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SMOOTHING EFFECT AND REGULARITY FOR LINEAR PARABOLIC EQUATIONS

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INTRODUCTION

Regularity of weak solutions for linear parabolic equations has been studied in the monographs by J. L. Lions, E. Magenes [7] (using the Fourier transformation technique and interpolation theory), O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva [6], A. Friedman [1], K. Rektorys [9] etc. Smoothing effect for linear parabolic equations has been considered by P. E. Sobolevskij [10], A. Friedman [1] (using semi-group theory), etc.

The aim of this paper is to obtain regularity of the weak solution in the interior of the domain by a simple technique using the regularity results of the elliptic equations theory. To this purpose we discretize the time variable and apply the technique of Rothe's method which allows to carry over the regularity results from elliptic to parabolic equations. Since our data for $t = 0$ are not regular, we prove some a priori estimates for $t > 0$ which imply the smoothing effect for the weak solution. The idea of deriving such a priori estimates is due to K. Rektorys – see Remark 3.2. The regularity results obtained are comparable with those in [7] restricted to the interior of the domain. The results on the smoothing effect are comparable with those obtained in [10] (see Remark 3.3). We assume a lesser degree of regularity of $\partial\Omega$ and of the coefficients a_{ij} (in the x variable) in the elliptic operator A, since we consider $A: V \to V^*$ instead of $A: D(A) \subset H \to H$ (see [10]). Under the regularity assumptions on the data for $t = 0$ we prove in Section 2 the regularity of $(4k, 2)$-type in (x, t)-variables (the elliptic operator being of the order $2k$). In Section 3 we establish the smoothing effect for $t > 0$. Using the procedure of Section 2 and the estimates of the smoothing effect we obtain in Section 4 regularity with respect to the (x, t)-variables. In Section 5 the regularity results are applied to the convergence of Rothe's method.

1. ASSUMPTIONS AND AUXILIARY RESULTS

Let V, H be real Hilbert spaces with the norms $\|\cdot\|$, $|\cdot|$ and the duals V^*, H^*, respectively. We assume the imbedding $V \hookrightarrow H$ to be dense and continuous. By (f, v) we denote the duality between $f \in V^*$, $v \in V$ which coincides with the scalar product in H provided $f \in H$. By $\rightarrow$ ($\rightharpoonup$) we denote strong (weak) convergence. The symbol C stands for nonnegative constants. We allow different values of C in the same discussion.

We consider $u \in L_2(I, V) \cap C(I, H)$ with $\mathrm{d}u/\mathrm{d}t \in L_2(I, V^*)$ to be a weak solution of the linear abstract parabolic equation

$$(1.1) \qquad \left(\frac{\mathrm{d}u(t)}{\mathrm{d}t}, v\right) + a(t; u, v) = (f(t), v) \quad \text{for all} \quad v \in V; \quad u(0) = u_0 \in H$$

where $a(t; u, v)$ is a continuous bilinear form in $u, v \in V$ for $t \in I \equiv (0, T)$, $T < \infty$ and $f(t) \in V^*$ for $t \in I$. The existence of such a weak solution is proved (under the corresponding regularity assumptions on $a(t; u, v)$ and $f(t)$ in t), e.g. in [6], [9], [1], [10], etc.

The identity (1.1) is an abstract formulation of the weak solution for linear parabolic initial-boundary value problems.

Example 1.1. Consider the equation

$$(1.2) \qquad \frac{\partial u}{\partial t} + A(t)\, u = f_0(x, t) \quad \text{for} \quad (x, t) \in Q = \Omega \times I$$

where $\Omega \subset R^N$ is a bounded domain with boundary $\partial\Omega$,

$$A(t)\, u = \sum_{|i|,|j| \leq k} (-1)^{|i|}\, D^i(a_{ij}(x, t)\, D^j u)\,, \quad a_{ij} \in L_\infty(Q)$$

with the corresponding boundary and initial conditions

$$(1.3) \qquad B_i u = g_i(t)\,, \quad i = 1, \ldots, \mu \quad \text{on} \quad \partial\Omega \times I\,,$$

$$(1.4) \qquad N_j(t)\, u = h_j(t)\,, \quad j = 1, \ldots, k - \mu \quad \text{on} \quad \partial\Omega \times I\,,$$

$$(1.5) \qquad u(x, 0) = u_0(x) \quad \text{on} \quad \Omega\,.$$

Let the conditions (1.3) be stable (of Dirichlet's type) and the conditions (1.4) (of Neumann's type) – see [8], [7], where B_i, $N_j(t)$ are linear differential operators (B_i, N_j are normal and cover A). A weak solution u of the corresponding elliptic problem $A(t)\, u = f_0(x, t)$ in Ω (t being a fixed parameter) with the boundary conditions (1.3), (1.4) is defined in the following way (see [8], [9]). Let $w(t) \in W_2^k(\Omega)$ satisfy (1.3) and let $V \subset W_2^k(\Omega)$ be a subspace where $V = \{v \in W_2^k(\Omega);\ B_i v = 0 \text{ on } \partial\Omega$ for $i = 1, \ldots, \mu\}$. A continuous bilinear form $b(t; u, v)$ on $W_2^k \times W_2^k$ (corresponding to (1.3), (1.4) and satisfying $b(t; u, v) = 0$ if u or v is from $\mathring{W}_2^k$) and $f(t) \in V^*$ can be constructed so that the weak solution u satisfies $u - w(t) \in V$ and

$$a(t; u, v) \equiv \sum_{|i|,|j| \leq k} \int_\Omega a_{ij}(x, t)\ D^j u D^i v\, \mathrm{d}x + b(t; u, v) = (f(t), v)$$

for all $v \in V$ where $(f(t), v) = \int_\Omega f_0(x, t)\, v \,\mathrm{d}x + (h(t), v)$ with $h(t) \in V^*$ $((h(t), v) = 0$ whenever $v \in \mathring{W}_2^k(\Omega))$.

Similarly, we define a weak solution u of (1.2)–(1.5) as $u = \bar{u} + w(t)$ where $\bar{u}$ satisfies

$$\left(\frac{\mathrm{d}\bar{u}(t)}{\mathrm{d}t}, v\right) + a(t; \bar{u}(t), v) = (f(t), v)\,, \quad \bar{u}(0) = u_0 - w(0)$$

for all $v \in V$ and a.e. $t \in I$ where

$$(f(t), v) = \int_\Omega f_0(x, t)\, v \,\mathrm{d}x - a(t; w(t), v) - \left(\frac{\mathrm{d}w(t)}{\mathrm{d}t}, v\right) + (h(t), v)\,,$$

which corresponds to (1.1) with $H = L_2(\Omega)$. If the weak solution u of (1.2)–(1.5) is sufficiently smooth then it is also a classical solution of (1.2)–(1.5) (for the details see [8]). In particular, if $\mu = k$, $V = \mathring{W}_2^k(\Omega)$ and $B_i = (\partial/\partial \nu)^i$, $i = 0, \ldots, k - 1$ (ν is the outward normal to $\partial\Omega$), $b \equiv 0$, $h(t) \equiv 0$ then we have the Dirichlet boundary conditions (1.3). If $k = 1$, $V = W_2^1(\Omega)$, $\mu = 0$,

$$b(t; u, v) = \int_{\partial\Omega} a(s, t)\, uv \,\mathrm{d}s \quad \text{and} \quad (f(t), v) = \int_\Omega f_0(x, t)\, v \,\mathrm{d}x +$$

$$+ \int_{\partial\Omega} \phi(s, t)\, v(s) \,\mathrm{d}s \quad (a, \phi \in L_\infty(\partial\Omega))$$

then we have Neumann's type boundary conditions (1.4),

$$\frac{\partial u}{\partial \nu_A} + a(s, t)\, u = \phi(s, t) \quad \text{on} \quad \partial\Omega \times I$$

where

$$\frac{\partial u}{\partial \nu_A} = \sum_{|j| \leqq 1, |i| = 1} a_{ij}(x, t) \cos(\nu, x_i) \frac{\partial u}{\partial x_j}\,, \quad (h(t), v) = \int_{\partial\Omega} \phi(s, t)\, v(s) \,\mathrm{d}s\,.$$

We shall assume that $a(t; u, v)$ is Lipschitz continuous in t, i.e.

