d) \in \sigma(\mathcal{P}_1) \times \ldots \times \sigma(\mathcal{P}_d)$ is defined by the equality

$$\int_{(A_1, \ldots, A_d)} (f_1, \ldots, f_d)\, \mathrm{d}\Gamma = \sum_{j_1=1}^{r_1} \ldots \sum_{j_d=1}^{r_d} \Gamma(A_1 \cap A_{1,j_1}, \ldots, A_d \cap A_{d,j_d})(x_{1,j_1}, \ldots, x_{d,j_d}).$$

If there is no danger of confusion we write $\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$, $\mathsf{X} S(\mathcal{P}_i, X_i)$ and $\mathsf{X}\sigma(\mathcal{P}_i)$.

Clearly, for a fixed $(f_i) \in \mathsf{X} S(\mathcal{P}_i, X_i)$ the mapping $\int_{(\cdot)} (f_i)\, \mathrm{d}\Gamma\colon \mathsf{X}\sigma(\mathcal{P}_i) \to Y$ is a vector d-polymeasure absolutely continuous with respect to Γ, and for a fixed $(A_i) \in \mathsf{X}\sigma(\mathcal{P}_i)$ the mapping $\int_{(A_i)} (\cdot)\, \mathrm{d}\Gamma\colon \mathsf{X} S(\mathcal{P}_i, X_i) \to Y$ is separately linear. Note that the finiteness of the semivariation $\hat{\Gamma}$ on (A_i) is just the requirement of continuity of this mapping. Note also that

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \int_{(A_i)} (f_i \cdot \chi_{A_i})\, \mathrm{d}\Gamma \text{ for each } (f_i) \in \mathsf{X} S(\mathcal{P}_i, X_i) \quad \text{and} \quad (A_i) \in \mathsf{X}\sigma(\mathcal{P}_i)\,.$$

Definition 3. *Let $\Gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ be an operator valued d-polymeasure separately countably additive in the strong operator topology. Then we define:*

a) *its semivariation $\hat{\Gamma}: \sigma(\mathscr{P}_1) \times \ldots \times \sigma(\mathscr{P}_d) \to [0, +\infty]$ by the equality*

$\hat{\Gamma}(A_1, \ldots, A_d) = \sup \{ |\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma|, \ f_i \in S(\mathscr{P}_i, X_i)$ *and* $\|f_i\|_{T_i} \leqq 1, \ i = 1, \ldots, d\} \cdot$

hence $\hat{\Gamma}(A_1, \ldots, A_d)$ is the $L^{(d)}(S(\mathscr{P}_1, X_1), \ldots, S(\mathscr{P}_d, X_d); Y)$ norm of the mapping $\int_{(A_i)} (\cdot) \, \mathrm{d}\Gamma$.

b) *its scalar semivariation $\|\Gamma\|: \sigma(\mathscr{P}_1) \times \ldots \times \sigma(\mathscr{P}_d) \to [0, +\infty]$ by the equality*

$\|\Gamma\| (A_1, \ldots, A_d) = \sup \{ |\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma|_{L^{(d)}}, \ f_i \in S(\mathscr{P}_i), \ \|f_i\|_{T_i} \leqq 1, \ i = 1, \ldots, d\}$

(*if* $f_i = \sum_{j_i=1}^{r_i} a_{i,j_i} \cdot \chi_{A_{i,j_i}}$ *with pairwise disjoint* $A_{i,j_i} \in \mathscr{P}_i$, $i = 1, \ldots,$ *then*

$\int_{(A_i)} (f_i) \, \mathrm{d}\Gamma = \sum_{j_1=1}^{r_1} \ldots \sum_{j_d=1}^{r_d} a_{1,j_1} \cdot \ldots \cdot a_{d,j_d} \cdot \Gamma(A_1 \cap A_{1,j_1}, \ldots, A_d \cap A_{d,j_d})$).

c) *its variation $v(\Gamma.(\ldots)): \sigma(\mathscr{P}_1) \times \ldots \times \sigma(\mathscr{P}_d) \to [0, +\infty]$ by the equality*

$$v(\Gamma, (A_1, \ldots, A_d)) = \sup \sum_{j_1=1}^{r_1} \ldots \sum_{j_d=1}^{r_d} |\Gamma(A_{1,j_1}, \ldots, A_{d,j_d})|_{L^{(d)}}, \quad A_{i,j_i} \in A_i \cap \mathscr{P}_i$$

are pairwise disjoint, $j_i = 1, \ldots, r_i$, $r_i = 1, 2, \ldots$, and $i = 1, \ldots, d\}$.

d) *for $\mathscr{P}_i$-measurable functions $f_i: T_i \to X_i$, or $f_i: T_i \to [0, +\infty)$, and sets $A_i \in \in \{\sigma(\mathscr{P}_i), T_i\}$ we define the multiple L_1-gauge $\hat{\Gamma}[(f_1, \ldots, f_d), (A_1, \ldots, A_d)]$ by the equality $\hat{\Gamma}[(f_1, \ldots, f_d), (A_1, \ldots, A_d)] = \sup \{ |\int_{(A_i)} (g_i) \, \mathrm{d}\Gamma|, \ g_i \in S(\mathscr{P}_i, X_i)$, $|g_i(t_i)| \leqq |f_i(t_i)|$ for each $t_i \in T_i$, $i = 1, \ldots, d\}$.*

Obviously $\bar{\Gamma}(A_i) \leqq \|\Gamma\| (A_i) \leqq \hat{\Gamma}(A_i) \leqq v(\Gamma, (A_i))$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, and $v(\Gamma, (A_i)) = 0$ for each Γ-null d-tuple $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. Further, for any $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, $\hat{\Gamma}(A_i) = \hat{\Gamma}[(\chi_{A_i}), (A_i)]$. We have

Theorem 3. *Let us have the notions from Definition 3. Then*

1) *$v(\Gamma, (\cdot)): \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ is a d-polymeasure;*

2) *$\hat{\Gamma}[(f_i), (A_i)] = \sup_{|y^*| \leqq 1} \widehat{y^*\Gamma}[(f_i), (A_i)]$, and*

$$\|\Gamma\| (A_i) = \sup_{|y^*| \leqq 1, |x_i| \leqq 1} \|y^*\Gamma(\cdot)(x_i)\| (A_i);$$

3) *$\hat{\Gamma}[(f_i), (A_i)] < +\infty$ if and only if $\widehat{y^*\Gamma}[(f_i), (A_i)] < +\infty$ for each $y^* \in Y^*$;*

4) *$\|\Gamma\| (A_i) \leqq 4^d \cdot \bar{\Gamma}(A_i)$ ($\leqq 2^d \cdot \bar{\Gamma}(A_i)$ in the real case). Hence $\|\Gamma\|$ is finite valued on $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$.*

5) *The set functions $\|\Gamma\|$, $\hat{\Gamma}$, and $\hat{\Gamma}[(f_i), (\cdot)]: \sigma(\mathscr{P}_1) \times \ldots \times \sigma(\mathscr{P}_d) \to [0, +\infty]$ are separately monotone and separately countably subadditive.*

6) *Let $a_i \in K$, and let $g_i: T_i \to X_i$ (or $g_i: T_i \to [0, +\infty)$ if $f_i: T_i \to [0, +\infty)$) be P_i-measurable, $i = 1, \ldots, d$. Then $\hat{\Gamma}[(a_i f_i), (A_i)] = |a_1| \cdot \ldots \cdot |a_d| \cdot \hat{\Gamma}[(f_i), (A_i)]$,*

$\hat{\Gamma}[(f_i + g_i), (A_i)] \leqq \hat{\Gamma}[(f_i), (A_i)] + \hat{\Gamma}[(g_i), (A_i)]$, *and* $\hat{\Gamma}[(g_i), (A_i)] \leqq \leqq \hat{\Gamma}[(f_i), (A_i)]$ *if* $|g_i| \leqq |f_i|$ *for each* $i = 1, \ldots, d$.

