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Jaromír Duda

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ARITHMETICITY AT 0

JAROMÍR DUDA, Brno

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1. CONGRUENCE PERMUTABILITY AT 0

The classical theorem of A. I. Mal'cev asserts that a variety $\boldsymbol{V}$ is congruence permutable iff there exists a ternary polynomial p such that the identities $p(y, x, x) = p(x, x, y) = y$ hold in $\boldsymbol{V}$, see [13]. Recently H.-P. Gumm and A. Ursini discovered that the identities $r(0, x, x) = 0$ and $r(x, x, y) = y$ (more precisely the equivalent identities $s(x, x) = 0$ and $s(x, 0) = x$; put $s(x, y) = r(0, y, x)$ and, conversely, $r(x, y, z) = s(z, s(y, x))$) play an important role in the theory of ideals of universal algebras having a constant 0. Since a number of nice structural properties follow from the original Mal'cev's ternary polynomial we first find a suitable structural characterization of the above identities $s(x, x) = 0$ and $s(x, 0) = x$.

Definition 1. An algebra A with a constant 0 is called *congruence permutable at* 0 whenever $[0]\, \Theta \circ \Psi = [0]\, \Psi \circ \Theta$ holds for any $\Theta, \Psi \in \operatorname{Con} A$.
A variety $\boldsymbol{V}$ with a constant 0 is called *congruence permutable at* 0 whenever each $\boldsymbol{V}$ algebra has this property.

Theorem 1. *Let $\boldsymbol{V}$ be a variety with a constant* 0. *The following conditions are equivalent*:

(1) *$\boldsymbol{V}$ is congruence permutable at* 0;

(2) *there exists a binary polynomial s such that $s(x, x) = 0$ and $s(x, 0) = x$ hold in $\boldsymbol{V}$.*

Proof. (1) $\Rightarrow$ (2). Let A be a $\boldsymbol{V}$ free algebra with free generators x and y. Take the principal congruences $\Theta(0, y)$ and $\Theta(x, y)$ on A. Then $x \in [0]\, \Theta(0, y) \circ \Theta(x, y)$ and so $x \in [0]\, \Theta(x, y) \circ \Theta(0, y)$, by hypothesis. In other words, we have $\langle 0, x\rangle \in \Theta(x, y) \circ \Theta(0, y)$ from which $\langle 0, s\rangle \in \Theta(x, y)$ and $\langle s, x\rangle \in \Theta(0, y)$ follow for some $s \in A$. Apparently s is the desired binary polynomial.

(2) $\Rightarrow$ (1). Conversely, we have to prove the equality $[0]\, \Theta \circ \Psi = [0]\, \Psi \circ \Theta$ for arbitrary $\Theta, \Psi \in \operatorname{Con} A$, $A \in \boldsymbol{V}$. Let $a \in [0]\, \Theta \circ \Psi$, then $\langle 0, b\rangle \in \Theta$ and $\langle b, a\rangle \in \Psi$ for some element $b \in A$. Applying the binary polynomial s from the hypothesis to these two statements we find that $\langle 0, s(a, b)\rangle = \langle s(b, b), s(a, b)\rangle \in \Psi$ and

$\langle s(a, b), a\rangle = \langle s(a, b), s(a, 0)\rangle \in \Theta$, i.e. $\langle 0, a\rangle \in \Psi \circ \Theta$ whence $[0]\ \Theta \circ \Psi \subseteq$ $\subseteq [0]\ \Psi \circ \Theta$. The converse inclusion can be handled similarly.

Next we want to show some other structural equivalents to the congruence permutability at 0. The following auxiliary result will be useful in the sequel.

Lemma 1. *Let A be an algebra with a constant* 0 *and let $R, S, T \subseteq A \times A$. Then*

(a) $[0]\ R \circ S = ([0]\ R]\ S$ (*here the symbol* $[X]\ T$ *denotes the union* $\bigcup_{x\in X} [x]\ T$ *for* $X \subseteq A,\ T \subseteq A \times A$);

(b) $[0]\ R \circ S \circ T = [0]\ S \circ R \circ T$ *whenever* $[0]\ R \circ S = [0]\ S \circ R$.

Proof. The first point is elementary

(b) Applying part (a) we successively get $[0]\ R \circ S \circ T = [[0]\ R \circ S]\ T =$ $= [[0]\ S \circ R]\ T = [0]\ S \circ R \circ T$.

Theorem 2. *Let A be an algebra with a constant* 0. *The following conditions are equivalent*:

(1) *A is congruence permutable at* 0;

(2) $[0]\ \Theta \vee \Psi = [0]\ \Theta \circ \Psi$ *for any* $\Theta, \Psi \in \operatorname{Con} A$.

