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LOCALLY ONE-TO-ONE MAPPINGS ON GRAPHS

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1. Preliminaries. All topological spaces considered in this paper are assumed to be metric, and all mappings are continuous. Exceptions are made at a few places (e.g. in § 3), where some other assumptions (which are explicitly stated there) are imposed on the spaces.

A continuum is understood as a compact connected metric space. A continuum that is locally connected and contains no simple closed curve is called a *dendrite*. By a finite dendrite we mean a dendrite having finitely many end points. An arc A, with end points a and b, contained in a continuum X, is said to be *free* provided $A \setminus \{a, b\}$ is an open subset of X. A connected set is called a *graph* provided it is the union of a finite sets of points, called *vertices*, and of a finite number of free arcs, called *edges*, so that both end points of each edge are vertices. All graphs considered here are assumed to be equipped with a convex metric.

We denote by $E(X)$ the set of all end points and by $R(X)$ the set of all ramification points of a space X, i.e., the sets of points of order 1 and of order greater than 2 in the sense of the Menger-Urysohn theory of order (see e.g. [3], Chapetr III, 1, p. 99). Observe that for every graph these sets are finite.

A continuum X is said to be the *union of finitely many arcs* if there are arcs $A_1, A_2, \ldots, A_n$ in X such that $X = \bigcup\{A_i: i \in \{1, 2, \ldots, n\}\}$. Note that the union of finitely many arcs need not be a graph: take the union of two arcs whose intersection has infinitely many components. It may even look as the curve in the figure (see § 5).

2. Locally one-to-one mappings. Let two topological (not necessarily metric) spaces X and Y be given. A mapping $f: X \to Y$ is said to be *locally one-to-one* provided that each point $x \in X$ has an open neighborhood $U \subset X$ such that the partial mapping $f \mid U: U \to f(U) \subset Y$ is one-to-one.

The following statements are quite elementary. Their proofs are therefore omitted.

Statement 1. *Let a locally one-to-one mapping $f: X \to Y$ of a continuum X onto Y be given and, for a point $x \in X$, let U denote an open neighborhood of x in X as in* *the definition of the locally one-to-one mapping. Then for every closed set $A \subset U$ the partial mapping $f \mid A: A \to f(A) \subset Y$ is a homeomorphism.*

Statement 2. *Let two mappings $f: X \to Y$ and $g: Y \to Z$ be locally one-to-one. Then their composite $gf: X \to Z$ is locally one-to-one as well.*

Proposition 1. *Every image of a graph under a locally one-to-one mapping is the union of finitely many arcs.*

Indeed, observe that a graph can be expressed as a finite union of such arcs that each of them is contained in an open set from the definition of a locally one-to-one mapping. Using Statement 1 we get the conclusion.

The next proposition is a converse to Proposition 1.

Proposition 2. *If a continuum is the union of finitely many arcs, then it is the image of a finite dendrite under a locally one-to-one mapping.*

Proof. Let a continuum be the union of a finite collection of arcs $A_1, A_2, \ldots, A_n$. Obviously we can assume that no member of this collection is contained in the union of the other ones and that $A_{j+1} \cap (A_1 \cup \ldots \cup A_j) \neq \emptyset$ for every $j \in \{1, 2, \ldots, n-1\}$. Put $Y_n = \bigcup\{A_j: j \in \{1, 2, \ldots, n\}\}$. We shall prove the proposition by induction with respect to n. More precisely, we shall prove that

(*) *for every natural number n there are a finite dendrite D_n and a locally one-to-one mapping $f_n: D_n \to Y_n$ of D_n onto Y_n.*

If $n = 1$, the assertion obviously holds. So take a number $n > 1$ and let $f_{n-1}: D_{n-1} \to Y_{n-1}$ be a locally one-to-one mapping of a finite dendrite D_{n-1} onto the union Y_{n-1} of $n - 1$ arcs A_j contained in Y_n. Consider two cases.

Case 1. The intersection $A_n \cap Y_{n-1}$ is connected. Thus this intersection is an arc, say y_2y_3, and we can label the end points of A_n as y_1 and y_4 in such a way that $y_2 \in y_1y_3$ (and, consequently, $y_3 \in y_2y_4$). Thus we have $Y_n = Y_{n-1} \cup y_1y_2 \cup y_3y_4$. Take two points x_2 and x_3 in the dendrite D_{n-1} such that $f_{n-1}(x_2) = y_2$ and $f_{n-1}(x_3) = y_3$. Define D'_n as the one-point union of D_{n-1} and of an arc $x_1x'_2$ with x_2 and x'_2 identified, and let D_n be the one-point union of D'_n and of an arc x'_3x_4 with x_3 and x'_3 identified. Extend f_{n-1} to a locally one-to-one mapping $f_n: D_n \to Y_n$ in a natural way: $f_n \mid D_{n-1} = f_{n-1}$; $f_n \mid x_1x_2$ and $f_n \mid x_3x_4$ are homeomorphisms onto y_1y_2 and y_3y_4, respectively.