7) $\inf_{t_1 \in T_1} |f_1(t_1)| \cdot \ldots \cdot \inf_{t_d \in T_d} |f_d(t_d)| \cdot \hat{\Gamma}(A_i) \leqq \hat{\Gamma}[(f_i), (A_i)] \leqq \|f_1\|_{A_1} \cdot \ldots \cdot \|f_d\|_{A_d} \cdot \hat{\Gamma}(A_i)$.

Hence we have a Tschebyscheff type inequality

$$\hat{\Gamma}(\{t_i \in A_i;\ |f_i(t_i)| \geqq a_i > 0\}) \leqq \frac{\hat{\Gamma}[(f_i), (A_i)]}{a_1 \cdot \ldots \cdot a_d}.$$

8) *In the* (VHSN)-*Theorem we have the inequalities*:

$$\|\Gamma\|\,(A_i) \leqq \liminf_n \|\Gamma_n\|\,(A_i) \leqq \limsup_n \|\Gamma_n\|\,(A_i) < +\infty$$

(*similarly for* $\bar{\Gamma}$),

$$\hat{\Gamma}(A_i) \leqq \liminf_n \hat{\Gamma}_n(A_i),$$

and similarly for $v(\Gamma, (A_i))$ *and* $\hat{\Gamma}[(f_i), (A_1)]$ *for any* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *and any* $\mathscr{P}_i$-*measurable functions* $f_i\colon T_i \to X_i$ (*or* $[0, +\infty)$), $i = 1, \ldots, d$.

Let us denote by $\mathrm{pm}(\mathsf{X}S_i, L^{(d)}(X_i; Y))$ the linear space of all operator valued d-polymeasures $\Gamma\colon \mathsf{X}\mathscr{S}_i \to L^{(d)}(X_i; Y)$ which are separately countably additive in the strong operator topology. Clearly $\Gamma \to \|\Gamma\|\,(T_i)$ defines a norm in which $\mathrm{pm}(\mathsf{X}\mathscr{S}_i, L^{(d)}(X_i; Y))$ is complete. By $\mathrm{pmbsv}(\mathsf{X}\mathscr{S}_i, L^{(d)}(X_i; Y))$ (by $\mathrm{pmbv}(\mathsf{X}\mathscr{S}_i, L^{(d)}(X_i; Y))$) we denote the linear subspace of $\mathrm{pm}(\mathsf{X}\mathscr{S}_i, L^{(d)}(X_i; Y))$ whose elements have bounded semivariation (variation). These spaces are also complete in the norms $\Gamma \to \hat{\Gamma}(T_i)$ and $\Gamma \to v(\Gamma, (T_i))$, respectively.

Let us note that there is an isometric isomorphism between the space $\mathrm{pm}(\mathscr{S}_1 \times \ldots \times \mathscr{S}_{d_1}, \mathrm{pm}(\mathscr{S}_{d_1+1} \times \ldots \times \mathscr{S}_d, Y))$, $1 \leqq d_1 < d$, and the subspace of $\mathrm{pm}(\mathscr{S}_1 \times \ldots \times \mathscr{S}_d, Y)$ whose elements in the coordinates $i = 1, \ldots, d_1$ are separately countably additive uniformly with respect to $(A_{d_1+1}, \ldots, A_d) \in \mathscr{S}_{d_1+1} \times \ldots \times \mathscr{S}_d$.

All these facts, together with those listed in Theorem 3 above, easily follow from definitions, or are analogues (with similar proofs) of the case $d = 1$, and so we omit their proofs.

We also note that $\hat{\Gamma}[(f_i), (\cdot)] = \hat{\Gamma}[(|f_i(\,)|), (\cdot)]$.

Theorem 4. *Let* Γ *be as in Definition* 3. *Then its multiple* L_1-*gauge* $\hat{\Gamma}[(\cdot), (A_i)]$, $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ *has the following Fatou property*:

$$\hat{\Gamma}[(f_{i,n}), (A_i)] \nearrow \hat{\Gamma}[(f_i), (A_i)],$$

whenever $f_{i,n}\colon T_i \to [0, +\infty)$, $n = 1, 2, \ldots$ *are* $\mathscr{P}_i$-*measurable, and* $f_{i,n} \nearrow f_i$, $i = 1, \ldots, d$.

Proof. For $d = 1$ we already have this result, see Lemma 1 in Part V = [11]. So let $d > 1$. If we prove the theorem in the special case when $f_{i,n} = f_i$ for each $i = 2, \ldots, d$ and each $n = 1, 2, \ldots$, then the general case will follow easily by using the assertion 6 of Theorem 3. Hence let us consider this special case. Suppose

$\hat{\Gamma}[(f_i), (A_i)] < +\infty$ (the case $= +\infty$ may be treated similarly), and let $\varepsilon > 0$. By the definition of the multiple L_1-gauge there are $u_i \in S(\mathscr{P}_i, X_i)$, $i = 1, \ldots, d$ such that

$$|u_i| \leqq |f_i|, \quad \text{and} \quad \left|\int_{(A_i)} (u_i)\, \mathrm{d}\Gamma\right| \geqq \hat{\Gamma}[(f_i), (A_i)] - \varepsilon .$$

Put

$$\varphi_{1,n}(t_1) = \frac{|f_{1,n}(t_1)| \wedge |u_1(t_1)|}{|u_1(t_1)|} \quad \text{if} \quad u_1(t_1) \neq 0 ,$$

and put $\varphi_{1,n}(t_1) = 0$ if $u_1(t_1) = 0$, $t_1 \in T_1$, $n = 1, 2, \ldots$. Then $\varphi_{1,n}: T_1 \to [0, 1]$, $n = 1, 2, \ldots$ are $\mathscr{P}_i$-measurable and $\varphi_{1,n} \nearrow \chi_{U_1}$, where $U_1 = \{t_1 \in T_1 \; u_1(t_1) \neq 0\} \in \sigma(P_1)$. Further, for $E_1 \in \sigma(\mathscr{P}_1) \cap A_1$ put $\mu_1(E_1) = \int_{(E_1, A_2, \ldots, A_d)} (u_i)\, \mathrm{d}\Gamma$. Then $\mu_1: \sigma(\mathscr{P}_1) \cap A_1 \to Y$ is a countably additive vector measure. Hence $\int_{E_1} \varphi_{1,n}\, \mathrm{d}\mu_1 \to \mu_1(U_1 \cap E_1)$ uniformly with respect to $E_1 \in \sigma(\mathscr{P}_1) \cap A_1$ by Theorem 17 in Part II = [8]. In particular,

$$\left|\int_{A_1} \varphi_{1,n}\, \mathrm{d}\mu_1\right| \to |\mu_1(U_1 \cap A_1)| = |\mu_1(A_1)| = \left|\int_{(A_i)} (u_i)\, \mathrm{d}\Gamma\right| \geqq \hat{\Gamma}[(f_i), (A_i)] - \varepsilon .$$