(1.6) $$|a(t; u, v) - a(t'; u, v)| \leqq C|t - t'|\, \|u\|\, \|v\| \quad \text{for all} \quad t, t' \in I\,, \quad u, v \in V\,.$$

We shall need a perturbed symmetry of $a(t; u, v)$ in u, v in the following form:

There exist $K \geqq 0$ and a continuous bilinear form $a_0(t; u, v)$ in $u, v \in V$, $t \in I$ satisfying $|a_0(t; u, v)| \leqq C\|u\|\, |v|$ and $|a_0(t; u, v) - a_0(t'; u, v)| \leqq C|t - t'|\, \|u\|\, |v|$ such that $\bar{a}(t; u, v) = a(t; u, v) + a_0(t; u, v) + K(u, v)$ is an equivalent scalar product in V uniformly for $t \in I$, i.e.,

(1.7) $$\bar{a}(t; u, v) = \bar{a}(t; v, u) \quad \text{and} \quad C_1\|u\|^2 \leqq \bar{a}(t; u, u) \leqq C_2\|u\|^2 \quad \text{for all} \quad u, v \in V\,, \quad t \in I\,.$$

By means of $a_0(t; u, v)$, we symmetrize the bilinear form $a(t; u, v)$ in lower order terms. If $a(t; u, v)$ satisfies (1.6) then we put $a_0 \equiv 0$, $K = 0$.

We shall use the functional spaces $L_2(I, V)$, $L_\infty(I, V)$ for measurable abstract functions $u: I \to V$ ($u \in L_2(I, V)$ if

$$\int_I \|u(t)\|^2 \, \mathrm{d}t < \infty \quad \text{and} \quad u \in L_\infty(I, V) \quad \text{if} \quad \operatorname*{ess\,sup}_{t \in I} \|u(t)\| < \infty$$

(for the details see [5]). The space of continuous functions $u: I \to V$ is denoted by $C(I, V)$. Let $q \geqq 0$ be an integer. Denote by $H^q(I, H)$ the B-space

$$H^q(I, H) = \left\{ f \in L_2(I, H);\ \frac{\mathrm{d}^i f}{\mathrm{d}t^i} \in L_2(I, H) \quad \text{for} \quad i = 1, \ldots, q \right\}$$

with the norm $\|f\|_{H^q(I,H)} = \left(\sum_{i=0}^{q} \|\mathrm{d}^i f/\mathrm{d}t^i\|^2_{L_2(I,H)} \right)^{1/2}$ where $\mathrm{d}^i f/\mathrm{d}t^i$ are taken in the sense of distributions on I with the values in H (i.e., $\mathrm{d}^i f/\mathrm{d}t^i \in \mathscr{D}'(I, H)$ and $\mathscr{D}'(I, H) = \mathscr{L}(\mathscr{D}(I), H)$ – see [7]).

To obtain the $(4k, 2)$-type regularity (in Q) we shall assume:

(1.8) $$f \in H^2(I, V^*)\,;$$

there exists $z_0 \in V$ such that

(1.9) $$(z_0, v) + a(0; u_0, v) = (f(0), v) \quad \text{for all} \quad v \in V$$

(u_0 is from (1.1)) and

(1.10) $$(f(t), v) = (F(t), v) \quad \text{for all} \quad v \in \mathscr{D}(\Omega) \quad \text{with} \quad F \in L_2(I, W_2^{2k})\,, \quad \frac{\mathrm{d}F}{\mathrm{d}t} \in L_2(I, L_2)$$

where $\mathscr{D}(\Omega)$ is the subset of functions in $C^\infty(\Omega)$ with a compact support in Ω.

Frequently we shall use the following regularity result from the elliptic equations theory (see [8]).

Theorem 1.1. *Let $l \geqq 1$ be an integer and let $t \in I$ be fixed. Let $a(t; u, u) \geqq C_1\|u\|^2 - C_2|u|^2$ for all $u \in \mathring{W}_2^k$ (see Example 1.1). Suppose that (1.11)–(1.13) are satisfied where*

(1.11) *$a_{ij}(x, t) \in C^{r_i,1}(\bar{\Omega})$ where $r_i = \max\{0, |i| + l - k - 1\}$ for all $|i|, |j| \leqq k$ (here $C^{r,1}(\bar{\Omega})$ is the set of all $v \in C(\bar{\Omega})$ for which $D^i v$ are Lipschitz continuous in $\bar{\Omega}$ for all $|i| \leqq r$);*

(1.12) *$a(t; u, v) = (G, v)$ for all $v \in \mathscr{D}(\Omega)$;*

(1.13) *$D^i G \in W_2^{-k+l}$ for all $|i| \leqq l - 1$ ($W_2^{-k} = (W_2^k)^*$).*

Then $u \in W_{2,\mathrm{loc}}^{k+l}$ and the estimate

$$\|u\|_{W_2^{k+l}(\Omega')} \leqq C(\Omega') \left(\|u\| + \sum_{|i| \leqq l-1} \|D^i G\|_{W_2^{-k+1}} \right)$$

holds for any $\Omega' \subset \Omega$ with $\bar{\Omega}' \subset \Omega$.

By $C^{r,1;s,1}(\bar{Q})$ we denote the set of all $v \in C(\bar{Q})$ for which $D_t^p D_x^j v$ are Lipschitz continuous in $\bar{Q}$ for all $|j| \leqq r$ and $0 \leqq p \leqq s$. We shall assume

(1.14) $a_{ij}(x, t) \in C^{r_{i,p},1;p,1}(\bar{Q})$ for all $|i|, |j| \leqq k$ and $p = 0, \ldots, q$ where $r_{i,p} = \max\{0, |i| + l_p - k - 1\}$ and $q \geqq 0$ is an integer.

The numbers l_p will be specified in the sequel.

To apply Theorem 1.1 to parabolic equations we use time discretization (1.1) in the form

$$\left(\frac{u - u_{i-1}}{h}, v\right) + a(t_i; u, v) = (f_i, v) \quad \text{for all} \quad v \in V \tag{1.15}$$

where $u = u_i$ is a (weak) solution of the elliptic equation, $h = T/n$,

$$f_i = \frac{1}{h}\int_{t_{i-1}}^{t_i} f(s)\,\mathrm{d}s$$

(Bochner's integral – see [2], [11]) and $i = 1, \ldots, n$ provided u_j $(j = 1, \ldots, i-1)$ are known. The existence of $u_i \equiv u_{i,n} \in V$ satisfying (1.15) is (for $h \leqq h_0$) a consequence of (1.7) and of the Lax-Milgram Lemma (see [8]). We denote

$$\delta_h u_i = \frac{u_i - u_{i-1}}{h} \; (i = 1, \ldots, n) \quad \text{and} \quad u_i^{(p)} \equiv \delta_h^p u_i = \delta_h^{p-1}(\delta_h u_i)\,, \quad \delta_h^0 u_i = u_i\,.$$

We construct

$$U_n^{(p)}(t) = u_{i-1}^{(p)} + (t - t_{i-1})\,h^{-1}(u_i^{(p)} - u_{i-1}^{(p)}) \quad \text{for} \quad t_{i-1} \leqq t \leqq t_i \tag{1.16}$$

$(i = p + 1, \ldots, n)$, and the corresponding step function

$$\bar{U}_n^{(p)}(t) = u_i \quad \text{for} \quad t_{i-1} < t \leqq t_{i-1}\,, \quad i = p + 1, \ldots, n\,. \tag{1.17}$$

In the case $p = 0$ we denote $u_n(t) = U_n^{(0)}(t)$ (Rothe's function) and $\bar{u}_n(t) \doteq \bar{U}_n^{(0)}(t)$. Similarly for $f \in L_2(I, V^*)$ (or $f \in L_2(I, H)$) we construct f_i, $f_n^{(p)}(t)$ and $\bar{f}_n^{(p)}(t)$. Frequently we shall make use of the following lemma.

Lemma 1.1. *If* $f \in H^q(I, H)$ $(q \geqq 0)$ *then*

(i) $\displaystyle\sum_{i=p+1}^{n} \|\delta_h^p f_i\|_H^2\, h \leqq C \int_I \left\|\frac{\mathrm{d}^p f(s)}{\mathrm{d}s^p}\right\|_H^2 \mathrm{d}s$ *for all* $p = 0, \ldots, q$;

(ii) $\displaystyle\int_{(p+1)\frac{T}{n}}^{T} \left\|\bar{f}_n^{(p)}(t) - \frac{\mathrm{d}^p f(t)}{\mathrm{d}t^p}\right\|_H^2 \mathrm{d}t \to 0$ *for* $n \to \infty$ *where* $p = 0, \ldots, q$.