Proof. (1) $\Rightarrow$ (2). Apparently it is sufficient to verify the inclusion $[0]\ \Theta \vee \Psi \subseteq$ $\subseteq [0]\ \Theta \circ \Psi$. So let $a \in [0]\ \Theta \vee \Psi$, then $\langle 0, a\rangle \in \Theta \vee \Psi$, i.e. $\langle 0, a\rangle \in (\Theta \circ \Psi)^n$ for some $n \geqq 1$. If $n > 1$ we have $[0]\ (\Theta \circ \Psi)^n = [0]\ \Theta \circ \Psi \circ (\Theta \circ \Psi)^{n-1} =$ $= [0]\ \Psi \circ \Theta \circ (\Theta \circ \Psi)^{n-1} = [0]\ \Psi \circ (\Theta \circ \Psi)^{n-1} = [0]\ \Psi \circ \Theta \circ \Psi \circ (\Theta \circ \Psi)^{n-2} =$ $= [0]\ \Theta \circ \Psi \circ (\Theta \circ \Psi)^{n-2} = [0]\ (\Theta \circ \Psi)^{n-1}$, by Lemma 1(b). Repeating this process one concludes $a \in [0]\ \Theta \circ \Psi$ which was to be proved.

The converse implication (2) $\Rightarrow$ (1) is evident.

In [18], H. Werner characterized the congruence permutable varieties in terms of compatible binary relations. For congruence permutable at 0 varieties Werner's results can be adapted e.g. in the following form (recall that the symbols $R(a, b)$ and $T(a, b)$ denote the smallest compatible reflexive binary relation and tolerance, respectively, containing the pair $\langle a, b\rangle$).

Theorem 3. *Let V be a variety with a constant* 0. *The following conditions are equivalent*:

(1) *V is congruence permutable at* 0;

(2) $R(x, 0) = R(0, x)$ *for any* $x \in A \in V$;

(3) $T(x, 0) = R(x, 0)$ *for any* $x \in A \in V$.

Proof. (1) $\Rightarrow$ (2). Apply the binary polynomial s from Theorem 1 (2) to the pairs $\langle x, x\rangle, \langle x, 0\rangle \in R(x, 0)$. Then $\langle 0, x\rangle = \langle s(x, x), s(x, 0)\rangle \in R(x, 0)$ and so $R(0, x) \subseteq$ $\subseteq R(x, 0)$. The converse inclusion follows by a symmetrical argument.

(2) $\Rightarrow$ (3). By hypothesis we find $R(x, 0) = R(0, x) = R^{-1}(x, 0)$, i.e. $R(x, 0)$ is a tolerance. Hence $T(x, 0) \subseteq R(x, 0)$, the other inclusion being trivial.

(3) $\Rightarrow$ (1). Let A be a V free algebra with one free generator x. By (3) we have $T(x, 0) = R(x, 0)$ and so $\langle 0, x\rangle \in R(x, 0)$. Then $0 = \tau(x)$, $x = \tau(0)$ for some unary

algebraic function τ over A, i.e. $0 = t(x, x)$, $x = t(0, x)$ for a suitable binary polynomial t of V. Theorem 1 completes the proof.

Remark 1. Further results on congruence permutability at 0 as well as on congruence n-permutability at 0 can be found in [5].

2. ARITHMETICITY AT 0

Another important Mal'cev's is formed by the congruence distributive varieties, see [12]. The concept of congruence distributivity has been recently weakened to that of congruence distributivity at 0. From [4] we adopt

Definition 2. An algebra A with a constant 0 is called *congruence distributive at* 0 whenever $[0](\Theta_1 \vee \Theta_2) \wedge \Psi = [0](\Theta_1 \wedge \Psi) \vee (\Theta_2 \wedge \Psi)$ holds for any congruences Θ_1, Θ_2 and Ψ on A.

A variety V with a constant 0 is called *congruence distributive at* 0 if each V algebra has the above property.

It is already known that also the congruence distributive at 0 varieties are definable by a Mal'cev condition. The characterizing identities can be found in [4].

The *arithmetical* (= congruence permutable and congruence distributive) algebras naturally combine the advantages of both attributes. Taking into account the preceding definitions the notion of arithmeticity at 0 readily follows:

Definition 3. An algebra A with a constant 0 is called *arithmetical at* 0 whenever A is congruence permutable at 0 and, simultaneously, congruence distributive at 0.

A variety V with a constant 0 is called *arithmetical at* 0 if each V algebra has the above property.

Before stating our main theorem we prove

Lemma 2. *Let A be a congruence permutable at* 0 *algebra. Then A is congruence distributive at* 0 (see Definition 2) *iff the dual equality* $[0](\Theta_1 \wedge \Theta_2) \vee \Psi = [0](\Theta_1 \vee \Psi) \wedge (\Theta_2 \vee \Psi)$ *holds for any* $\Theta_1, \Theta_2, \Psi \in \operatorname{Con} A$.