Now, according to Theorem 9 in Part V = [11],

$$\left|\int_{A_1} \varphi_{1,n}\, \mathrm{d}\mu_1\right| = \left|\int_{(A_i)} (\varphi_{1,n} \,.\, u_1, u_2, \ldots, u_d)\, \mathrm{d}\Gamma\right| \leqq$$
$$\leqq \hat{\Gamma}[(\varphi_{1,n} \,.\, u_1, u_2, \ldots, u_d), (A_i)] \leqq \hat{\Gamma}[(f_{1,n}, f_2, \ldots, f_d), (A_i)] .$$

Hence $\lim_{n\to\infty} \hat{\Gamma}[(f_{1,n}, f_2, \ldots, f_d), (A_i)] = \hat{\Gamma}[(f_i), (A_i)]$, which we wanted to show. The theorem is proved.

Corollary 1. *The set functions $\hat{\Gamma}$, $\|\Gamma\|: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ have the Fatou property.*

Corollary 2. *The set functions $\|\Gamma\|$, $\hat{\Gamma}$, and $\hat{\Gamma}[(f_i), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ ($f_i: T_i \to [0, +\infty)$ $\mathscr{P}_i$-measurable, $i = 1, \ldots, d$) are continuous at $(\emptyset, \ldots, \emptyset)$ if and only if they are exhaustive (for these notions see the proof of Lemma* 1).

Corollary 3. *Let $f_i: T_i \to X_i$ (or to $[0, +\infty)$) be $\mathscr{P}_i$-measurable and let $A_i \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$. Then $\hat{\Gamma}[(f_i), (A_i)] = 0$ if and only if $\hat{\Gamma}(A_i \cap S_{f_i}) = 0$, where $S_{f_i} = \{t_i \in T_i;\ f_i(t_i) \neq 0\}$. ($A_{i,n} = \{t_i \in A_i, |f_i(t_i)| > 1/n\} \nearrow A_i \cap S_{f_i}$).*

Now similarly as the $*$-Theorem in Part I = [7] and Theorem 5 in Part II = [8] we have

Theorem 5. *Let Y contain no isomorphic copy of c_0, for example let Y be a weakly sequentially complete Banach space,* see pp. 160–161 in [2], or [5], *and let Γ be as in Definition* 3. *Then:*

1) If $\hat{\Gamma}$ is finite valued on $\mathsf{X}\mathscr{P}_i$, then $\hat{\Gamma}: \mathsf{X}\mathscr{P}_i \to [0, +\infty)$ is continuous at $(\emptyset, \ldots, \emptyset)$ from above, and
2) If $f_i: T_i \to X_i$ (or to $[0, +\infty)$) are $\mathscr{P}_i$-measurable, $i = 1, \ldots, d$, and $\hat{\Gamma}[(f_i), (A_i)] <$

$< +\infty$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, then $\hat{\Gamma}[(f_i), (T_i)] < +\infty$, and the set function $\hat{\Gamma}[(f_i), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty)$ is continuous at $(\emptyset, \ldots, \emptyset)$.

Using the Saks decomposition of submeasures, see Theorem 8 in [16], we easily obtain the following generalization of Corollary of Theorem 5 in Part II = [8].

Theorem 6. *Let Γ be as in Definition 3, let $f_i: T_i \to X_i$ (or to $[0, +\infty)$) be $\mathscr{P}_i$-measurable, and let $\hat{\Gamma}[(f_i), (\cdot)]: \mathsf{X}\sigma(\mathscr{P}_i) \to [0, +\infty]$ be separately continuous, i.e., let it be a separately subadditive submeasure in the sense of Definition 1 in [16]. Then $\hat{\Gamma}[(f_i), (T_i)] < +\infty$. (If $S_{f_i} = \{t_i \in T_i,\ f_i(t_i) \neq 0\}$, $i = 1, \ldots, d$, then $\hat{\Gamma}[(f_i), (T_i)] = \hat{\Gamma}[(f_i), (S_{fi})]$).*

Corollary. *The set of those operator valued d-polymeasures in* pmbsv$(\mathsf{X}S_i, L^{(d)}(X_i\ Y))$ *whose semivariation $\hat{\Gamma}$ is separately continuous on $\mathsf{X}\mathscr{S}_i$ is a closed linear subspace in* pmbsv$(\mathsf{X}\mathscr{S}_i, L^{(d)}(X_i; Y))$, *containing* pmbv$(\mathsf{X}\mathscr{S}_i, L^{(d)}(X_i; Y))$.

2. UNIFORM POLYMEASURES

First we note that the set of all uniform polymeasures $\gamma: \mathsf{X}\mathscr{S}_i \to Y$, which we denote by upm$(\mathsf{X}\mathscr{S}_i, Y)$, is clearly a closed linear subspace of pm$(\mathsf{X}\mathscr{S}_i, Y)$. We now deduce some pleasant properties of uniform polymeasures. For the proof of the next theorem we need two preparatory results. The next lemma may be proved in just the same way as Lemma 1.

Lemma 2. *Let a family $v_{\mathfrak{d}}: \mathscr{S} \to [0, +\infty)$, $\mathfrak{d} \in \mathfrak{J}$, of subadditive submeasures in the sense of Definition 1 in [16] be uniformly exhaustive, let $A_n \in \mathscr{S}$, $n = 1, 2, \ldots$, and let $A_n \to A$. Then $\lim_{n\to\infty} v_{\mathfrak{d}}(A_n) = v_{\mathfrak{d}}$ uniformly with respect to $\mathfrak{d} \in \mathfrak{J}$.*

The following fact is evident.

Lemma 3. *Let $v_n: \mathscr{R} \to [0, +\infty]$, $n = 1, 2, \ldots$ be monotone and continuous from above at $\emptyset$, and let $\lim_{n\to\infty} v_n(A) = v(A) \in [0, +\infty]$ exist for each $A \in \mathscr{R}$. Then v_n, $n = 1, 2, \ldots$ are uniformly continuous from above at $\emptyset$ if and only if v is continuous from above at $\emptyset$.*

Now, similarly as Theorem 1, using Lemma 2 instead of Lemma 1 and Lemma 3 instead of (VHSN)-Theorem, we can prove

Theorem 7. *Let $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ be a locally uniform vector d-polymeasure, let $A_{i,n} \in \mathscr{S}_i$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$, and let $A_{i,n} \to A_i$ for each $i = 1, \ldots, d$. Then*

$$\lim_{n_1,\ldots,n_d\to\infty} \bar{\gamma}(A_{1,n_1}, \ldots, A_{d,n_d}) = \bar{\gamma}(A_1, \ldots, A_d)\,.$$

In particular, $\bar{\gamma}: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to [0, +\infty)$ *is a separately subadditive submeasure in the sense of Definition* 1 *in* [16].