Proof. From the properties of Bochner's integral (see [2], [11]) we have

$$\delta_h f_i = \frac{1}{h^2}\int_{t_{i-1}}^{t_i}\int_{s-h}^{s} \frac{\mathrm{d}f(z)}{\mathrm{d}z}\,\mathrm{d}z\,\mathrm{d}s\,, \quad i = 2, \ldots, n$$

and

$$\delta_h^p f_i = h^{-(p+1)}\int_{t_{i-1}}^{t_i}\int_{s_1-h}^{s_1}\ldots\int_{s_p-h}^{s_p} \frac{\mathrm{d}^p f(z)}{\mathrm{d}z^p}\,\mathrm{d}z\,\mathrm{d}s_p \ldots \mathrm{d}s_1$$

for $n \geqq i \geqq p + 1$. Hence we obtain

$$\|\delta_h^p f_i\|_H \leqq h^{-(p+1)} \int_{t_{i-1}-ph}^{t_i} \int_{t_{i-1}-ph}^{s_1} \dots \int_{t_{i-1}-ph}^{s_p} \left\| \frac{d^p f(z)}{dz^p} \right\|_H dz\, ds_p \dots ds_1 =$$

$$= h^{-(p+1)} \int_{t_{i-1}-ph}^{t_i} \frac{(t_i - z)^p}{p!} \left\| \frac{d^p f(z)}{dz^p} \right\|_H dz \leqq C \left(\frac{1}{h} \int_{t_{i-1}-ph}^{t_i} \left\| \frac{d^p f(z)}{dz^p} \right\|_H^2 dz \right)^{1/2}$$

which implies Assertion (i). Similarly as above, we have

$$\bar{f}_n^{(p)}(t) - \frac{d^p f(t)}{dt^p} = h^{-(p+1)} \int_{t_{i-1}}^{t_i} \int_{s_1-h}^{s_1} \dots \int_{s_p-h}^{s_p} \left(\frac{d^p f(z)}{dz^p} - \frac{d^p f(t)}{dt^p} \right) dz\, ds_p \dots ds_1$$

for $t \in (t_{i-1}, t_i)$, $i = p + 1, \dots, n$ and

$$\int_{(p+1)\frac{T}{n}}^{T} \left\| \bar{f}_n^{(p)}(t) - \frac{d^p f(t)}{dt^p} \right\|_H^2 dt \leqq C \sup_{|z| \leqq (p+1)h} \int_0^T \left\| \frac{d^p f(t)}{dt^p} - \frac{d^p f(t+z)}{dt^p} \right\|_H^2 dt \to 0$$

as $n \to \infty$ since $\|u(t + h) - u(t)\|_{L_2(I,H)} \to 0$ with $h \to 0$ for $u \in L_2(I, H)$ (see [2]). Thus, the proof is complete.

Also we shall use the following modification of the Arzelà-Ascoli Theorem (see e.g. [5], [4]).

Lemma 1.2. *Let $V \subset\subset H$ be compact. If the estimates*

$$\left| \frac{du_n(t)}{dt} \right| \leqq C \quad \left(\text{or} \int_I \left| \frac{du_n(t)}{dt} \right|^2 dt \leqq C \right); \quad \|\bar{u}_n(t)\| \leqq C \quad (\forall t \in I)$$

hold for all $n \geqq n_0 > 0$ then there exist $u \in L_\infty(I, V) \cap C(I, H)$ with

$$\frac{du}{dt} \in L_\infty(I, H) \quad \left(\frac{du}{dt} \in L_2(I, H)\,, \ \textit{respectively} \right)$$

and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \to u \ \textit{in} \ C(I, H)\,, \quad u_{n_k}(t) \rightharpoonup u(t)\,, \quad \bar{u}_{n_k}(t) \rightharpoonup u(t) \ \textit{in} \ V$$

for all $t \in I$, and $(d/dt)\, u_{n_k} \rightharpoonup (du/dt)$ in $L_2(I, H)$. Moreover, $u: I \to H$ is strongly differentiable for a.e. $t \in I$.

The above assumed a priori estimates guarantee the equiboundedness of $\{u_n\}$ in V (and hence in H) and equicontinuity in H. Applying the Arzelà-Ascoli Theorem and the reflexivity of V, H, $L_2(I, H)$ we obtain the required assertions. Differentiability of $u: I \to H$ follows from

$$u(t) = u_0 + \int_0^t g(s)\, ds = u_0 + \int_0^t \frac{du(s)}{ds}\, ds \quad \left(g = \frac{du}{dt} \ \text{in} \ L_2(I, H) \right)$$

(du/dt is in the sense of $\mathscr{D}'(I, H)$) and from the properties of Bochner's integral (see [2], [11]).

In this section we prove the regularity $u \in L_2(I, W^{4k}_{2,\mathrm{loc}})$, $\mathrm{d}u/\mathrm{d}t \in L_2(I, W^{2k}_{2,\mathrm{loc}})$, $\mathrm{d}^2u/\mathrm{d}t^2 \in L_2(I, L_2)$ for the weak solution of (1.1) where $V, H, f, a(t; u, v)$ are the same as in Example 1.1.

Remark 2.1. By means of $a_0(t; u, v)$ in (1.7) we symmetrize the form $a(t; u, v)$ in some lower order terms. So we can assume that the coefficients $a^*_{ij}(t, x)$ in the form $a_0(t; u, v)$ coincide with some $a_{\alpha\beta}(t, x)$ in the form $a(t; u, v)$. Thus, in the sequel, we assume that $a^*_{ij}(x, t)$ satisfy (1.14), whence a_{ij} satisfy (1.14).

Theorem 2.1. *Suppose* (1.7)–(1.10) *and* (1.14) *for* $q = 1$, $l_0 = 3k$, $l_1 = k$. *If* $u_0 \in W^{4k}_2(\Omega)$, *then the solution* u *of* (1.1) *satisfies*

$$u \in L_2(I, W^{4k}_{2,\mathrm{loc}}) \cap L_\infty(I, W^k_2)\,, \quad \frac{\mathrm{d}u}{\mathrm{d}t} \in L_2(I, W^{2k}_{2,\mathrm{loc}}) \cap L_\infty(I, W^k_2) \ \textit{and} \ \frac{\mathrm{d}^2u}{\mathrm{d}t^2} \in L_2(I, L_2)\,.$$

Proof. From (1.15), where $u = u_i$, $\delta_h u_i = h^{-1}(u_i - u_{i-1})$ we obtain

$$(\delta^2_h u_i, v) + a(t_i; \delta_h u_i, v) + \delta_h a(t_i; u_{i-1}, v) = (\delta_h f_i, v) \tag{2.1}$$

for all $i = 2, \ldots, n$, $v \in V$, where $t_i = ih$,

$$f_i = \frac{1}{h}\int_{t_{i-1}}^{t_i} f(s)\,\mathrm{d}s$$

and $\delta_h a(t; u, v) = h^{-1}(a(t; u, v) - a(t - h; u, v))$. Using (1.9) with $z_0 = \delta_h u_0$ and (1.15) (for $u = u_1$, $i = 1$), we obtain (2.1) also for $i = 1$. Now, we put $v \approx \delta_h u_i - \delta_h u_{i-1}$ in (1.15) and then we sum it up for $i = 1, \ldots, j$. Owing to (1.7) we have

$$\bar{a}(t; v, v - w) \geqq \tfrac{1}{2}\bar{a}(t; v, v) - \tfrac{1}{2}\bar{a}(t; w, w)$$

and hence

$$\sum_{i=1}^{j} |\delta^2_h u_i|^2 h + \sum_{i=1}^{j} (\bar{a}(t_i; \delta_h u_i, \delta_h u_i) - \bar{a}(t_{i-1}; \delta_h u_{i-1}, \delta_h u_{i-1})) \leqq$$

$$\leqq C_1 \sum_{i=0}^{j} \|\delta_h u_i\|^2 h + C_2 \sum_{i=1}^{j} \|\delta^2_h f_i\|^2_* h + C_3 \sum_{i=1}^{j} |\delta_h u_i|\, |\delta_h u_i - \delta_h u_{i-1}| +$$

$$+ \sum_{i=1}^{j} a_0(t_i; \delta_h u_i, \delta_h u_i - \delta_h u_{i-1}) + \sum_{i=1}^{j} \delta_h a(t_i; u_{i-1}, \delta_h u_i - \delta_h u_{i-1}) + C_4\,.$$