Proof. Suppose that A is congruence distributive at 0. Then $[0](\Theta_1 \vee \Psi) \wedge (\Theta_2 \vee \Psi) = [0](\Theta_1 \wedge (\Theta_2 \vee \Psi)) \vee (\Psi \wedge (\Theta_2 \vee \Psi)) = [0](\Theta_1 \wedge (\Theta_2 \vee \Psi)) \vee \Psi = [0](\Theta_1 \wedge (\Theta_2 \vee \Psi)) \circ \Psi = [[0]\, \Theta_1 \wedge (\Theta_2 \vee \Psi)]\, \Psi = [[0](\Theta_1 \wedge \Theta_2) \vee (\Theta_1 \wedge \Psi)]\, \Psi = [0](\Theta_1 \wedge \Theta_2) \circ (\Theta_1 \wedge \Psi) \circ \Psi = [0](\Theta_1 \wedge \Theta_2) \vee \Psi$, by Lemma 1 and Theorem 2.

The converse implication holds on any algebra with 0.

Theorem 4. *Let V be a variety with a constant* 0. *The following conditions are equivalent*:

(1) *V is arithmetical at* 0;

(2) $[0](\Theta_1 \wedge \Theta_2) \circ \Psi = [0]\,\Psi \circ \Theta_1 \wedge [0]\,\Psi \circ \Theta_2$ *for any* $\Theta_1, \Theta_2, \Psi \in \operatorname{Con} A$, $A \in V$;

(3) *there exists a binary polynomial* b *such that* $b(x, x) = b(0, x) = 0$, $b(x, 0) = x$ *hold in* V.

Proof. The implication (1) ⇒ (2) is a direct consequence of Theorem 2 and Lemma 2.

(2) ⇒ (3). Let A be a V free algebra with free generators x, y. Take the congruences $\Theta_1 = \Theta(0, x)$, $\Theta_2 = \Theta(x, y)$ and $\Psi = \Theta(0, y)$ on A. Since $x \in [0]\,\Psi \circ \Theta_1 \wedge [0]\,\Psi \circ \Theta_2$, we have $x \in [0](\Theta_1 \wedge \Theta_2) \circ \Psi$ as well. Now $\langle 0, x\rangle \in (\Theta_1 \wedge \Theta_2) \circ \Psi$ gives $\langle 0, b\rangle \in \Theta(0, x) \wedge \Theta(x, y)$ and $\langle b, x\rangle \in \Theta(0, y)$ for some binary polynomial b of V. The first statement implies $b(x, x) = b(0, y) = 0$, the second one yields $b(x, 0) = x$.

(3) ⇒ (1). The congruence permutability at 0 is ensured by the identities $b(x, x) = 0$, $b(x, 0) = x$, see Theorem 1.

To prove the inclusion $[0](\Theta_1 \vee \Theta_2) \wedge \Psi \subseteq [0](\Theta_1 \wedge \Psi) \vee (\Theta_2 \wedge \Psi)$, $\Theta_1, \Theta_2, \Psi \in \operatorname{Con} A$, $A \in V$, observe first that the binary polynomial u defined by $u(x, y) = b(x, b(x, y))$ satisfies the identities $u(x, x) = x$, $u(x, 0) = u(0, x) = 0$. Now let $x \in [0](\Theta_1, \vee \Theta_2) \wedge \Psi$, then $\langle 0, x\rangle \in \Theta_1 \vee \Theta_2$ and $\langle 0, x\rangle \in \Psi$. In virtue of Theorem 2 the first statement can be rewritten to $\langle 0, x\rangle \in \Theta_1 \circ \Theta_2$, i.e., we find that $\langle 0, a\rangle \in \Theta_1$ and $\langle a, x\rangle \in \Theta_2$ for some element $a \in A$. Applying the above mentioned binary polynomial u we get that $\langle 0, u(x, a)\rangle = \langle u(x, 0), u(x, a)\rangle \in \Theta_1$ and $\langle u(x, a), x\rangle = \langle u(x, a), u(x, x)\rangle \in \Theta_2$. On the other hand, $\langle 0, x\rangle \in \Psi$ implies $\langle 0, u(x, a)\rangle = \langle u(0, a), u(x, a)\rangle \in \Psi$ and so $\langle u(x, a), x\rangle \in \Psi$, by transitivity. Altogether we have $\langle 0, u(x, a)\rangle \in \Theta_1 \wedge \Psi$, $\langle u(x, a), x\rangle \in \Theta_2 \wedge \Psi$ from which the desired conclusion $\langle 0, x\rangle \in (\Theta_1 \wedge \Psi) \vee (\Theta_2 \wedge \Psi)$ readily follows.