Case 2. The intersection $A_n \cap Y_{n-1}$ is not connected. Then $Y_n = Y_{n-1} \cup A_n$ contains a simple closed curve C such that $C \setminus Y_{n-1} \neq \emptyset$. Take a point $c \in C \setminus Y_{n-1} \subset A_n$ and let y_0 be the first (with respect to a fixed circular order on C) point of C which is after c and which belongs to Y_{n-1}. Next, take a point $y_2 \in C \setminus A_n$ and let y_3 be the last (with respect to the same order) point of C which is before y_2 and which belongs to A_n (y_3 may coincide with y_0). Denote by y_1 and y_4 the end points of A_n. Take a point $x_0 \in D_{n-1}$ such that $f_{n-1}(x_0) = y_0$ and define D_n as the one-point union of D_{n-1} and of a simple triod with the center x_3 and with end points x'_0, x_1 and x_4, where x_0 and x'_0 are identified. Pick an interior point x_2 of the arc x_0x_3

and extend f_{n-1} to a locally one-to-one mapping f_n: $D_n \to Y_n$ as follows: $f_n \mid D_{n-1} = f_{n-1}$; $f_n \mid x_0x_2$, $f_n \mid x_2x_3$ and $f_n \mid x_1x_3x_4$ are homeomorphisms onto the arcs y_0cy_2, y_2y_3 and A_n, respectively, with $f_n(x_i) = y_i$ for $i \in \{0, 1, 2, 3, 4\}$. The construction may even be simpler provided some of the considered subarcs of A_n are degenerate. Thus (*) is shown, and the proof is complete.

Combining Propositions 1 and 2 we get

Theorem 1. *The following assertions are equivalent for a continuum X*:
(i) *X is the union of finitely many arcs*;
(ii) *X is a locally one-to-one image of a finite dendrite*;
(iii) *X is a locally one-to-one image of a graph.*

Corollary 1. *The image of a continuum being the union of finitely many arcs under a locally one-to-one mapping is the union of finitely many arcs as well.*

Indeed, this follows from Theorem 1 and Statement 2, or by a straightforward argument as in the proof of Proposition 1.

Remark that the class Δ of finite dendrites not only generates the class Γ of the unions of finitely many arcs (in the sense of Proposition 2), but it is even the smallest one (in the sense of inclusion) generating Γ. To prove this, note that if the image of an element of Γ under a locally one-to-one mapping is in Δ, then the mapping is (globally) one-to-one, i.e., it is a homeomorphism.

3. Local isometries, local expansions, local contractions. Relations between the three kinds of mappings listed in the subtitle will be needed in a further part of the paper. The relations are discussed in the present section, the whole contents of which is due to A. Całka.

Let X and Y be topological (not necessarily metric) spaces, and let $f: X \to Y$ be a mapping from X onto Y. If the spaces X and Y are metric with metrics d_X and d_Y, respectively, the mapping f is said to be (a) *local isometry*, (b) *local expansion*, (c) *local contraction* provided that for each point $x \in X$ there exist a neighborhood U of x and a number M with (a) $M = 1$, (b) $M > 1$, (c) $0 < M < 1$, such that for every two points $y, z \in U$ we have

(a) $$d_Y(f(y), f(z)) = M \cdot d_X(y, z),$$
(b) $$d_Y(f(y), f(z)) \geqq M \cdot d_X(y, z),$$
(c) $$d_Y(f(y), f(z)) \leqq M \cdot d_X(y, z),$$

respectively. Obviously, each local isometry or a local expansion is a locally one-to-one mapping.

The following proposition is proved in [1].

Proposition 3. *Let a continuous mapping f of a compact metrizable space X into a compact metrizable space Y be given. Then the following assertions are equivalent*:
(i) *f is locally one-to-one*;
(ii) *there exist metrics d_X and d_Y on X and Y which are compatible with the topo-*

logies on X and Y, respectively, and such that f is a local isometry of (X, d_X) into (Y, d_Y);

(iii) *for every metric d_Y on Y which is compatible with the topology there exists a metric d_X on X, compatible with the topology and such that f is a local isometry of (X, d_X) into (Y, d_Y).*

Remark 1. The metric d_X on X satisfying (iii) of Proposition 3 is determined "locally uniquely". Namely, if two such metrics d_X and d_X^* are given, then they are locally identical in the sense that the identity mapping id_X is a local isometry of (X, d_X) into (X, d_X^*) and of (X, d_X^*) into (X, d_X). For the proof see [1].