Hence and from the estimates

$$\Big|\sum_{i=1}^{j} a_0(t_i; \delta_h u_i, \delta_h u_i - \delta_h u_{i-1})\Big| \leqq C \sum_{i=1}^{j} \|\delta_h u_i\|\, |\delta^2_h u_i|\, h \leqq$$

$$\leqq \frac{C\varepsilon}{2} \sum_{i=1}^{j} |\delta^2_h u_i|^2 h + \frac{C}{2\varepsilon} \sum_{i=1}^{j} \|\delta_h u_i\|^2 h\,,$$

$$\sum_{i=1}^{j} \delta_h a(t_i; u_{i-1}, \delta_h u_i - \delta_h u_{i-1}) = -\sum_{i=1}^{j-1} (\delta_h a(t_{i+1}; \delta_h u_i, \delta_h u_i) +$$

$$+ \delta_h^2 a(t_{i+1}; u_{i-1}, \delta_h u_i)) h + \delta_h a(t_j; u_{j-1}, \delta_h u_j) - \delta_h a(t_1; u_0, \delta_h u_0),$$

$$|\delta_h a(t_j; u_{j-1}, \delta_h u_j)| \leqq \frac{C}{2\varepsilon} \|u_{j-1}\|^2 + \frac{C\varepsilon}{2} \|\delta_h u_j\|^2,$$

$$|\delta_h a(t_{i+1}; \delta_h u_i, \delta_h u_i)| + |\delta_h^2 a(t_{i+1}; u_{i-1}, \delta_h u_i)| \leqq C(\|\delta_h u_i\|^2 + \|u_{i-1}\|^2)$$

$$\|u_l\|^2 \leqq C(\|u_0\|^2 + \sum_{i=1}^{l} \|\delta_h u_i\|^2 h)$$

and

$$|(\delta_h u_i, \delta_h u_i - \delta_h u_{i-1})| \leqq |\delta_h u_i| \, |\delta_h^2 u_i| \, h \leqq \frac{\varepsilon}{2} |\delta_h^2 u_i|^2 h + \frac{1}{2\varepsilon} |\delta_h u_i|^2 h$$

we conclude

$$\sum_{i=1}^{j} |\delta_h^2 u_i|^2 h + \|\delta_h u_j\|^2 \leqq C_1 + C_2 \sum_{i=1}^{j} \|\delta_h u_i\|^2 h$$

because of Lemma 1.1, $\|\delta_h u_0\| = \|z_0\| < \infty$. Thus, Gronwall's Lemma yields

$$\sum_{i=1}^{n} |\delta_h^2 u_i|^2 h + \|\delta_h u_i\| \leqq C \quad \text{for all} \quad n \geqq n_0, \quad i = 1, \ldots, n. \tag{2.2}$$

Now we construct $U_n^{(1)}(t)$ and $\bar{U}_n^{(1)}(t)$ by means of $z_i = \delta_h^1 u_i$ (see (1.16), (1.17)). We have $(d/dt)\, U_n^{(1)}(t) = \delta_h^2 u_i$ for $t_{i-1} < t < t_i$, $i = 1, \ldots, n$. Thus, Lemma 1.2 implies $U_n^{(1)}(t) \to Z$ in $C(I, L_2)$, $U_n^{(1)}(t) \rightharpoonup Z(t)$, $\bar{U}_n^{(1)}(t) \rightharpoonup Z(t)$ in V for all $t \in I$ $(d/dt)\, U_n^{(1)}(t) \rightharpoonup dZ/dt$ in $L_2(I, L_2)$ and $Z \in L_\infty(I, V)$ since $V \subset L_2$ is compact. Moreover, the estimate

$$|U_n^{(1)}(t) - U_n^{(1)}(t')| \leqq |t - t'|^{1/2} \left(\int_I \left| \frac{d}{ds} U_n^{(1)}(s) \right|^2 ds \right)^{1/2} \leqq$$

$$\leqq |t - t'|^{1/2} \left(\sum_{i=1}^{n} |\delta_h^2 u_i|^2 h \right)^{1/2} \leqq C |t - t'|^{1/2}$$

implies $\bar{U}_n^{(1)}(t) \to Z(t)$ in L_2 and $|\bar{U}_n^{(1)}(t)| \leqq C$ for all $t \in I$. Thus $\bar{U}_n^{(1)} \to Z$ in $L_2(I, L_2)$. On the other hand,

$$\bar{U}_n^{(1)}(t) = \frac{d}{dt} u_n(t) \quad \text{for a.e.} \quad t \in I$$

and

$$\frac{d}{dt} u_n \rightharpoonup \frac{du}{dt} \quad \text{in} \quad L_2(I, L_2)$$

(see Lemma 1.2) which implies $Z = du/dt$. Thus we have

$$\frac{du}{dt} \in L_\infty(I, W_2^k) \quad \text{and} \quad \frac{d^2 u}{dt^2} \in L_2(I, L_2).$$

The regularity of u with respect to x will be proved by means of Theorem 1.1 in the following way (see [4]). From (1.15) where $u = u_i$, $|\delta_h u_i| \leqq C$ and from Theorem

1.1 for $l = k$ we obtain

$$\|u_i\|^2_{W^{2k}_{2,\mathrm{loc}}} \leqq C(\Omega')\left(\|u_i\|^2 + |\delta_h u_i|^2 + |F_i|^2 \leqq C(\Omega')\right) \tag{2.3}$$

for all $n \geqq n_0$, $i = 1, \ldots, n$, where F is from (1.10) and $F_i = \int_{t_{i-1}}^{t_i} F \,\mathrm{d}s$. Due to (1.14) ($q = 1$, $l_0 = 3k$, $l_1 = k$) we have

$$\delta_h a(t_i; u_{i-1}, v) = (z_{i-1}, v) \quad \text{for all} \quad v \in \mathscr{D}(\Omega'),$$

where $z_{i-1} \in L_2(\Omega')$ and $|z_{i-1}|_{L_2} \leqq C(\Omega')$ for all n, $i = 1, \ldots, n$ because of Green's formula and (2.3). Then (2.1) and Theorem 1.1 yield

$$\|\delta_h u_i\|^2_{W^{2k}_{2,\mathrm{loc}}} \leqq C(\Omega'')\left(\|\delta_h u_i\|^2 + |z_{i-1}|^2_{L_2(\Omega')} + |\delta_h^2 u_i|^2 + |\delta_h F_i|^2\right)$$

where $\Omega'' \subset \Omega'$ with $\overline{\Omega''} \subset \Omega'$, $i = 1, \ldots, n$. Hence and from (2.2) we conclude

$$\int_I \|\bar{U}_n^{(1)}(t)\|^2_{W^{2k}_{2,\mathrm{loc}}} \,\mathrm{d}t \leqq C(\Omega'') \quad \text{for all} \quad n \geqq n_0\,. \tag{2.4}$$

Similarly as above, from (1.15) where $u = u_i$, $\delta_h u_i \in W_2^{2k}(\Omega'')$ and from Theorem 1.1 for $l = 3k$ we conclude

$$\|u_i\|^2_{W^{4k}_{2,\mathrm{loc}}} \leqq C(\Omega''')\left(\|u_i\|^2 + \|F_i\|^2_{W_2^{2k}} + \|\delta_h u_i\|^2_{W_2^{2k}(\Omega'')}\right)$$

for $i = 1, \ldots, n$, which implies

$$\int_I \|\bar{u}_n(t)\|^2_{W^{4k}_{2,\mathrm{loc}}} \,\mathrm{d}t \leqq C(\Omega''') \quad \text{for all} \quad n \geqq n_0 \tag{2.5}$$

because of (2.4) and Lemma 1.1. Assumption (1.10) implies $F_i \in W_2^{2k}$ for all $i = 1, \ldots, n$. The a priori estimate (2.5) implies $\bar{u}_n \rightharpoonup g$ in $L_2(I, W_2^{4k}(\Omega'''))$. On the other hand, $\bar{u}_n \to u$ in $L_2(I, L_2(\Omega'''))$ and hence $u = g$. Thus $u \in L_2(I, W^{4k}_{2,\mathrm{loc}})$. Similarly, from (2.4) and from $\bar{U}_n^{(1)} \to Z = \mathrm{d}u/\mathrm{d}t$ in $L_2(I, L_2(\Omega))$ we obtain $\mathrm{d}u/\mathrm{d}t \in L_2(I, W^{2k}_{2,\mathrm{loc}})$. Thus, Theorem 2.1 is proved.