Remark 2. (1) One easily sees that the condition (3) from Theorem 4 can be replaced by: $t_1(x, x, 0) = t_1(0, x, 0) = 0$, $t_1(x, 0, 0) = x$ (or, equivalently, by $t_1(x, x, 0) = t_1(0, x, 0) = 0$, $t_1(x, y, y) = x$) for some ternary polynomial t_1. Notice that the original Pixley's result (see [16]) states: *A variety* V *is arithmetical iff* $t(y, y, x) = t(x, y, x) = t(x, y, y) = x$ *for some ternary polynomial* t *of* V.

(2) The congruence distributivity at 0 follows directly from the existence of the binary polynomial u (see [4; Thm. 1 for $n = 2$]) in the proof of implication (3) ⇒ (1). However, the full version of this proof will be useful in Section 3.

Examples 1. (1) Any BCK-algebra is arithmetical at 0. Recall first (e.g. from [11; p. 423]) that a set A with a binary operation $*$ and a constant 0 is called a BCK-algebra if the following conditions hold:

(i) $(x * y) * (x * z) \leqq z * y$,

(ii) $x * (x * y) \leqq y$,

(iii) $x \leqq x$,

(iv) $0 \leqq x$,

(v) $x \leqq y$ and $y \leqq x$ imply $x = y$,
(vi) $x \leqq y$ iff $x * y = 0$.

In particular, conditions (i), ..., (vi) give $x * 0 = x$, see [11; p. 424] again. Since the identities $x * x = 0 * x = 0$ are evident the binary polynomial $b(x, y) = x * y$ ensures the arithmeticity at 0 for any BCK-algebra. Notice that BCK-algebras are in general not congruence permutable, see [6] and references there.

(2) Any pseudocomplemented $\wedge$-semilattice (i.e. an algebra $\langle A, \wedge, *, 0, 1\rangle$ of type $\langle 2, 1, 0, 0\rangle$ where a^* denotes the pseudocomplement of $a \in A$: $x \wedge a = 0$ iff $x \leqq a^*$, see [2] for the details) is arithmetical at 0 since the binary polynomial $b(x, y) = x \wedge y^*$ satisfies the identities $b(x, x) = x \wedge x^* = 0$, $b(0, x) = 0 \wedge x^* = 0$, and $b(x, 0) = x \wedge 0^* = x$.

(3) The complementary semigroups form a variety arithmetical at a constant. Recall from [3] (see also [19; p. 37]) that a complementary semigroup is a commutative semigroup $\langle S, \cdot\rangle$ with an additional binary operation $*$ for which

(i) $x(y * y) = x$,
(ii) $x(x * y) = y(y * x)$, and
(iii) $x * (y * z) = xy * z$ hold.

Then $\langle S, \leqq\rangle$ (here $x \leqq y$ means: x divides y) is a $\vee$-semilattice with the join $x \vee y = x(x * y)$ and the smallest element $1 = x * x$ which is the unit of the semigroup $\langle S, \cdot\rangle$.

Taking $b(x, y) = y * x$ we get that $b(x, x) = x * x = 1$, $b(1, x) = x * 1 = (x \vee 1) * 1 = x(x * 1) * 1 = (x * 1) x * 1 = (x * 1) * (x * 1) = 1$, and $b(x, 1) = 1 * x = 1(1 * x) = 1 \vee x = x$.

3. ARITHMETICITY AT 0 AND THE EXTENDED CRT

In the proof of Theorem 4 a binary polynomial u satisfying the identities $u(x, x) = x$, $u(x, 0) = u(0, x) = 0$ was used. The same polynomial can be found in [4; Thm. 1 for $n = 2$]. Since the presence of the above polynomial can be equivalently expressed by a ternary polynomial v such that $v(x, x, 0) = x$, $v(x, 0, 0) = v(0, x, 0) = 0$ (put $v(x, y, z) = u(x, y)$ and $u(x, y) = v(x, y, 0)$) we now introduce the concept of the near 0-unanimity polynomial as follows:

Definition 4. Let A be an algebra with a constant 0, u an n-ary ($n > 1$) polynomial of A. u is called a near 0-*unanimity* polynomial whenever $u(x, \ldots, x) = x$, $u(x, 0, 0, \ldots, 0) = u(0, x, 0, \ldots, 0) = \ldots = u(0, \ldots, 0, x) = 0$ hold in A.