We now apply this theorem to obtain (for $d = 1$ see [25, § 17, Exercise 3]

Theorem 8. (Exhaustion of locally uniform vector d-polymeasures). *Let* $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ *be a locally uniform vector d-polymeasure. Then there are sets* $Q_i \in \mathscr{S}_i$, $i = 1, \ldots, d$ *such that* $\gamma(A_1, \ldots, A_d) = \gamma(A_1 \cap Q_1, \ldots, A_d \cap Q_d)$ *and* $\bar{\gamma}(A_1, \ldots, A_d) = \bar{\gamma}(A_1 \cap Q_1, \ldots, A_d \cap Q_d)$ *for each* $(A_i) \in \mathsf{X}\mathscr{S}_i$. *In particular,* γ *is a uniform vector d-polymeasure, and* $\bar{\gamma}(T_1, \ldots, T_d) = \bar{\gamma}(Q_1, \ldots, Q_d) < +\infty$.

Proof. For any $(A_i), (Q'_i) \in \mathsf{X}\mathscr{S}_i$, obviously

$$A_1 \times \ldots \times A_d - Q'_1 \times \ldots \times Q'_d =$$

$$= [(A_1 - Q'_1) \cup (A_1 \cap Q'_1)] \times \ldots \times [(A_d - Q'_d) \cup (A_d \cap Q'_d)] - Q'_1 \times \ldots \times Q'_d \subset$$

(2) $\quad - \subset (A_1 - Q'_1) \times \ldots \times (A_d - Q'_d) \cup$

(3) $\quad - \cup Q'_1 \times (A_2 - Q'_2) \times \ldots \times (A_d - Q'_d) \cup \ldots \cup (A_1 - Q'_1) \times \ldots$
$\ldots \times (A_{d-1} - Q'_{d-1}) \times Q'_d \cup$

. . . .

(d + 1) $- \cup Q'_1 \times \ldots \times Q'_{d-1} \times (A_d - Q'_d) \cup \ldots \cup (A_1 - Q'_1) \times Q'_2 \times \ldots \times Q'_d$.

1st step. We take $(Q_i^0) \in \mathsf{X}\mathscr{S}_i$ so that $\bar{\gamma}(Q_i^0) = \bar{\gamma}(T_i)$. (If $(Q_{i,n}) \in \mathsf{X}\mathscr{S}_i$, $n = 1, 2, \ldots$, are such that $\bar{\gamma}(Q_{i,n}) \nearrow \bar{\gamma}(T_i)$, then $Q_i^0 = \bigcup_{n=1}^{\infty} Q_{i,n}$ $i = 1, \ldots, d$, have the reqiured property.)

2nd step. We take $Q_i^1 \in \mathscr{S}_i$, $i = 1, \ldots, d$ so that $Q_i^1 \supset Q_i^0$ for each $i = 1, \ldots, d$, and $\bar{\gamma}(T_1 - Q_1^1, \ldots, T_d - Q_d^1) = 0$. If $a_0 = \bar{\gamma}(T_1 - Q_1^0, \ldots, T_d - Q_d^0) > 0$, then we take $Q_{i,1}^0 \in (T_i - Q_i^0) \cap \mathscr{S}_i$, $i = 1, \ldots, d$, so that $a_0 = \bar{\gamma}(Q_{1,1}^0, \ldots, Q_{d,1}^0)$. If $a_1 = \bar{\gamma}(T_1 - (Q_1^0 \cup Q_{1,1}^0), \ldots, T_d - (Q_d^0 \cup Q_{d,1}^0)) > 0$, then we take $Q_{i,2}^0 \in (T_i - (Q_i^0 \cup Q_{i,1}^0)) \cap \mathscr{S}_i$, $i = 1, \ldots, d$, so that $a_1 = \bar{\gamma}(Q_{1,2}^0, \ldots, Q_{d,2}^0)$. Continuing in this way we either arrive at a $k_0 \in N$ such that $a_{k_0} = 0$, or $a_k > 0$ for each $k = 0, 1, 2, \ldots$. In the first case the sets $Q_i^1 = \bigcup_{k=0}^{k_0} Q_{i,k}^0$, where $Q_{i,0}^0 = Q_i^0$, $i = 1, \ldots, d$, have the required properties. In the second case $a_k \to 0$ by Theorem 7, hence the sets $Q_i^1 = \bigcup_{k=0}^{\infty} Q_{i,k}^0$, $i = 1, \ldots, d$, have the required properties.

3rd step. We first apply the analogue of the 2nd step for the first term in (3) with fixed $Q'_1 = Q_1^1$ to obtain $(Q_2^{2,1}, \ldots, Q_d^{2,1}) \in \mathscr{S}_2 \times \ldots \times \mathscr{S}_d$ such that $Q_i^{2,1} \supset Q_i^1$, $i = 2, \ldots, d$, and $(*)$ $- \bar{\gamma}(Q_1^1, T_2 - Q_2^{2,1}, \ldots, T_d - Q_d^{2,1}) = 0$. It is important to observe that for any sets $Q'_1 \in \mathscr{S}_1$, $Q'_1 \supset Q_1^1$, $Q'_2 \in \mathscr{S}_2$, $Q'_2 \supset Q_2^{2,1}, \ldots, Q'_d \in \mathscr{S}_d$, $Q'_d \supset Q_d^{2,1}$ we have $\bar{\gamma}(Q'_1, T_2 - Q'_2, \ldots, T_d - Q'_d) = 0$ $(0 \leq \bar{\gamma}(Q'_1, T_2 - Q'_2, \ldots, T_d - Q'_d) \leq \bar{\gamma}(Q'_1 - Q_1^1, T_2 - Q'_2, \ldots, T_d - Q'_d) + \bar{\gamma}(Q_1^1, T_2 - Q'_2, \ldots, T_d - Q'_d) = 0$ (by the 2nd step) $+ 0$ (by $(*)$ and the monotonocity of $\bar{\gamma}$)). Having $(Q_1^1, Q_2^{2,1}, \ldots, Q_d^{2,1})$ we apply the same argument to the second term in (3), and obtain $(Q_1^{2,1}, Q_2^{2,1}, Q_3^{2,2}, \ldots, Q_d^{2,2})$. Proceeding in this way we arrive at the last term in (3) with $(Q_1^{2,d-1}, \ldots, Q_d^{2,d-1})$. Hence, if $Q_i^2 = Q_i^{2,d-1}$, $i = 1, \ldots, d$, then for $Q_i' = Q_i^2$, $i = 1, \ldots, d$, the value of $\bar{\gamma}$ on all terms in (3) is equal to 0 for any $(A_i) \in \in \mathsf{X}\mathscr{S}_i$.