Example 2.1. Consider the initial boundary value problem (1.2)–(1.5). Suppose $w \in L_2(I, W_2^{4k})$, $\mathrm{d}w/\mathrm{d}t \in L_2(I, W_2^{2k})$ and $\mathrm{d}^2 w/\mathrm{d}t^2 \in L_2(I, L_2)$ and let $f_0(t): I \to L_2(\Omega)$ ($f_0(t) = f_0(x, t)$) satisfy

$$f_0 \in L_2(I, W_2^{2k}), \quad \frac{\mathrm{d}f_0}{\mathrm{d}t} \in L_2(I, L_2)\,.$$

We assume that $a_{ij} = a_{ji}$ for all $|i|, |j| \leqq k$ and that (1.14) takes place for $q = 1$, $l_0 = 3k$, $l_1 = k$. If $u_0 \in W_2^{4k}$, h, $\mathrm{d}h/\mathrm{d}t \in L_2(I, V^*)$ and

$$\sum_{|i|,|j|=k} a_{ij}(x, t)\, \xi_i \xi_j \geqq C|\xi|^2\,,$$

then the weak solution u of (1.2)–(1.5) satisfies

$$u \in L_2(I, W^{4k}_{2,\mathrm{loc}})\,, \quad \frac{\mathrm{d}u}{\mathrm{d}t} \in L_2(I, W^{2k}_{2,\mathrm{loc}})\,, \quad \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} \in L_2(I, L_2)\,.$$

In particular, if $(h(t), v) = \int_{\partial\Omega} g(t, s)\, v(s) \,\mathrm{d}s$ then g, $\mathrm{d}g/\mathrm{d}t \in L_2(I, L_2(\partial\Omega))$ implies h, $\mathrm{d}h/\mathrm{d}t \in L_2(I, V^*)$.

In this section, we study higher order regularity of the solution u of (1.1 with) respect to t assuming the corresponding regularity of $a(t; u, v)$, $f(t)$ with respect to t. Further, we assume $u_0 \in H$ where V, H, f, a are the same as in Section 1.

In obtaining the higher order regularity of u in t, the procedure of Theorem 2.1 is limited by the assumptions for $t = 0$ (see (1.9)). In the proof of the following theorem, a priori estimates $\|\delta_h^p u_i\|^2 \leqq C\varepsilon^{-(2p+1)}$ will be obtained for all $n \geqq n_0$ $i = j_0, \ldots, n$ where $\varepsilon > 0$ $(\varepsilon < T)$, $j_0 \equiv j_0(n)$ satisfies $\frac{1}{2}\varepsilon \leqq K j_0 h < \varepsilon$ for all n $(K > 0$ is a suitable constant) and $C = C(p, |u_0|_H, \|f\|_{H^{p+1}(I,V^*)})$. Then we use the procedure of Theorem 2.1 (with respect to the interval (ε, T)) and obtain

$$\frac{\mathrm{d}^p u}{\mathrm{d}t^p} \in L_\infty(\langle\varepsilon, T\rangle, V) \quad (p \geqq 0 \text{ is an integer}).$$

The idea of deriving the above estimates is due to K. Rektorys [9] – see Remark 3.2.

Theorem 3.1. *Let $q \geqq 0$ be an integer and let $V \subset H$ be compact. Suppose* (1.7), *$u_0 \in H$ and let u be a (weak) solution of* (1.1). *Denote $g_{v,w}(t) = a(t; v, w)$, $g^*_{v,w}(t) = a_0(t; v, w)$ for all $t \in I$, $v, w \in V$ (see* (1.7)). *Let $g_{v,w}(t)$, $g^*_{v,w}(t) \in C^{q-1,1}(\langle 0, T\rangle)$ and let*

$$\left|\frac{\mathrm{d}^p}{\mathrm{d}t^p} g_{v,w}(t) - \frac{\mathrm{d}^p}{\mathrm{d}t^p} g_{v,w}(t')\right| \leqq C|t - t'| \, \|v\| \, \|w\| ,$$

$$\left|\frac{\mathrm{d}^p}{\mathrm{d}t^p} g^*_{v,w}(t) - \frac{\mathrm{d}^p}{\mathrm{d}t^p} g^*_{v,w}(t')\right| \leqq C|t - t'| \, |w| \, \|v\|$$

for all $t, t' \in I$, $p = 0, \ldots, q - 1$ and $v, w \in V$. If $f \in H^{q+1}(I, V^*)$, or if $f \in H^q(I, H)$, then

$$t^{p+1/2} \frac{\mathrm{d}^p u}{\mathrm{d}t^p} \in L_\infty(I, V) \quad \text{and} \quad t^{p+\gamma} \frac{\mathrm{d}^{p+1}}{\mathrm{d}t^{p+1}} \in L_2(I, H) \quad \text{for all} \quad p = 0, \ldots, q$$

and $\gamma > 1/2$.

Proof. First we prove some auxiliary inequalities (see (3.9), (3.11)) from which we then obtain the a priori estimates

$$(3.1) \qquad \|\delta_h^p u_i\|^2 \leqq C\varepsilon^{-(2p+1)}, \quad \sum_{i=j_0+1}^{n} |\delta_h^{p+1} u_i|^2 \, h \leqq C\varepsilon^{-(2p+1)}$$

for all $p = 0, \ldots, q$ where $\varepsilon > 0$ $(\varepsilon < T)$ is arbitrary, $j_0 \leqq i \leqq n$, $j_0 \equiv j_0(n)$, $\frac{1}{2}\varepsilon < 3 \cdot 4^{q-1}(j_0 - 2p) \, h < \varepsilon$, $p = 0, \ldots, q$, $n \geqq n_0$, $h \equiv h_n = T/n$ and $C = C(q, |u_0|, \|f\|_{H^{q+1}(I,V^*)})$. Finally we deduce the required assertions.

Let $0 \leqq p \leqq q$. From (1.15) $(u = u_i)$, we successively obtain the formula

$$(3.2) \qquad (\delta_h^{p+1} u_i, v) + a(t_i; \delta_h^p u_i, v) + \sum_{j=0}^{p-1} \binom{j}{p} \delta_h^{p-j} a(t_i; \delta_h^j u_{i+j-p}, v) = (\delta_h^p f_i, v) \quad \text{for all} \quad v \in V \quad \text{and} \quad p + 1 \leqq i \leqq n .$$

(For $p = 0$ the third term vanishes.) Let us put $v = \delta_h^p u_i - \delta_h^p u_{i-1}$. In virtue of (1.7), similarly as in the proof of Theorem 2.1 we estimate

$$\sum_{i=r+1}^{s} |\delta_h^{p+1} u_i|^2 h + \tfrac{1}{2}\bar{a}(t_s; \delta_h^p u_s, \delta_h^p u_s) \leqq \tfrac{1}{2}\bar{a}(t_r; \delta_h^p u_r, \delta_h^p u_r) +$$

$$+ C_\lambda \sum_{i=r}^{s} \|\delta_h^p u_i\|^2 h + \sum_{i=r+1}^{s} (\delta_h^p f_i, \delta_h^p u_i - \delta_h^p u_{i-1}) + \lambda \sum_{i=r+1}^{s} |\delta_h^{p+1} u_i|^2 h +$$

$$+ \sum_{i=r+1}^{s} a_0(t_i; \delta_h^p u_i, \delta_h^p u_i - \delta_h^p u_{i-1}) +$$

$$+ \Big| \sum_{i=r+1}^{s} \sum_{j=0}^{p-1} \binom{j}{p} \delta_h^{p-j} a(t_i; \delta_h^j u_{i+j-p}, \delta_h^p u_i - \delta_h^p u_{i-1}) \Big|$$

where $p + 1 \leqq r < s \leqq n$ and λ is sufficiently small. Here we use the estimates

$$\sum_{i=r+1}^{s} |a_0(t_i; \delta_h^p u_i, \delta_h^p u_i - \delta_h^p u_{i-1})| \leqq \sum_{i=r+1}^{s} \left(\frac{\lambda}{2} |\delta_h^{p+1} u_i|^2 + \frac{C}{2\lambda} \|\delta_h^p u_i\|^2 \right) h ,$$

$$\sum_{i=r+1}^{s} \sum_{j=0}^{p-1} \binom{j}{p} \delta_h^{p-j} a(t_i; \delta_h^j u_{i+j-p}, \delta_h^p u_i - \delta_h^p u_{i-1}) =$$

$$= -h \sum_{i=r+1}^{s-1} \sum_{j=0}^{p-1} \binom{j}{p} (\delta_h^{p-j} a(t_i; \delta_h^{j+1} u_{i+j-p+1}, \delta_h^p u_i) +$$

$$+ \delta_h^{p-j+1} a(t_{i+1}; \delta_h^j u_{i+j-p+1}, \delta_h^j u_i)) +$$

$$+ \sum_{j=0}^{p-1} \binom{j}{p} (\delta_h^{p-j} a(t_s; \delta_h^j u_{s+j-p}, \delta_h^p u_s) - \delta_h^{p-j} a(t_{r+1}; \delta_h^j u_{r+1+j-p}, \delta_h^p u_r)) ,$$

$$|\delta_h^{p-j} a(t_s; \delta_h^j u_{s+j-p}, \delta_h^p u_s)| \leqq C(\|\delta_h^j u_{s+j-p}\|^2 + \|\delta_h^p u_s\|^2) ,$$

$$\delta_h^j u_m = \delta_h^j u_r + \sum_{i=r+1}^{m} \delta_h^{j+1} u_i h (\|\delta_h^j u_m\|^2 \leqq C(\|\delta_h^j u_r\|^2 + \sum_{i=r+1}^{m} \|\delta_h^{j+1} u_i\|^2 h))$$

for $r \leqq m \leqq s$. In the case $f \in H^q(I, H)$, we use the estimate

$$J_{r,s} \equiv \sum_{i=r+1}^{s} (\delta_h^p f_i, \delta_h^p u_i - \delta_h^p u_{i-1}) \leqq \lambda \sum_{i=r+1}^{s} |\delta_h^{p+1} u_i|^2 h + C_\lambda \sum_{i=r+1}^{s} |\delta_h^p f_i|^2 h .$$