It is well-known that the classical near unanimity polynomials (recall that $v(x, y, \ldots, y) = v(y, x, y, \ldots, y) = \ldots = v(y, \ldots, y, x) = y$ for any near unanimity polynomial v) characterize the varieties whose congruences satisfy the extended Chinese remainder theorem, see [1, 17] for the details. This fact motivates

Theorem 5. *Let V be a variety with a constant* 0. *The following conditions are* *equivalent for an integer* $n > 1$:

(1) (extended CRT at 0) $\bigwedge_{i=1}^{n} [0]\, \Theta_i \wedge [a]\, \Psi \neq \emptyset$ *whenever* $\bigwedge_{\substack{i=1 \\ i \neq j}}^{n} [0]\, \Theta_i \wedge [a]\, \Psi \neq \emptyset$, $1 \leqq j \leqq n$, *for* $a \in A \in V$ *and* $\Theta_1, \ldots, \Theta_n$, $\Psi \in \mathrm{Con}\, A$;

(2) $[0]\, (\bigwedge_{i=1}^{n} \Theta_i) \circ \Psi = \bigwedge_{j=1}^{n} [0]\, (\bigwedge_{\substack{i=1 \\ i \neq j}}^{n} \Theta_i) \circ \Psi$ *for any* $\Theta_1, \ldots, \Theta_n$, $\Psi \in \mathrm{Con}\, A$, $A \in V$;

(3) *There exists an n-ary near* 0-*unanimity polynomial in* V.

Proof. (1) $\Rightarrow$ (3). Let A be a V free algebra with n ($n > 1$) free generators $x_1, \ldots, x_n$. Take the congruences

$$\begin{aligned}
\Psi &= \Theta(x_1, x_2, \ldots, x_{n-1}, x_n),\\
\Theta_1 &= \Theta(x_1, x_2, \ldots, x_{n-1}, 0),\\
\Theta_2 &= \Theta(x_1, x_2, \ldots, 0, x_n),\\
&\ldots\\
\Theta_n &= \Theta(0, x_2, \ldots, x_{n-1}, x_n) \text{ on } A.
\end{aligned}$$

Since the hypothesis of (1) is fulfilled for $x_1 = a$ we conclude that $\langle x_1, u\rangle \in \Psi$ and $\langle 0, u\rangle \in \Theta_i$, $1 \leqq i \leqq n$, for some $u \in A$. Apparently u is the desired near 0-unanimity polynomial of V.

(3) $\Rightarrow$ (1). Conversely, let u be an n-ary near 0-unanimity polynomial of V. We have to prove that (1) holds for arbitrary congruences $\Theta_1, \ldots, \Theta_n$, $\Psi \in \mathrm{Con}\, A$. To do this take elements $c_1, \ldots, c_n \in A$ such that $c_j \in \bigwedge_{\substack{i=1 \\ i \neq j}}^{n} [0]\, \Theta_i \wedge [a]\, \Psi$, $1 \leqq j \leqq n$. We claim that $u(c_1, \ldots, c_n) \in \bigwedge_{i=1}^{n} [0]\, \Theta_i \wedge [a]\, \Psi$. Clearly $u(c_1, \ldots, c_n) \in [a]\, \Psi$ since $\langle a, c_j\rangle \in \Psi$ for $1 \leqq j \leqq n$. Further, $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n \in [0]\, \Theta_j$ implies $\langle u(0, \ldots, 0, c_j, 0, \ldots, 0), u(c_1, \ldots, c_n)\rangle \in \Theta_j$, i.e. we find that $u(c_1, \ldots, c_n) \in [0]\, \Theta_j$ for $1 \leqq j \leqq n$. Altogether, we have $u(c_1, \ldots, c_n) \in \bigwedge_{i=1}^{n} [0]\, \Theta_i \wedge [a]\, \Psi$ as required.

(2) $\Rightarrow$ (3). Take the same V free algebra and the same congruences as in the proof of (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (2). This implication can be verified similarly as (3) $\Rightarrow$ (1); we omit the details.

Remark 3. It is evident that any n-ary ($n > 1$) near 0-unanimity polynomial u_n is also an m-ary near 0-unanimity polynomial u_m for each $m \geqq n$.

Corollary 1. *Let V be a congruence permutable at* 0 *variety. Then V is congruence distributive at* 0 *iff there exists a binary near* 0-*unanimity polynomial in V.*

Proof. Apply Theorem 2, Lemma 2, and Theorem 5.

Corollary 2. *A near* 0-*unanimity polynomial ensures the congruence distributivity at* 0 *on any algebra with a constant* 0.