Further steps (if $d > 2$). Similarly as above we obtain $(Q_1^3, \ldots, Q_d^3)$ such that $Q_i^3 \supset Q_i^2$, $i = 1, \ldots, d$, and $\bar{\gamma}$ on all terms in (2), (3) and (4) with $Q_i' = Q_i^3$, $i = 1, \ldots \ldots, d$, is equal to 0 for any $(A_i) \in \mathsf{X}\mathscr{S}_i$. Continuing in this way we finally obtain sets $Q_i^{d-1} \supset Q_i^{d-2}$, $i = 1, \ldots, d$, such that for $Q_i' = Q_i^{d-1}$, $i = 1, \ldots, d$, $\bar{\gamma}$ at all terms on the right hand side, i.e., at all terms in (2), …, (d + 1), is equal to 0 for any $(A_i) \in \mathsf{X}\mathscr{S}_i$.

The theorem is proved.

Corollary. *Let each X_i, $i = 1, \ldots, d$, be a separable Banach space and let Γ: $\mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ be an operator valued d-polymeasure such that $\Gamma(\ldots)(x_1, \ldots, x_d)$: $\mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ is a uniform (equivalently, a locally uniform) vector d-polymeasure for each $(x_i) \in \mathsf{X}X_i$. Then there are $Q_i \in \mathscr{S}_i$, $i = = 1, \ldots, d$, such that $\Gamma(A_1, \ldots, A_d) = \Gamma(A_1 \cap Q_1, \ldots, A_d \cap Q_d)$ and $\bar{\Gamma}(A_1, \ldots, A_d) = = \bar{\Gamma}(A_1 \cap Q_1, \ldots, A_d \cap Q_d)$ for each $(A_i) \in \mathsf{X}S_i$.*

Proof. Let $x_{i,n}$, $n = 1, 2, \ldots$ be a dense sequence in X_i, $i = 1, \ldots, d$. Since the family $(x_{1,n_1}, \ldots, x_{d,n_d})$, $n_1, \ldots, n_d = 1, 2, \ldots$ is countable, we may write it as a sequence $(x'_{1,n}, \ldots, x'_{d,n})$, $n = 1, 2, \ldots$. Now for each uniform vector d-polymeasure $\Gamma(\ldots)(x'_{1,n}, \ldots, x'_{d,n})$: $\mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$, $n = 1, 2, \ldots$ take $(Q_{1,n}, \ldots, Q_{d,n})$ according to the theorem, and put $Q_i = \bigcup_{n=1}^{\infty} Q_{i,n}$, $i = 1, \ldots, d$. Then clearly (Q_i) has the required properties.

It remains an open problem whether the preceding theorem remains valid for arbitrary vector d-polymeasures.

Theorem 9. *Let γ: $\mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ be a uniform vector d-polymeasure, and let $1 \leqq d_1 < d$. Let further $A_{i,n} \in \mathscr{S}_i$, $n = 1, 2, \ldots$, $i = 1, \ldots, d_1$, and let $A_{i,n} \to \to A_i$ for $i = 1, \ldots, d_1$. Then:*

1) $\bar{\gamma}(A_{1,n} \Delta A_1, T_2, \ldots, T_d) \to 0, \ldots, \bar{\gamma}(T_1, \ldots, T_{d_1-1}, A_{d_1,n} \Delta A_{d_1}, T_{d_1+1}, \ldots, T_d) \to 0$,

2) $$\lim_{n_1, \ldots, n_{d1}, \to \infty} \gamma(A_{1,n_1}, \ldots, A_{d_1,n_{d1}}, A_{d_1+1}, \ldots, A_d) = \gamma(A_1, \ldots, A_d)$$
uniformly with respect to $(A_{d_1+1}, \ldots, A_d) \in \mathscr{S}_{d_1+1} \times \ldots \times \mathscr{S}_d$, and

3) *the analogue of* 2) *holds for $\bar{\gamma}$.*

Proof. 1) is an immediate consequence of Theorems 7 and 8.

2) First we note that for an additive set function v: $\mathscr{R} \to Y$ we have the inequalities: $|v(A) - v(B)| = |v(A - B) - v(B - A)| \leqq |v(A - B)| + |v(B - A)| \leqq \bar{v}(A - B) + + \bar{v}(B - A) \leqq 2\bar{v}(A \Delta B)$. Hence

$$|\gamma(A_{1,n_1}, \ldots, A_{d_1,n_{d1}}, A_{d_1+1}, \ldots, A_d) - \gamma(A_1, \ldots, A_d)| \leqq$$
$$\leqq |\gamma(A_{1,n_1}, \ldots, A_{d_1,n_{d1}}, A_{d_1+1}, \ldots, A_d) - \gamma(A_1, A_{2,n_2}, \ldots)| +$$

$$+\ |\gamma(A_1, A_{2,n_2}, \ldots) - \gamma(A_1, A_2, \ldots)| + \ldots$$

$$\ldots \leqq 2\bar{\gamma}(A_{1,n_1} \Delta A_1, T_2, \ldots, T_d) + 2\bar{\gamma}(T_1, A_{2,n_2} \Delta A_2, T_3, \ldots, T_d) + \ldots .$$

Thus 2) follows from 1).

3) For a monotone and subadditive set function $\nu: \mathscr{R} \to [0, +\infty)$ we have the inequality: $|\nu(A) - \nu(B)| \leqq \nu(A \Delta B)$. Using this inequality we obtain 3) as a consequence of 1).

From Theorem 8 and Theorem IV.9.2 in [22], or Theorem 3.10 in [24], or Theorem 7 in [17] we immediately obtain

Theorem 10. *Let $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ be a uniform vector d-polymeasure. Then there are countably additive measures $\lambda_i: \mathscr{S}_i \to [0, +\infty)$, $i = 1, \ldots, d$, such that*

1) *$\lambda_1(A_1) \leqq \bar{\gamma}(A_1, T_2, \ldots, T_d)$ for each $A_1 \in \mathscr{S}_1$, and the vector measures $\gamma(\cdot, A_2, \ldots, A_d): \mathscr{S}_1 \to Y$, $(A_2, \ldots, A_d) \in \mathscr{S}_2 \times \ldots \times \mathscr{S}_d$, are uniformly absolutely continuous with respect to λ_1*

. . . .

d) *$\lambda_d(A_d) \leqq \bar{\gamma}(T_1, \ldots, T_{d-1}, A_d)$ for each $A_d \in \mathscr{S}_d$, and the vector measures $\gamma(A_1, \ldots, A_{d-1}, \cdot): \mathscr{S}_d \to Y$, $(A_1, \ldots, A_{d-1}) \in \mathscr{S}_1 \times \ldots \times \mathscr{S}_{d-1}$, are uniformly absolutely continuous with respect to λ_d.*

3. EXISTENCE OF CONTROL POLYMEASURES

Definition 4. We say that an operator valued d-polymeasure $\Gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ separately countably additive in the strong operator topology has a control d-polymeasure, if there are countably additive measures $\lambda_i: \mathscr{P}_i \to [0, +\infty)$, $i = 1, \ldots, d$, such that Γ is absolutely continuous with respect to $\lambda_1 \times \ldots \times \lambda_d$ on $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$.