In the case $f \in H^{q+1}(I, V^*)$, we estimate

$$J_{r,s} = - \sum_{i=r+1}^{s} h(\delta_h^{p+1} f_i, \delta_h^p u_{i-1}) + (\delta_h^p f_s, \delta_h^p u_s) - (\delta_h^p f_r, \delta_h^p u_r)$$

and hence (see Lemma 1.1)

$$|J_{r,s}| \leqq C_\lambda \|f\|^2_{H^{p+1}(I,V^*)} + \lambda \|\delta_h^p u_s\|^2 + \|\delta_h^p u_r\|^2) + C \sum_{i=r}^{s} \|\delta_h u_i\|^2 h .$$

Then we conclude

$$\sum_{i=r+1}^{s} |\delta_h^{p+1} u_i|^2 h + \|\delta_h^p u_s\|^2 \leqq C(1 + \sum_{i=0}^{p} \sum_{j=0}^{p} \|\delta_h^j u_{r-i}\|^2 + \sum_{i=r+1-p}^{s} \sum_{j=0}^{p} \|\delta_h^j u_i\|^2 h) .$$

Hence we have

$$\sum_{j=0}^{p}\Big(\sum_{i=r+1}^{s}|\delta_h^{j+1}u_i|^2 h + \|\delta_h^j u_s\|^2\Big) \leqq C\sum_{j=0}^{p}\Big(1 + \sum_{i=0}^{p}\|\delta_h^j u_{r-i}\|^2 + \sum_{i=r+1-p}^{s}\|\delta_h^j u_i\|^2 h\Big).$$

In virtue of Gronwall's Lemma we estimate

$$\sum_{j=0}^{p}\|\delta_h^j u_l\|^2 \leqq C\sum_{j=0}^{p}\Big(1 + \sum_{i=0}^{p}\|\delta_h^j u_{r-i}\|^2 + \sum_{i=r+1-p}^{r}\|\delta_h^j u_i\|^2 h\Big)$$

for all $r + 1 \leqq l \leqq s$ and hence

$$\sum_{j=0}^{p}\Big(\sum_{i=r+1}^{s}|\delta_h^{j+1}u_i|^2 h + \|\delta_h^j u_s\|^2\Big) \leqq \tag{3.3}$$

$$\leqq C\sum_{j=0}^{p}\Big(1 + \sum_{i=0}^{p}\|\delta_h^j u_{r-i}\|^2 + \sum_{i=r+1-p}^{r}\|\delta_h^j u_i\|^2 h\Big).$$

Similarly, from (3.2) for $i = l + 1, \ldots, s$ and $v = \delta_h^p u_i$ we obtain

$$|\delta_h^p u_s|^2 + \sum_{i=l+1}^{s}\|\delta_h^p u_i\|^2 h \leqq$$

$$\leqq C\Big(1 + |\delta_h^p u_l|^2 + \sum_{j=0}^{p-1}\sum_{i=l+1}^{s}\|\delta_h^j u_{i+j-p}\|^2 h + C_1\sum_{i=l+1}^{s}|\delta_h^p u_i|^2 h\Big)$$

because of $\sum_{i=l+1}^{s}\|\delta_h^p f_i\|_*^2 h \leqq C$, $H \subset V^*$ and (1.7). In virtue of Gronwall's Lemma, we may put $C_1 = 0$. Then we obtain

$$\sum_{j=0}^{p}\Big(|\delta_h^j u_s|^2 + \sum_{i=l+1}^{s}\|\delta_h^j u_i\|^2 h\Big) \leqq \tag{3.4}$$

$$\leqq C\Big(1 + \sum_{j=0}^{p}|\delta_h^j u|^2 + \sum_{j=0}^{p-1}\sum_{i=l+1}^{s}\|\delta_h^j u_{i+j-p}\|^2 h\Big).$$

In the case $p = 0$, (3.2) immediately yields

$$|u_s|^2 + \sum_{i=l+1}^{s}\|u_i\|^2 h \leqq C\Big(|u_0|^2 + \sum_{i=l+1}^{s}\|f_i\|_*^2 h\Big). \tag{3.5}$$

In virtue of (3.4), (3.5), successively for $p = 1$, $l = r - m$; $p = 2$, $l = r - m + 1$; $\ldots$; $p = m$, $l = r$ we conclude

$$\sum_{j=0}^{m}\Big(|\delta_h^j u_s|^2 + \sum_{i=r+1}^{s}\|\delta_h^j u_i\|^2\Big) h \leqq C\Big(1 + \sum_{j=0}^{m}\sum_{i=0}^{m}|\delta_h^j u_{r-i}|^2\Big). \tag{3.6}$$

Inserting (3.6) (for $m = p$ and with r, $r - p$ instead of s, r, respectively) into (3.3) we obtain

$$\sum_{j=0}^{p}\Big(\sum_{i=r+1}^{s}|\delta_h^{j+1}u_i|^2 h + \|\delta_h^j u_s\|^2\Big) \leqq C\sum_{j=0}^{p}\Big(1 + \sum_{i=0}^{p}\|\delta_h^j u_{r-i}\|^2 + \sum_{i=0}^{2p}|\delta_h^j u_{r-i}|^2\Big). \tag{3.7}$$

The crucial point of proving (3.9), (3.11) consists in the following estimates. Let j_0, $j_0 > 2q$ be fixed. Consider (3.7) (the first term being omitted) for $j_0 + 2p \leqq r \leqq$

$\leqq 2j_0 - l = s$ ($l \in \{0, \ldots, p\}$ is fixed). We have

$$\sum_{r=j_0+2p-l+1}^{2j_0-l} \sum_{j=0}^{p} \|\delta_h^j u_{2j_0-l}\|^2 h \leqq C((j_0 - 2p) h + \sum_{j=0}^{p} (\sum_{r=j_0+p-l+1}^{2j_0-l} \|\delta_h^j u_r\|^2 h + \sum_{i=j_0-l+1}^{2j_0-l} |\delta_h^j u_i|^2 h) \quad \text{for all } l = 0, \ldots, p .$$

Hence and from (3.6) (where $m = p$, $r = j_0 + p - l$ and the first term is omitted) we conclude

$$(3.8) \quad \sum_{j=0}^{p} \|\delta_h^j u_{2j_0-l}\|^2 \leqq \frac{C}{(j_0 - 2p) h} (1 + \sum_{j=0}^{p} (\sum_{i=j_0-l}^{2j_0-l} |\delta_h^j u_i|^2 h + \sum_{i=j_0-l}^{j_0+p-l} |\delta_h^j u_i|^2)) \quad \text{for all } l = 0, \ldots, p .$$