Proof. This can be shown in a similar way as the implication (3) $\Rightarrow$ (1) in the proof of Theorem 4. So, let u be an n-ary near 0-unanimity polynomial. From $x \in [0](\Theta_1 \vee \Theta_2) \wedge \Psi$ (see the proof of Theorem 4) we get

$\langle 0, x \rangle \in \Psi$ and
$\langle a_i, a_{i+1} \rangle \in \Theta_1$, i even,
$\langle a_i, a_{i+1} \rangle \in \Theta_2$, i odd,

for some elements $0 = a_0, \ldots, a_k = x$. Consider the sequence

$$\begin{aligned} 0 &= u(x, 0, 0, \ldots, 0), u(x, a_1, 0, \ldots, 0), u(x, a_2, 0, \ldots, 0), \ldots, u(x, a_k, 0, \ldots, 0) = \\ &= u(x, x, 0, \ldots, 0), u(x, x, a_1, \ldots, 0), u(x, x, a_2, \ldots, 0), \ldots, u(x, x, a_k, \ldots, 0) = \ldots \\ &\ldots \\ \ldots &= u(x, x, \ldots, x, 0), u(x, x, \ldots, x, a_1), u(x, x, \ldots, x, a_2), \ldots, u(x, x, \ldots, x, a_k) = \\ &= u(x, x, \ldots, x, x) = x\,. \end{aligned}$$

Now it is a routine to verify that $x \in [0](\Theta_1 \wedge \Psi) \vee (\Theta_2 \wedge \Psi)$, as claimed.

Example 2. Apparently $u(x, y) = x \wedge y$ is a near 0-unanimity polynomial in the variety of the $\wedge$-semilattices having the least element 0. Consequently, the $\wedge$-semilattices with 0 are congruence distributive at 0. For the sake of completeness notice that the $\wedge$-semilattices with 0 are in general not congruence distributive; consider e.g. the direct product of two element chains $\mathbf{2} \times \mathbf{2}$. See [15] for detailed information.

4. ARITHMETICITY AT 0 IN 0-REGULAR VARIETIES

Recall that an algebra A with a constant 0 is *0-regular* if the congruence class $[0]\,\Theta$ uniquely determines the whole congruence Θ on A. A variety $\boldsymbol{V}$ with a constant 0 is *0-regular* whenever each $\boldsymbol{V}$ algebra has this property. H.-P. Gumm and A. Ursini discovered in [10] that the 0-regular and congruence permutable at 0 varieties (briefly called the *ideal determined varieties* in [10]) form a very suitable material for a study of universal algebra ideals.

Theorem 6. *Let $\boldsymbol{V}$ be a variety with a constant* 0. *The following conditions are equivalent*:

(1) *$\boldsymbol{V}$ is ideal determined (i.e. 0-regular and congruence permutable at* 0);

(2) *there exist an integer $m \geqq 1$ and binary polynomials $s, d_1, \ldots, d_m$ such that*
$s(x, 0) = x$,
$s(x, x) = d_1(x, x) = \ldots = d_m(x, x) = 0$,
and the implication
$d_1(x, y) = \ldots = d_m(x, y) = 0 \Rightarrow x = y$
holds in $\boldsymbol{V}$.

(3) *There exist an integer* $m \geqq 1$, *binary polynomials* $s, d_1, \ldots, d_m$, *and ternary polynomials* $q_1, \ldots, q_m$ *such that*

$$s(x, 0) = x,$$
$$s(x, x) = d_1(x, x) = \ldots = d_m(x, x) = 0,$$
$$q_1(x, y, 0) = x,$$
$$q_i(x, y, d_i(x, y)) = q_{i+1}(x, y, 0),\ 1 \leqq i < m,$$
$$q_m(x, y, d_m(x, y)) = y$$

hold in $\boldsymbol{V}$;

(4) *There exist integers* $n > m \geqq 1$, *binary polynomials* $d_1, \ldots, d_m$, *and* $(3 + m)$-*ary polynomials* $p_1, \ldots, p_n$ *such that*

$$p_1(x, y, 0, 0, \ldots, 0) = 0,$$
$$p_1(x, y, y, d_1(x, y), \ldots, d_m(x, y)) = x,$$
$$p_i(x, y, y, d_1(x, y), \ldots, d_m(x, y)) = p_{i+1}(x, y, y, 0, \ldots, 0),\ 1 \leqq i < n,$$
$$p_n(x, y, y, d_1(x, y), \ldots, d_m(x, y)) = y$$

hold in $\boldsymbol{V}$.

Proof. [8] and [10] together give (1) ⇔ (2). Combining [7] and [10] we find that (1) ⇔ (3). Hence it remains to verify the equivalence (1) ⇔ (4):

(4) ⇔ (1). It is a routine to check that $d_1(x, y) = \ldots = d_m(x, y) = 0$ imply $x = y$ which proves the 0-regularity of $\boldsymbol{V}$. Congruence permutability at 0 is ensured by the binary polynomial $s(x, y) = p_1(x, y, 0, d_1(x, y), \ldots, d_m(x, y))$.