Proposition 3 remains true after replacing "f is a local isometry" by "f is a local expansion". To see this it is enough to multiply the metrics by adequate constants. Thus as an immediate consequence of Proposition 3 we have

Statement 3. *Let a locally one-to-one mapping f of a compact metrizable space X into a compact metric space (Y, d_Y) be given. Then there exists a metric d_X on X which is compatible with the topology and such that f is a local expansion of (X, d_X) into (Y, d_Y). Moreover, for every number $M > 1$ the metric d_X can be chosen so that f is a local expansion with the coefficient M for all points $x \in X$.*

Similarly, we have

Statement 4. *Let a locally one-to-one mapping f of a compact metrizable space X into a compact metric space (Y, d_Y) be given. Then there exists a metric d_X on X which is compatible with the topology and such that f is a local contraction of (X, d_X) into (Y, d_Y). Moreover, for every number M with $0 < M < 1$ the metric d_X can be chosen so that f is a local contraction with the coefficient M for all points $x \in X$.*

In fact, by (iii) of Proposition 3 there exists a metric d_X^* on X compatible with the topology and such that f is a local isometry of (X, d_X^*) into (Y, d_Y). For a fixed number M with $0 < M < 1$ define a metric d_X on X by putting $d_X(a, b) = d_X^*(a, b)/M$ for all $a, b \in X$ and note that d_X satisfies the assertion.

4. Applications. It follows from Proposition 3 that distinguishing between the three kinds of locally one-to-one mappings (i.e. between local expansions, local isometries and local contractions) is not interesting from the topological point of view: all their topological properties are the same. Therefore, using Proposition 3 and Statement 3 we see that Theorem 1 can be reformulated in the following way which gives several characterizations of continua that are unions of finitely many arcs in terms of various kinds of mappings. Namely, we have

Theorem 2. *The following assertions are equivalent for a continuum X:*

I. *X is the union of a finite family of arcs;*

II. *X is the image of a finite dendrite under either* (i) *a locally one-to-one mapping, or* (ii) *a local isometry, or* (iii) *a local expansion;*

III. *X is the image of a graph under either* (i) *a locally one-to-one mapping, or* (ii) *a local isometry, or* (iii) *a local expansion.*

Note that we cannot add "or (iv) a local contraction" in Parts II and III of the above theorem: see Remark 2 below (in § 5).

As an application of Statement 3 to Propositions 4 and 5 of [2] we obtain the following assertions.

Statement 5. *If a space Y is the image of a space X under a locally one-to-one mapping, then* card $E(Y) \leqq$ card $E(X)$; *in particular, if X is a graph, then the set* $E(Y)$ *of end points of the curve Y is finite.*

Statement 6. *A graph is the image of an arc* (*of a simple closed curve*) *under a locally one-to-one mapping if and only if it has at most two* (*it has none, respectively*) *end points.*

To see that Statement 6 cannot be generalized to continua which are the unions of finitely many arcs, consider the union of $S = \{(t^{-1}\cos t, t^{-1}\sin t): t \in [2\pi, +\infty)\}$ and of the arc $[0, 1] \times \{0\}$ (the union of S and of the arc $[-1/3\pi, 1/2\pi] \times \{0\}$, respectively). Observe that the inverse set of the point $(0, 0)$ should consist of at least two (of at least one) end points.

5. Construction of an example. A continuum is said to be a *regular curve* in the sense of the theory of order provided each its point has arbitrarily small neighborhoods whose boundaries are finite sets. If a point p of a continuum has arbitrarily small neighbourhoods with finite boundaries and, moreover, the cardinality of these boundaries tends to infinity when diameters of the neighborhoods tend to zero, then p is said to be of *order* ω.

It is known that each connected union of finitely many arcs is a regular curve ([3], p. 179), whence we have by Proposition 2

Corollary 2. *Every image of a graph under a locally one-to-one mapping is a regular curve.*

Remark 2. The inverse is not true, as can be seen from Statement 5: the one-point union of countably many arcs with diameters tending to zero is a regular curve and has countably many end points, so it is not the image of a graph under a locally one-to-one mapping. Note also that this gives an example of a continuum which is the image of an arc under a local contraction, but (in view of Theorem 2) it is not the image of a graph under a locally one-to-one mapping.