We use this estimate and (3.6) in (3.3) (where $s \geqq 2j_0$, $r = 2j_0$). Finally, we obtain

$$(3.9) \quad \sum_{j=0}^{p} (\sum_{i=2j_0+1}^{s} |\delta_h^{j+1} u_i|^2 h + \|\delta_h^j u_s\|^2) \leqq$$

$$\leqq \frac{C}{(j_0 - 2p) h} (1 + \sum_{j=0}^{p} (\sum_{i=2j_0-2p}^{2j_0} |\delta_h^j u_i|^2 + \sum_{i=j_0-p}^{j_0+p} |\delta_h^j u_i|^2) + \sum_{j=0}^{p} \sum_{i=j_0-l}^{2j_0-l} |\delta_h^j u_i|^2 h)$$

for all $2j_0 \leqq s \leqq n$. Now, we prove (3.11). Subtract (3.2) for $i = j$ and $i = j - 1$ where $v = \delta_h^{p+1} u_j$. Then summing it up for $j = r + 1, \ldots, s$ and using (1.7) we conclude

$$|\delta_h^{p+1} u_s|^2 + C_1 \sum_{j=r+1}^{s} \|\delta_h^{p+1} u_j\|^2 h \leqq |\delta_h^{p+1} u_r|^2 + \lambda \sum_{i=r+1}^{s} \|\delta_h^{p+1} u_i\|^2 h + C_\lambda \sum_{i=r+1}^{s} |\delta_h^{p+1} u_i|^2 h + C_2 \sum_{i=r+1}^{s} \sum_{j=0}^{p} \sum_{l=0}^{p} \|\delta_h^j u_{i-l}\|^2 h + C_3 \sum_{i=r+1}^{s} \|\delta_h^{p+1} f_i\|_*^2 h)$$

where λ is sufficiently small. The third term on the right-hand side can be handled by Gronwall's Lemma. Then, in virtue of (3.6) and Lemma 1.1 we estimate

$$(3.10) \quad |\delta_h^{p+1} u_s|^2 + \sum_{i=r+1}^{s} \|\delta_h^{p+1} u_j\|^2 h \leqq C(1 + |\delta_h^{p+1} u_r|^2 + \sum_{j=0}^{p} \sum_{i=0}^{2p} |\delta_h^j u_{r-i}|^2) .$$

Hence, similarly as in (3.8) we estimate ($s = 3j_0$, $2j_0 \leqq r \leqq 3j_0$)

$$|\delta_h^{p+1} u_{3j_0}|^2 \leqq \frac{C}{j_0 h} (1 + \sum_{r=2j_0}^{3j_0} |\delta_h^{p+1} u_r|^2 h + \sum_{j=0}^{p} \sum_{i=2j_0-2p}^{3j_0} |\delta_h^j u_i|^2 h) \leqq$$

$$\leqq \frac{C}{j_0 h} (1 + \sum_{j=0}^{p} (\sum_{i=2j_0-p}^{2j_0} \|\delta_h^j u_i\|^2 + \sum_{i=2j_0-2p}^{2j_0} |\delta_h^j u_i|^2) + \sum_{j=0}^{p} \sum_{i=2j_0-2p}^{3j_0} |\delta_h^j u_i|^2 h)$$

because of (3.7). Here we use the estimate (3.8) and then put it into (3.10), where

$r = 3j_0$ and $s \geqq 3j_0$. Finally, we have

$$(3.11)\qquad |\delta_h^{p+1} u_s|^2 \leqq \frac{C}{((j_0 - 2p)\,h)^2}\,(1 + \sum_{j=0}^{p}(\sum_{i=3j_0-2p}^{3j_0} |\delta_h^j u_i|^2 + \sum_{i=2j_0-2p}^{2j_0} |\delta_h^j u_i|^2 + \sum_{i=j_0-p}^{j_0-p} |\delta_h^j u_i|^2) + \sum_{j=0}^{p} \sum_{i=j_0-p}^{3j_0} |\delta_h^j u_i|^2\, h)$$

for all $s \geqq 3j_0$ where $C = C(p, |u_0|_H, \|f\|_{H^{p+1}(I,V^*)})$.

The a priori estimates (3.1) are obtained from (3.9) and (3.11) in the following way. Let us take $j_0(n)$ such that $\frac{1}{2}\varepsilon \leqq 3 \cdot 4^{q-1}(j_0(n) - 2p)\, h_n < \varepsilon$ for all $p = 0, \ldots, q$ and $n \geqq n_0$ where $\varepsilon > 0$ is arbitrary (fixed) and n_0 is sufficiently large. In virtue of (3.5) and (3.9), we have (3.1) for $p = 0$, where $n \geqq i \geqq 3j_0(n)$. Then we take $4j_0(n)$ instead of j_0 in (3.11). Since $4j_0 - 2p > 3j_0(n)$ $(j_0(n) > 2p)$, we can use the estimate

$$|\delta_h u_s|^2 \leqq \frac{C}{((j_0 - 2p)\, h)^2} \leqq \frac{C_1}{\varepsilon^2}$$

for all $n \geqq s \geqq 3j_0(n)$ from the case $p = 0$. Thus we obtain

$$|\delta_h^2 u_s|^2 \leqq \frac{C_k}{\varepsilon^4} \quad \text{for all} \quad n \geqq s \geqq 3 \cdot 4j_0(n)\,.$$

Repeating the above procedure we obtain

$$|\delta_h^p u_s|^2 \leqq \frac{C_q}{\varepsilon^{2p}} \quad \text{for all} \quad p = 0, \ldots, q \quad \text{and} \quad s \geqq 3 \cdot 4^{q-1}\, j_0(n)\,.$$

Hence and from (3.9) we obtain (3.1). Now we prove the estimates (3.13). We rewrite (3.1) in the form

$$(3.12)\qquad \|\overline{U}_n^{(p)}(t)\|^2 \leqq \frac{C}{\varepsilon^{2p+1}}, \quad \int_\varepsilon^T \left|\frac{\mathrm{d}U_n^{(p)}(t)}{\mathrm{d}t}\right|^2 \mathrm{d}t \leqq \frac{C}{\varepsilon^{2p+1}}$$

for all $p = 0, \ldots, q$, $t \in (\varepsilon, T)$ and for all $n \geqq n_0$, where $U_n^{(p)}(t)$, $\overline{U}_n^{(p)}(t)$ are constructed by means of $\delta_h^p u_i$ (see (1.16), (1.17)). These functions are defined in (ε, T) since $\varepsilon > 3 \cdot 4^{q-1}\, j_0(n)\, h_n$ for $n \geqq n_0$. Ih virtue of (3.12) and Lemma 1.2 we have

$$U_n^{(p)} \to U^{(p)} \quad \text{in} \quad C(\langle \varepsilon, T \rangle, H) \quad \text{for all} \quad p = 0, \ldots, q \quad \text{and}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\, U_n^{(q)} = \overline{U}_n^{(q+1)} \rightharpoonup U^{(q+1)} \quad \text{in} \quad L_2(\langle \varepsilon, T \rangle, H)\,.$$

Taking the limit as $n \to \infty$ in (3.2) for $p = 0$, we conclude that $U^{(0)}$ satisfies (1.1) for $t \in (\varepsilon, T)$. By the uniqueness argument we have $U^{(0)}(t) = u(t)$ in (ε, T). Moreover, by a standard argument we have $(d/\mathrm{d}t)\, U^{(p)} = U^{(p+1)}(p = 0, \ldots, q)$ and hence $(\mathrm{d}^p/\mathrm{d}t^p)\, u = U^{(p)}$ for all $p = 0, \ldots, q + 1$. In virtue of (3.12) and $\overline{U}_n^{(p)}(t) \rightharpoonup U^{(p)}(t) = \mathrm{d}^p u(t)/\mathrm{d}t^p$ in V for all $t \in \langle \varepsilon, T \rangle$ we have

$$(3.13)\qquad \left\|\frac{\mathrm{d}^p u}{\mathrm{d}t^p}\right\|^2_{L_\infty(\langle \varepsilon, T \rangle, H)} \leqq \frac{C}{\varepsilon^{2p+1}} \quad \text{and} \quad \int_\varepsilon^T \left|\frac{\mathrm{d}^{p+1} u(s)}{\mathrm{d}s^{p+1}}\right|^2 \mathrm{d}s \leqq \frac{C}{\varepsilon^{2p+1}}$$

for all $\varepsilon > 0$ ($\varepsilon < T$) and $p = 0, 1, \ldots, q$. Hence we obtain

$$\int_0^T s^{2p+\alpha} \left|\frac{d^{p+1}u(s)}{dt^{p+1}}\right|^2 ds \leqq C < \infty (p = 0, \ldots, q) \quad \text{for} \quad \alpha > 1$$

because of the equality

$$\int_\varepsilon^T s^{2p+\alpha} g(s)\, ds = -\left\{s^{2p+\alpha} \int_s^T g(t)\, dt\right\}_\varepsilon^T + (2p + \alpha) \int_\varepsilon^T t^{2p+\alpha-1} \int_t^T g(s)\, ds\, dt$$

where $g(s) = |d^{p+1}u(s)/ds^{p+1}|^2$ and $\varepsilon \to 0$. Thus, Theorem 3.1 is proved.

In particular, considering the weak solution of (1.2)–(1.5) where $F \in L_2(I, L_2)$ (see (1.10)) we obtain also the $W^{2k}_{2,\text{loc}}$-regularity of $d^p u(t)/dt^p$ ($p = 0, \ldots, q$). Here, $\mathring{W}^k_2 \subset V \subset W^k_2$.