(2) ⇒ (4). Put $n = m + 1$ and $p_j(x, y, z, u_1, \ldots, u_m) = q_{j-1}(x, y, u_{j-1})$, $1 < j \leqq n$. Further, let A be a $\boldsymbol{V}$ free algebra with free generators x and y. As shown in [10], the elements 0 and x belong to the ideal generated by a subset $\{y, d_1(x, y), \ldots, d_m(x, y)\}$ of A. Consequently, there exists a $(3 + m)$-ary polynomial p_1 (an ideal polynomial in the terminology of [10]) such that

$$p_1(x, y, 0, 0, \ldots, 0) = 0 \text{ and}$$
$$p_1(x, y, y, d_1(x, y), \ldots, d_m(x, y)) = x.$$

The proof is complete.

Examples 3. (1) Any variety of BCK-algebras is ideal determined. We have already proved that any BCK-algebra is congruence permutable at 0. Now define $d_1(x, y) = x * y$ and $d_2(x, y) = y * x$. Then $d_1(x, x) = d_2(x, x) = x * x = 0$. Conversely, $x * y = 0$ and $y * x = 0$ imply $x \leqq y$ and $y \leqq x$, i.e. $x = y$.

(2) Following W. H. Cornish [6; p. 484], a 0-regular variety with characterizing polynomials $d_1, \ldots, d_m$ (see Theorem 7) is called *strongly* 0-*regular* whenever $d_j(x, 0) = x$ for some $j \in \{1, \ldots, m\}$. Clearly, any strongly 0-regular variety is ideal determined.

Apparently, congruence equalities are closely related to their variations at 0 on 0-regular algebras:

Lemma 3. *Let A be a 0-regular algebra, π_1, π_2 two lattice polynomials on* Con A. *Then $\pi_1 = \pi_2$ is satisfied in* Con A *iff $[0]\,\pi_1 = [0]\,\pi_2$ holds on A.*

Proof. Suppose $[0]\,\pi_1 = [0]\,\pi_2$. Then $[0]\,\pi_1 \times [0]\,\pi_1 = [0]\,\pi_2 \times [0]\,\pi_2$ and so $\pi_1 = \Theta([0]\,\pi_1 \times [0]\,\pi_1) = \Theta([0]\,\pi_2 \times [0]\,\pi_2) = \pi_2$. The converse implication is trivial.

Remark 4. Notice that Lemma 3 does not hold whenever the operation of the relation product is admitted in π_1, π_2. Counterexample: BCK-algebras are congruence permutable at 0 but not congruence permutable in general, see e.g. [6] and references there.

Now we want to apply some previous results to ideal lattices of algebras from ideal determined varieties. Recall that in this case the ideals coincide with the 0-classes of congruences. Hence the meet of ideals $[0]\,\Psi$, $[0]\,\Phi$ is given by an evident formula $[0]\,\Psi \wedge [0]\,\Phi = [0]\,\Psi \wedge \Phi$. The dual statement does not hold, consider e.g. the maximal lattice congruences on the three element chain. The description of $[0]\,\Psi \vee \vee\, [0]\,\Phi$ is given in

Lemma 4. *Let A be an algebra with a constant* 0 *and let $\Psi, \Phi \in$* Con A. *Then*

(a) $[0]\,\Psi \vee [0]\,\Phi = [0]\,\Theta([0]\,\Psi \times [0]\,\Psi \cup [0]\,\Phi \times [0]\,\Phi)$;

(b) $[0]\,\Psi \vee [0]\,\Phi = [0]\,\Psi \vee \Phi$ *whenever A is* 0*-regular.*

Proof. (a) For the sake of brevity put $B = [0]\,\Psi$ and $C = [0]\,\Phi$. Denote by Ξ the congruence on A for which $B \vee C = [0]\,\Xi$. Then $\Xi \supseteq (B \vee C) \times (B \vee C) \supseteq \supseteq (B \cup C) \times (B \cup C) \supseteq B \times B \cup C \times C$ and so $\Xi \supseteq \Theta(B \times B \cup C \times C)$. In this way we find that $\Xi \supseteq \Theta([0]\,\Psi \times [0]\,\Psi \cup [0]\,\Phi \times [0]\,\Phi)$; the converse inclusion is trivial.

(b) It suffices to prove the inclusion $[0]\,\Psi \vee \Phi \subseteq [0]\,\Psi \vee [0]\,\Phi$. By hypothesis, $\Psi = \Theta([0]\,\Psi \times [0]\,\Psi)$ and $\Phi = \Theta([0]\,\Phi \times [0]\,\Phi)$. Consequently, $\Psi \vee \Phi \subseteq \subseteq \Theta([0]\,\Psi \times [0]\,\Psi \cup [0]\,\Phi \times [0]\,\Phi)$ and so $[0]\,\Psi \vee \Phi \subseteq [0]\,\Theta([0]\,\Psi \times [0]\,\Psi \cup \cup\, [0]\,\Phi \times [0]\,\Phi) = [0]\,\Psi \vee [0]\,\Phi$, by part (a) of this lemma. The proof is complete.