Note that in this definition we do not require the absolute continuity of $\lambda_1 \times \ldots \times \lambda_d$ with respect to Γ. According to Theorem 10 each uniform vector d-polymeasure $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ has a control d-polymeasure. The solution of the following problem is of great importance for the theory of integration with respect to operator valued polymeasures, with which we will start in Part IX.

Problem. (Control polymeasure problem.) *Does every vector d-polymeasure $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ have locally control d-polymeasures?*

In this section we give important partial positive results concerning this problem.

Lemma 4. *Let $\gamma_k: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$, $k = 1, 2$, be two vector d-polymeasures, and let $(A_1, \ldots, A_d) \in \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d)$. Then there are $(A_{1,n}, \ldots$*

$\ldots, A_{d,n}) \in \mathscr{R}_1 \times \ldots \times \mathscr{R}_d$, $n = 1, 2, \ldots$ *such that* $\gamma_k(A_1, \ldots, A_k) = \lim_{n\to\infty} \gamma_k(A_{1,n}, \ldots$ $\ldots, A_{d,n})$ *for both* $k = 1, 2$. *The analogue holds if* $\sigma(\mathscr{R}_i)$ *is replaced by* $\delta(\mathscr{R}_i)$.

Proof. Since $\gamma_k(\cdot, A_2, \ldots, A_d): \sigma(\mathscr{R}_1) \to Y$, $k = 1, 2$, are countably additive vector measures, they have countably additive control measures $\lambda_{1,k}: \sigma(\mathscr{R}_1) \to$ $\to [0, +\infty)$, see the end of Introduction in Part III = [9]. Put $\lambda_1 = \lambda_{1,1} + \lambda_{1,2}$. By the well known result, see Theorem D in § 13 in [25], there is a sequence $A_{1,n_1} \in \mathscr{R}_1$, $n_1 = 1, 2, \ldots$ such that $\lambda_1(A_1 \Delta A_{1,n_1}) \to 0$. Hence $\gamma_k(A_1, \ldots, A_d) =$ $= \lim_{n_1\to\infty} \gamma_k(A_{1,n_1}, A_2, \ldots, A_d)$ for both $k = 1, 2$ by the absolute continuity of $\gamma_k(\cdot, A_2, \ldots, A_d)$ with respect to λ_1. Similarly, let λ_{2,k,n_1} be a control measure for the vector measure $\gamma_k(A_{1,n_1}, \cdot, A_3, \ldots, A_d): \sigma(\mathscr{R}_2) \to Y$, $k = 1, 2$, and $n_1 = 1, 2, \ldots$, and put

$$\lambda_2 = \sum_{n_1=1}^{\infty} \frac{1}{2^{n_1}} \frac{\lambda_{2,1,n_1} + \lambda_{2,2,n_1}}{1 + (\lambda_{2,1,n_1} + \lambda_{2,2,n_1})(T_2)}.$$

Then there is a sequence $A_{2,n_2} \in \mathscr{R}_2$, $n_2 = 1, 2, \ldots$ such that $\gamma_k(A_{1,n_1}, A_2, \ldots, A_d) =$ $= \lim_{n_2\to\infty} \gamma_k(A_{1,n_1}, A_{2,n_2}, A_3, \ldots, A_d)$ for both $k = 1, 2$ and each $n_1 = 1, 2, \ldots$. Continuing in this way we finally arrive at a sequence $A_{d,n_d} \in \mathscr{R}_d$, $n_d = 1, 2, \ldots$ such that $\gamma_k(A_{1,n_1}, \ldots, A_{d-1,n_{d-1}}, A_d) = \lim_{n_d\to\infty} \gamma_k(A_{1,n_1}, \ldots, A_{d,n_d})$ for both $k = 1, 2$ and each $n_1, \ldots, n_{d-1} = 1, 2, \ldots$. Hence $\gamma_k(A_1, \ldots, A_d) = \lim_{n_1\to\infty} \lim_{n_2\to\infty} \ldots \lim_{n_d\to\infty} \gamma_k(A_{1,n_1}, \ldots, A_{d,n_d})$ for both $k = 1, 2$, and the assertion of the lemma is evident.

Finally, the case of $\delta(\mathscr{R}_i)$ is a corollary of the case of $\sigma(\mathscr{R}_i)$ just proved, since obviously $\delta(\mathscr{R}_i) = \bigcup_{R_i \in \mathscr{R}_i} R_i \cap \sigma(\mathscr{R}_i)$.

Corollary. *If two vector d-polymeasures* $\gamma_k: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$ (*or* γ_k: $\delta(\mathscr{R}_1) \times \ldots \times \delta(\mathscr{R}_d) \to Y$), $k = 1, 2$, *are equal on* $\mathscr{R}_1 \times \ldots \times \mathscr{R}_d$, *then they are identical.*

Theorem 11. *Let each* $\mathscr{S}_i$, $i = 1, \ldots, d$, *be generated by a countable family of sets. Then every vector d-polymeasure* $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ *has a control d-polymeasure.*

Proof. Without loss of generality, see Theorem C in § 5 in [25], we may suppose that each $\mathscr{S}_i$ is generated by a countable ring $\mathscr{R}_i$, $i = 1, \ldots, d$. Put $R_{i,\sigma} = \bigcup_{R_i \in \mathscr{R}_i} R_i \in$ $\in \mathscr{R}_{i,\sigma}$, $i = 1, \ldots, d$. Then clearly $\mathscr{S}_i = \sigma(\mathscr{R}_i) = R_{i,\sigma} \cap \sigma(\mathscr{R}_i)$, $i = 1, \ldots, d$. Since $\gamma(\cdot, R_2, \ldots, R_d): \mathscr{S}_1 \to Y$ is a countably additive vector measure for each $(R_2, \ldots, R_d) \in$ $\in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$, it has a control measure $\lambda_{1,(R_2,\ldots,R_d)}: \mathscr{S}_1 \to [0, +\infty)$. Since $\mathscr{R}_2 \times \ldots \times \mathscr{R}_d$ is a countable family, we obtain a corresponding countable family $\lambda_{1,n}$, $n = 1, 2, \ldots$ of control measures. For $A_1 \in \mathscr{S}_1$ define

$$\lambda_1(A_1) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_{1,n}(A_1)}{1 + \lambda_{1,n}(R_{1,\sigma})}.$$

If now $N_1 \in \mathscr{S}_1$ and $\lambda_1(N_1) = 0$, then clearly $\gamma(N_1, R_2, \ldots, R_d) = 0$ for each $(R_2, \ldots, R_d) \in \mathscr{R}_2 \times \ldots \times \mathscr{R}_d$. Hence $\gamma(N_1, A_2, \ldots, A_d) = 0$ for each $(A_2, \ldots, A_d) \in \mathscr{S}_2 \times \ldots \times \mathscr{S}_d$ by Lemma 4. By symmetry in the coordinates there are similar λ_i's for $i = 2, \ldots, d$. Now clearly $\lambda_1 \times \ldots \times \lambda_d$ is a control d-polymeasure for γ.