K. Menger in [3], p. 179 gives an example of a curve that is the union of two arcs and contains a point of order ω. A modification of his example leads to a curve which is the image of the interval $[0, 1]$ under a local expansion, it is the union of two arcs, and contains a point of order ω (see [2], Proposition 3, p. 78). Further modification shows that this result can be strengthened as follows.

Proposition 4. *There exists a continuum such that*:

1) *it is the image of the closed unit interval under a local expansion*;
2) *it is the union of two arcs*;
3) *it lies in the plane*;
4) *the set of all its ramification points consists of three countable subsets*: *of points of order* 3, *of order* 4 *and of order* ω;
5) *it has only one end point.*

Proof. Let (x, y) be a point in the plane R^2 equipped with a rectangular coordinate system. Put $p = (0, 0)$, $q = (1, 1)$, $a_0 = (1, 0)$, and, for each $n \in \{1, 2, \ldots\}$, put

$$a_n = (2^{-n}, 0) \quad \text{and} \quad b_n = (2^{-n}, 2^{1-[(n+1)/2]}),$$

where $[t]$ denotes the integral part of a real number t. Denote by ab the straight line segment joining a and b in the plane. Let $A_0 = qa_0 \cup a_0a_1$ and, for each $k \in \in \{1, 2, \ldots\}$, let

$$A_{2k-1} = a_{2k-1}b_{2k-1} \cup b_{2k-1}b_{2k} \cup b_{2k}a_{2k}$$

and

$$A_{2k} = a_{2k}a_{2k+1} \,.$$

Putting

$$A = A_0 \cup \bigcup\{A_n : n \in \{1, 2, \ldots\}\} \cup \{p\}$$

we see that A is an arc in the plane joining p with q. Let A^* denote the image of A under the symmetry with respect to the line $y = x$. So the continuum

$$Q = A \cup A^*$$

is the union of two arcs, and $\text{ord}_p Q = \omega$.

Now for every $i \in \{1, 2, \ldots\}$ we consider three transformations of the plane onto itself: a homothetic transformation s_i, a parallel displacement t_i and the composite u_i of the two previous ones, defined consecutively as follows:

$$s_i(x, y) = (x \,.\, 2^{-i}, y \,.\, 2^{-i}),$$
$$t_i(x, y) = (x + 2^{-i}, y + 2^{-i}),$$
$$u_i(x, y) = t_i(s_i(x, y)),$$

where $(x, y) \in R^2$. We see that u_i is a contraction with the ratio $M = 2^{-i}$.

Put $Y = \{p\} \cup \bigcup\{u_i(Q): i \in \{1, 2, \ldots\}\}$ (see the figure). Note that $p \in \text{cl}(\bigcup\{u_i(Q): i \in \{1, 2, \ldots\}\})$ and $u_i(p) = (2^{-i}, 2^{-i}) = u_{i+1}(q)$, whence $u_i(Q) \cap u_{i+1}(Q)$ is just a one-point set. This implies that Y is connected. Moreover, Y is the union of two arcs, $\{p\} \cup \bigcup\{u_i(A): i \in \{1, 2, \ldots\}\}$ and $\{p\} \cup \bigcup\{u_i(A^*): i \in \{1, 2, \ldots\}\}$, each of which joins the points p and q. Thus Y is a continuum satisfying 2) and 3). We show that it satisfies all other conclusions of the proposition.