Since the 0-regular varieties are congruence modular, the ideal lattice of any algebra from an ideal determined variety satisfies the modular law. Combining Theorem 4 (3) with [8] we get a characterization of the ideal determined varieties whose algebras have distributive ideal lattices.

Theorem 7. *Let V be a variety with a constant* 0. *The following conditions are equivalent*:

(1) *V is ideal determined and each $A \in V$ has a distributive ideal lattice*;

(2) *V is* 0*-regular and arithmetical at* 0;

(3) *there exist an integer $m \geqq 1$ and binary polynomials $b, d_1, \ldots, d_m$ such that*
$b(x, 0) = x$,
$b(0, x) = b(x, x) = d_1(x, x) = \ldots = d_m(x, x) = 0$,

and the implication

$d_1(x, y) = \ldots = d_m(x, y) = 0 \Rightarrow x = y$

holds in V.

Examples 4. (1) Any variety of BCK-algebras has all the properties mentioned in Theorem 7, see Example 1 (1) and Example 3 (1).

(2) Only the trivial variety of Abelian groups is arithmetical at 0 (and hence arithmetical): The binary polynomial b from Theorem 4 (3) can be written in the form $b(x, y) = mx + ny$ for some $m, n \in \boldsymbol{Z}$. Then $b(x, x) = mx + nx = 0$, $b(0, x) = nx = 0$ and $b(x, 0) = mx = x$ imply $x = 0$, as claimed.

References

[1] *Baker, K. A., Pixley, A. F.:* Polynomial interpolation and the Chinese remainder theorem for algebraic systems. Math. Z. *143* (1975), 165—174.

[2] *Birkhoff, G.:* Lattice Theory, 3rd ed., American Math. Soc., Providence, 1979.

[3] *Bosbach, B.:* Komplementäre Halbgruppen, ein Beitrag zur Idealtheorie kommutativer Halbgruppen. Math. Ann. *161* (1965), 279—295.

[4] *Chajda, I.:* Congruence distributivity in varieties with constants. Arch. Math. (Brno), *22* (1986), 121—124.

[5] *Chajda, I.:* Localization of some congruence conditions in varieties with nullary operations. Preprint.

[6] *Cornish, W. H.:* 3-permutability and quasicommutative BCK-algebras. Math. Japonica *25*, (1980), 477—496.

[7] *Duda, J.:* On two schemes applied to Mal'cev type theorems. Ann. Univ. Sci. Budapest, Sectio Mathematica, *26* (1983), 39—45.

[8] *Fichtner, K.:* Eine Bemerkung über Mannigfaltigkeiten universeller Algebren mit Idealen. Monatsh. d. Deutsch. Akad. d. Wiss. Berlin *12* (1970), 21—25.

[9] *Grätzer, G.:* Universal Algebra, 2nd expanded ed., Springer-Verlag, Berlin—Heidelberg—New York 1979.

[10] *Gumm, H.-P., Ursini, A.:* Ideals in universal algebras. Algebra Univ. *19* (1984), 45—54.

[11] *Iséki, K.:* Some problems of BCK-algebras and Griss algebras. Universal Algebra and applications, Banach Center Publ. *9* (1982), 423—430.

[12] *Jónsson, B.:* Algebras whose congruence lattices are distributive. Math. Scand. *21* (1967), 110—121.

[13] *Mal'cev, A. I.:* On the general theory of algebraic systems (Russian). Math. Sbornik (New Series) *35* (77), (1954), 3—20.

[14] *Mitschke, A.:* Near unanimity identities and congruence distributivity in equational classes. Algebra Univ. *8* (1978), 29—32.

[15] *Papert, D.:* Congruence relations in semilattices. J. London Math. Soc. *39* (1964), 723—729.

[16] *Pixley, A. F.:* Distributivity and permutability of congruence relations in equational classes of algebras. Proc. Amer. Math. Soc. *14* (1963), 105—109.

[17] *Pixley, A. F.:* Characterizations of arithmetical varieties. Algebra Univ. *9* (1979), 87—98.

[18] *Werner, H.:* A Mal'cev condition for admissible relations. Algebra Univ. *3* (1973), 263.

[19] *Werner, H.:* Discriminator Algebras, Studien zur Algebra und ihre Anwendungen 6, Akademie-Verlag, Berlin 1978.

Author's address: 616 00 Brno 16, Kroftova 21, Czechoslovakia.