Corollary. *Let each $\mathscr{P}_i$, $i = 1, \ldots, d$, be generated by a countable family of sets. Then each vector d-polymeasure $\gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to Y$ has a control d-polymeasure.*

Proof. Without loss of generality we may suppose that $\mathscr{P}_i$ is generated by a countable ring $\mathscr{R}_i$, $i = 1, \ldots, d$. For each i put $R_{i,\sigma} = \bigcup_{R_i \in \mathscr{R}_i} R_i$, and take $R_{i,n} \in \mathscr{R}_i$, $n = 1, 2, \ldots$ so that $R_{i,n} \nearrow R_{i,\sigma}$. Clearly $\mathscr{P}_i = \delta(\mathscr{R}_i) = \bigcup_{n=1}^{\infty} R_{i,n} \cap \sigma(\mathscr{R}_i)$. According to the previous theorem, for the restrictions $\gamma: (R_{1,n} \cap \sigma(\mathscr{R}_1) \times \ldots \times (R_{d,n} \cap \sigma(\mathscr{R}_d)) \to Y$, $n = 1, 2, \ldots$ there are control d-polymeasures $\lambda_{1,n} \times \ldots \times \lambda_{d,n}$. For $i = 1, \ldots, d$ and $A_i \in \mathscr{P}_i$ put

$$\lambda_i(A_i) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_{i,n}(R_{i,n} \cap A_i)}{1 + \lambda_{i,n}(R_{i,n})}.$$

Then evidently $\lambda_1 \times \ldots \times \lambda_d: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to [0, 1]$ is a control d-polymeasure for γ.

We now give a few applications of control polymeasures.

Theorem 12. *Let $\gamma: \mathscr{S}_1 \times \ldots \times \mathscr{S}_d \to Y$ be a vector d-polymeasure, and let $\lambda_1 \times \ldots \times \lambda_d$ be its control d-polymeasure. Let further $A_i, A_{i,n} \in \mathscr{S}_i$, $i = 1, \ldots, d$ and $n = 1, 2, \ldots$, and let $\lambda_i(A_i \Delta A_{i,n}) \to 0$ as $n \to \infty$ for each $i = 1, \ldots, d$. Then*

$$\lim_{n_1, \ldots, n_d \to \infty} \gamma(A_{1,n_1}, \ldots, A_{d,n_d}) = \gamma(A_1, \ldots, A_d).$$

Proof. Suppose the contrary. Then there is an $\varepsilon > 0$ and integers $n_{i,j}$, $i = 1, \ldots, d$ and $j = 1, 2, \ldots$ such that $n_{i,j} < n_{i,j+1}$ for each $i = 1, \ldots, d$ and $j = 1, 2, \ldots$, and $|\gamma(A_{1,n_{1,j}}, \ldots, A_{d,n_{d,j}}) - \gamma(A_1, \ldots, A_d)| > \varepsilon$ for each $j = 1, 2, \ldots$. Since $\lambda_i(A_i \Delta A_{j,n_{i,j}}) = \int_{T_i} |\chi_{A_i} - \chi_{A_{i,n_{i,j}}}| \, d\lambda_i \to 0$ as $j \to \infty$ for each $i = 1, \ldots, d$, there is a subsequence $\{j_k\} \subset \{j\}$ and sets $N_i \in \mathscr{S}_i$, $i = 1, \ldots, d$, such that $\lambda_i(N_i) = 0$ and $\chi_{A_{i,n_{i,j_k}}}(t_i) \to \chi_{A_i}(t_i)$ for $t_i \in T_i - N_i$ for each $i = 1, \ldots, d$. But then $\gamma(A_{1,n_{1,j_k}}, \ldots, A_{d,n_{d,j_k}}) = \gamma((A_{1,n_{1,j_k}} - N_1), \ldots, A_{d,n_{d,j_k}}) \to \gamma((A_1 - N_1), \ldots, (A_d - N_d)) = \gamma(A_1, \ldots, A_d)$ by the absolute continuity of γ with respect to $\lambda_1 \times \ldots \times \lambda_d$ and by Theorem 1, a contradiction. The theorem is proved.

Theorem 13. *Let $\gamma_k: \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d) \to Y$, $k = 1, 2, \ldots$ be vector d-polymeasures, and let $(A_1, \ldots, A_d) \in \sigma(\mathscr{R}_1) \times \ldots \times \sigma(\mathscr{R}_d)$. Then there are $A_{i,n} \in \mathscr{R}_i$, $i = 1, \ldots, d$ and $n = 1, 2, \ldots$, such that*

$$\lim_{n_1, \ldots, n_d \to \infty} \gamma_k(A_{1,n_1}, \ldots, A_{d,n_d}) = \gamma_k(A_1, \ldots, A_d)$$

for each $k = 1, 2, \ldots$. The analogue holds if $\sigma(\mathscr{R}_i)$ is replaced by $\delta(\mathscr{R}_i)$.

Proof. According to Theorem D in § 5 in [25] for each A_i, $i = 1, \ldots, d$, there is a countable family of sets $\mathscr{R}_i' \subset \mathscr{R}_i$ such that $A_i \in \sigma(\mathscr{R}_i')$. Hence using also Theorem C in § 5 in [25] we may suppose that each $\mathscr{R}_i$, $i = 1, \ldots, d$, is countable. Now by Theorem 11 for each of the vector d-polymeasures γ_k, $k = 1, 2, \ldots$ there are control d-polymeasures $\lambda_{1,k} \times \ldots \times \lambda_{d,k}$. If we put

$$\lambda_i = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\lambda_{i,k}}{1 + \lambda_{i,k}(T_i)}, \quad i = 1, \ldots, d,$$

and use Theorem D in § 13 in [25] and Theorem 12, we obtain the assertion of the theorem. The case of $\delta(\mathscr{R}_i)$ follows similarly as in Lemma 4.

Using the well known properties of regular Borel measures on locally compact Hasudorff spaces, see [25] and [19], similarly as Theorem 13 we can prove

Theorem 14. *Let T_i, $i = 1, \ldots, d$, be locally compact Hausdorff topological spaces, and let $\gamma_k: \sigma(\mathscr{C}_1) \times \ldots \times \sigma(\mathscr{C}_d) \to Y$, $k = 1, 2, \ldots$ be separately regular Borel vector d-polymeasures. Then for each $(A_1, \ldots, A_d) \in \sigma(\mathscr{C}_1) \times \ldots \times \sigma(\mathscr{C}_d)$ there are compact G_δ sets $C_{0,i,n} \in \mathscr{C}_{0,i}$, $i = 1, \ldots, d$, $n = 1, 2, \ldots$ such that*

$$\lim_{n_1, \ldots, n_d \to \infty} \gamma_k(C_{0,1,n_1}, \ldots, C_{0,d,n_d}) = \gamma_k(A_1, \ldots, A_d)$$

for each $k = 1, 2, \ldots$.