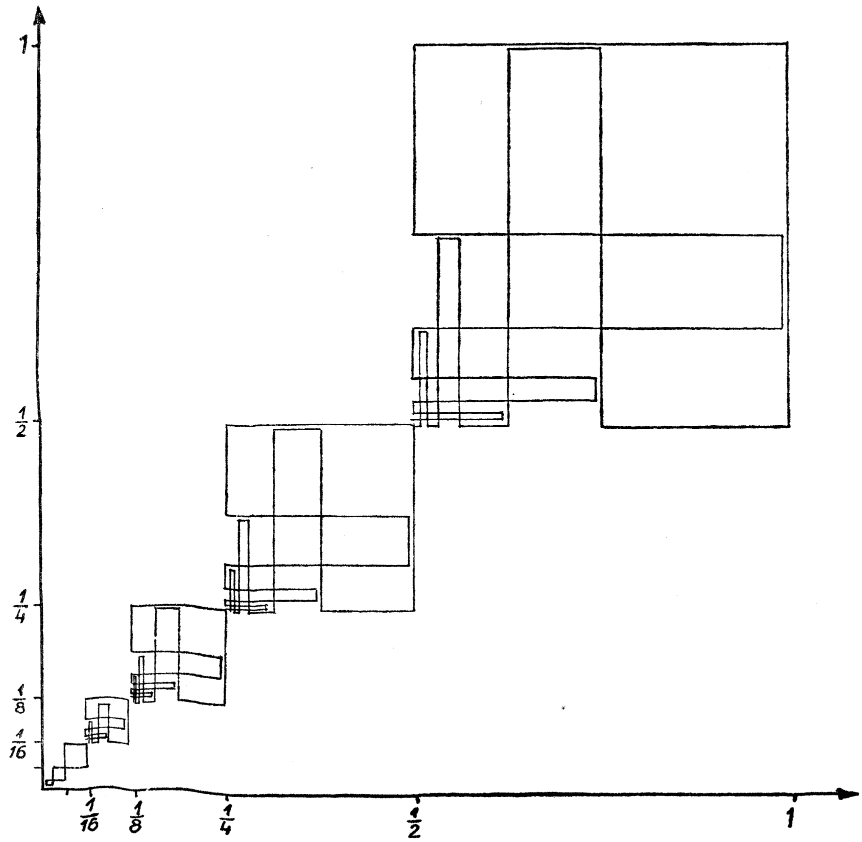
In fact, note that p is an end point, and the points $u_i(p)$, where $i \in \{1, 2, \ldots\}$, are of order ω in Y. The other part of 4) can be easily seen from the construction.

To show 1) let $X = pa_0$. Thus X is the unit segment containing the points a_n for each $n \in \{1, 2, \ldots\}$ and having a_1 as its midpoint. Define an auxiliary mapping $g: pa_1 \to A$ from pa_1 onto A as follows: $g(p) = p$; $g(a_1) = q$, $g(a_n) = a_{n-1}$ for each

$n \in \{2, 3, \ldots\}$; and we assume that $g \mid a_n a_{n+1}: a_n a_{n+1} \to A_{n-1}$ is a linear surjection for each $n \in \{1, 2, \ldots\}$. Hence $g(pa_1) = A$. It is quite elementary to verify that g is a local expansion with the constant $M = 3/2$.

Let $h_n: a_n a_{n+1} \to pa_1$, where $n \in \{1, 2, \ldots\}$, be a linear, order preserving mapping. In particular, $h_n(a_{n+1}) = p$, $h_n(a_n) = a_1$, and we see that h_n expands $a_n a_{n+1}$ to pa_1 with ratio 2^n.

To define a local expansion $f: X \to Y$ from X onto Y we note that $X = pa_1 \cup a_1 a_0$ and define f separately on each of these two segments. Since $pa_1 = \{p\} \cup \bigcup\{a_n a_{n+1}: n \in \{1, 2, \ldots\}\}$, we can define $f \mid pa_1: pa_1 \to \{p\} \cup \bigcup\{u_n(A): n \in \{1, 2, \ldots\}\}$ putting $f(p) = p$ and $f(c) = u_n(g(h_n(c)))$ for any point $c \in a_n a_{n+1}$, where $n \in \{1, 2, \ldots\}$. The reader can verify in a routine way that this definition is correct, f is continuous and maps pa_1 onto the arc $\{p\} \cup \bigcup\{u_n(A): n \in \{1, 2, \ldots\}\}$. Observe that, since h_n expands each interval $a_n a_{n+1}$ 2^n-times while u_n is a contraction of the plane with ratio 2^{-n}, the mapping $f \mid pa_1$ acts as g does; so we conclude that $f \mid pa_1$ is a local expansion with the coefficient $M = 3/2$.



Now we define $f \mid a_1 a_0$. Given a point $(x, 0) \in a_1 a_0$ (i.e., such that $1/2 \leqq x \leqq 1$) we put $f((x, 0)) = [f((1 - x, 0))]^*$, where the asterisk denotes the symmetry $a = (x, y) \to a^* = (y, x)$ with respect to the line $y = x$. So $f \mid a_1 a_0 \colon a_1 a_0 \to \{p\} \cup \bigcup \{u_0(A^*) \colon n \in \{1, 2, \ldots\}\}$ is again a local expansion with the same coefficient $M = 3/2$. Thus $f \colon X \to Y$ is well-defined.

To inspect local expansibility of the mapping f it is enough to state that both partial mappings $f \mid pa_1$ and $f \mid a_1 a_0$ are local expansions and to observe that for the point a_1 we can choose an open interval U of length $1/6$ with a_1 as its midpoint, which satisfies the condition mentioned in the definition of a local expansion. So we have shown $f \colon X \to Y$ is a local expansion, and $M = 3/2$ is the coefficient for each point of X. The proof is complete.

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