Corollary. *If two separately regular Borel vector d-polymeasures $\gamma_k: \sigma(\mathscr{C}_1) \times \ldots \times \sigma(\mathscr{C}_d) \to Y$, $k = 1, 2$, are equal on the products of compact G_δ sets $\mathscr{C}_{0,1} \times \ldots \times \mathscr{C}_{0,d}$, then they are identical.*

Let us remark that in the preceding theorem and its corollary we may replace $\sigma(\mathscr{C}_i)$ by $\delta(\mathscr{C}_i)$, but also $\mathscr{C}_i$ by $\mathscr{U}_i$, where $\mathscr{U}_i$ denotes the family of all open subsets of T_i (then $\sigma(\mathscr{U}_i)$ is the σ-algebra of all weakly Borel subsets of T_i (recently often called Borel), see [19]).

Let us now give further results on existence of control polymeasures. We shall need

Definition 5. We say that a set function $v: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to [0, +\infty]$ is *σ-finite*, if there are δ-rings $\mathscr{P}_i' \subset \mathscr{P}_i$, $i = 1, \ldots, d$, such that $\sigma(\mathscr{P}_i') \supset \mathscr{P}_i$ for each i, and the restriction of v to $\mathscr{P}_1' \times \ldots \times \mathscr{P}_d'$ is finite valued.

The next theorem may be proved in a way similar to the proof of Corollary of Theorem 11.

Theorem 15. *Let $\Gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ be an operator valued d-polymeasure separately countably additive in the strong operator topology, let $\mathscr{P}_i' \subset \mathscr{P}_i$, $i = 1, \ldots, d$, be δ-rings such that $\sigma(\mathscr{P}_i') \supset \mathscr{P}_i$ for each i, and let Γ have locally control d-polymeasures on $\mathscr{P}_1' \times \ldots \times \mathscr{P}_d'$. Then Γ has locally control d-polymeasures on $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$.*

The following corollary is evident.

Corollary. *Let $\gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to Y$ be a vector d-polymeasure and let its variation $v(\gamma,(\ldots))$ be σ-finite on $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$. Then γ has locally control d-polymeasures on $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$.*

Let us note that the previous theorem and the following two theorems are generalizations of the assertions of Theorem 13 in Part III = [9].

Theorem 16. *Let Y have a countable norming set, for example let Y be a separable or be a dual of separable Banach space, see Definition 2.8.1 in [26], let $\gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to Y$ be a vector d-polymeasure, and let the scalar d-polymeasure $y^*\gamma$ have locally control d-polymeasures for each $y^* \in Y^*$. Then γ has locally control d-polymeasures.*

Proof. Let $(A_1, \ldots, A_d) \in \mathscr{P}_1 \times \ldots \times \mathscr{P}_d$, and let $y_n^* \in Y^*$, $n = 1, 2, \ldots$ be a norming sequence for Y. By assumption for each $n = 1, 2, \ldots$ there is a control d-polymeasure $\lambda_{1,n} \times \ldots \times \lambda_{d,n}: (A_1 \cap \mathscr{P}_1) \times \ldots \times (A_d \cap \mathscr{P}_d) \to [0, +\infty)$ for the restriction $y_n^*\gamma: (A_1 \cap \mathscr{P}_1) \times \ldots \times (A_d \cap \mathscr{P}_d) \to K$. For $i = 1, \ldots, d$ and $E_i \in A_i \cap \mathscr{P}_i$ put

$$\lambda_i(E_i) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda_{i,n}(E_i)}{1 + \lambda_{i,n}(A_i)}.$$

If $N_1 \in A_1 \cap \mathscr{P}_1$ and $\lambda_1(N_1) = 0$, then clearly $y_n^*\gamma(N_1, E_2, \ldots, E_d) = 0$ for each $n = 1, 2, \ldots$ and each $(E_2, \ldots, E_d) \in (A_2 \cap \mathscr{P}_2) \times \ldots \times (A_d \cap \mathscr{P}_d)$. Hence $|\gamma(N_1, E_2, \ldots, E_d)| = \sup_n |y_n^*\gamma(N_1, E_2, \ldots, E_d)| = 0$ for each $(E_2, \ldots, E_d) \in (A_2 \cap \mathscr{P}_2) \times \ldots \times (A_d \cap \mathscr{P}_d)$. By symmetry in the coordinates, analogues hold for $i = 2, \ldots, d$. Thus $\lambda_1 \times \ldots \times \lambda_d: (A_1 \cap \mathscr{P}_1) \times \ldots \times (A_d \cap \mathscr{P}_d) \to [0, 1]$ is a control d-polymeasure for the restriction $\gamma: (A_1 \cap \mathscr{P}_1) \times \ldots \times (A_d \cap \mathscr{P}_d) \to Y$. Since $(A_i) \in \mathsf{X}\, \mathscr{P}_i$ were arbitrary, the theorem is proved.

Using the result of K. Ylinen, see (Y) at the beginning, we immediately have

Corollary 1. *Let Y have a countable norming set. Then any vector bimeasure $\beta: \mathscr{S}_1 \times \mathscr{S}_2 \to Y$ has a control bimeasure.*

Using Corollary of Theorem 15 we have

Corollary 2. *Let Y have a countable norming set, let $\gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to Y$ be a vector d-polymeasure and let $v(y^*\gamma, (\ldots))$ be σ-finite on $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$ for each $y^* \in Y^*$. Then γ has locally control d-polymeasures.*

Similarly as the preceding theorem one can prove

Theorem 17. *Let each X_i, $i = 1, \ldots, d$, be a separable Banach space, let $\Gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ be an operator valued d-polymeasure separately countably additive in the strong operator topology, and suppose that $\Gamma(\ldots)(x_1, \ldots, x_d): \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to Y$ has a control d-polymeasure for each $(x_1, \ldots, x_d) \in X_1 \times \ldots \times X_d$. Then Γ has a control d-polymeasure.*

From this theorem and from the corollaries of Theorems 11 and 16 we immediately have

Theorem 18. a) *Let each $\mathscr{P}_i$, $i = 1, \ldots, d$, be generated by a countable family of sets, and let each X_i, $i = 1, \ldots, d$, be a separable Banach space. Then any operator valued d-polymeasure $\Gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ separately countably additive in the strong operator topology has a control d-polymeasure.*

b) *Let X_1 and X_2 be separable Banach spaces and let Y have a countable norming set. Then every operator valued bimeasure $B: \mathscr{S}_1 \times \mathscr{S}_2 \to L^{(2)}(X_1, X_2; Y)$ separately countably additive in the strong operator topology has a control bimeasure.*

Using the Fatou property of the supremation $\bar{\Gamma}$, see Corollary 1 of Theorem 2, similarly as Corollary of Theorem 11 we obtain our final.

Theorem 19. *Let $\Gamma: \mathscr{P}_1 \times \ldots \times \mathscr{P}_d \to L^{(d)}(X_1, \ldots, X_d; Y)$ be an operator valued d-polymeasure separately countably additive in the strong operator topology and let Γ have locally control polymeasures on $\mathscr{P}_1 \times \ldots \times \mathscr{P}_d$. Then its supremation $\bar{\Gamma}$ (equivalently: scalar semivariation $\|\Gamma\|$, semivariation $\hat{\Gamma}$, variation $v(\Gamma, (\ldots))$) has locally control d-polymeasures on $\sigma(\mathscr{P}_1) \times \ldots \sigma(\mathscr{P}_d)$ in the sense of Definition* 4.

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