Theorem 3.2. *Let $q \geqq 0$ be an integer. Suppose* (1.7), (1.14) *for $l_p = k$ ($p = 0, \ldots, q$; see* Remark 2.1), *$f \in H^{q+1}(I, V^*)$ or $f \in H^q(I, L_2)$), $u_0 \in L_2(\Omega)$, $F \in H^q(I, L_2)$ (F is from (1.10)) and let u be a (weak) solution of (1.2)–(1.5) (see (1.1)). Then*

$$t^{p+\gamma} \frac{d^p u}{dt^p} \in L_2(I, W^{2k}_{2,\text{loc}}) \quad \textit{for all} \quad p = 0, \ldots, q \quad \textit{and} \quad \gamma > 1/2\,.$$

Proof. We shall make use of the a priori estimates from the proof of Theorem 3.1 since its assumptions are satisfied. Consider (3.2) for $p = 0$ (where the third term is omitted). Hence and from Theorem 1.1 ($l = k$, $G = F$) we have

$$\|u_i\|^2_{W^{2k}_{2,\text{loc}}} \leqq C(\Omega') (\|u_i\| + |\delta_h u_i|^2 + |F_i|) \quad \text{for all} \quad j_0(n) \leqq i \leqq n$$

and $n \geqq n_0$ where

$$F_i = \frac{1}{h} \int_{t_{i-1}}^{t_i} F(s)\, ds\,.$$

In virtue of (3.1) (for $p = 0$), we then conclude

$$\text{(3.14)} \qquad \sum_{i=j_0}^n \|u_i\|^2_{W^{2k}_{2,\text{loc}}} h \leqq \frac{C(\Omega')}{\varepsilon} \quad \text{or} \quad \int_\varepsilon^T \|\bar{u}_n(s)\|^2_{W^{2k}_{2,\text{loc}}} ds \leqq \frac{C(\Omega')}{\varepsilon}$$

because of Lemma 1.1. Analogously as in Theorem 3.1, in virtue of (3.14) we conclude $t^\gamma u \in L_2(I, W^{2k}_{2,\text{loc}})$ which is our statement for $p = 0$. Let us suppose that our theorem is proved for $p = 1, \ldots, r < q$ with the estimates

$$\text{(3.14')} \qquad \sum_{i=j_0}^n \|\delta^p_h u_i\|^2_{W^{2k}_{2,\text{loc}}} h \leqq \frac{C(\Omega')}{\varepsilon^{2p+1}}\,; \quad \int_\varepsilon^T \|\bar{U}^{(p)}_n(t)\|^2_{W^{2k}_{2,\text{loc}}} dt \leqq \frac{C}{\varepsilon^{2p+1}}$$

$p = 1, \ldots, r$. Then, from (3.2) for $p = r + 1$ and from (3.14') we obtain

$$\text{(3.15)} \qquad \|\delta^{r+1}_h u_i\|^2_{W^{2k}_{2,\text{loc}}} \leqq C(\Omega') (\|\delta^{r+1}_h u_i\|^2 + \sum_{j=0}^r |z_{i+j-r-1}|^2 + |\delta^{r+1}_h F_i|^2 + |\delta^{r+2}_h u_i|^2)$$

for all $i \geqq j_0$ where $\delta_h^{r+1-j} a(t_i; \delta_h^j u_{i+j-r-1}, v) = (z_{i+j-r-1}, v)$, $z_{i+j-r-1} \in L_2(\Omega')$ ($\Omega'' \subset \Omega'$ with $\overline{\Omega''}(\Omega')$ for all $v \in \mathscr{D}(\Omega)$ because of Green's Theorem, and $\delta_h^j u_{i+j-r-1} \in \in W_2^{2k}(\Omega')$ for all $j = 0, \ldots, r$. Moreover, the estimate

$$|z_{i+j-r-1}|^2_{L_2(\Omega')} \leqq C \|\delta_h^j u_{i+j-r-1}\|^2_{W_2^{2k}(\Omega')} \leqq \frac{C(\Omega')}{\varepsilon^{2j+1}}$$

takes place for all $i \geqq j_0$ and $j = 0, \ldots, r$. Then, (3.14'), (3.1) and (3.15) imply

$$\sum_{i=j_0}^{n} \|\delta_h^{r+1} u_i\|^2_{W^{2k}_{2,\mathrm{loc}}} \leqq \frac{C(\Omega'')}{\varepsilon^{2r+3}} \quad \text{or} \quad \int_\varepsilon^T \|\overline{U}_n^{(r+1)}(t)\|^2_{W^{2k}_{2,\mathrm{loc}}} \, \mathrm{d}t \leqq \frac{C(\Omega'')}{\varepsilon^{2r+3}}$$

for all $n \geqq n_0$ which is (3.14') for $p = r + 1$. Similarly as above, the last inequality implies

$$t^{r+1+\gamma} \frac{\mathrm{d}^{r+1} u}{\mathrm{d}t^{r+1}} \in L_2(I, W^{2k}_{2,\mathrm{loc}}),$$

and Theorem 3.2 is proved.

In deriving (3.9) and (3.11), the estimates

$$\sum_{i=1}^{n} \|\delta_h^p f_i\|^2_* \, h \leqq C \|f\|_{H^p(I,V^*)} \leqq C(f)$$

have been used. In fact, the estimate (3.9) can be written in a more detailed form where $\|f\|_{H^{p+1}(I,V^*)}$ and $|u_0|$ stand in the place of 1. Similarly, in (3.11) we have $\|f\|_{H^{p+1}(I,V^*)}$ instead of 1. Then, taking account of (3.5), by the same arguments as in Theorem 3.1 and Theorem 3.2 we obtain

Theorem 3.3. (i) *If the assumptions of Theorem* 3.1 *are satisfied, then*

$$\left\|\frac{\mathrm{d}^p u(t)}{\mathrm{d}t^p}\right\|_{W_2^k} \leqq C t^{-p-1/2} (|u_0|_{L_2} + \|f\|_{H^{p+1}(I,V^*)})$$

for all $t \in I$,

$$\left\|t^{p+\gamma} \frac{\mathrm{d}^{p+1} u}{\mathrm{d}t^{p+1}}\right\|_{L_2(I,L_2)} \leqq C(|u_0|_{L_2} + \|f\|_{H^{p+1}(I,V^*)})$$

for all $p = 0, \ldots, q$ *and* $\gamma > 1/2$;

(ii) *If the assumptions of Theorem* 3.2 *are satisfied and* $\gamma > 1/2$ *then*

$$\left\|t^{p+\gamma} \frac{\mathrm{d}^p u(t)}{\mathrm{d}t^p}\right\|_{L_2(I,W_2^{2k}(\Omega'))} \leqq C(\Omega') (|u_0|_{L_2} + \|f\|_{H^{p+1}(I,V^*)})$$

for all $t \in I$ *and for all* $p = 0, \ldots, q$. *If* $f \in H^q(I, H)$ *then, in the above estimates,* $\|f\|_{H^{p+1}(I,V^*)}$ *can be replaced* $\|f\|_{H^p(I,H)}$.

Remark 3.1. Theorem 3.1 holds true without the assumption that $V \hookrightarrow H$ is compact. Indeed, the a priori estimates (3.12) and the uniqueness of the weak solution of (1.1) imply

$$U_n^{(p)}(t) \rightharpoonup U^{(p)}(t), \quad \overline{U}_n^{(p)}(t) \rightharpoonup U^{(p)}(t) \quad \text{in } H \text{ and also in } V$$

for all $p = 0, \ldots, q$ and $t \in \langle \varepsilon, T \rangle$. This is sufficient to establish (3.13).

Remark 3.2. In [9] a special case is considered: $a(t; u, v) \equiv a(u, v)$; $a(u, v) = a(v, u)$; $a(u, u) \geqq C\|u\|^2$; $(f(t), v) = \int_\Omega f(x)\, v \,\mathrm{d}x$ with $f \in L_2(\Omega)$ and $u_0 \equiv 0$. It is proved that $\mathrm{d}^p u/\mathrm{d}t^p \in C(\langle \varepsilon, T\rangle, V)$ for all $p = 0, \ldots$ (see [9], Theorem 12.2).

Remark 3.3. In the paper [10], the semi-group theory has been employed. Homogeneous Dirichlet boundary conditions are considered and more regular coefficients $a_{ij}(x, t)$ (in x) are supposed. Assertion (i) in Theorem 3.3 corresponds to the estimate (2.37) in [10] for $\beta = \frac{1}{2}$ (where $\|A^{1/2}v\| \simeq \|v\|_{W_2^k}$, see [3]). The a priori estimates of the smoothing effect can be used in the error analysis for the approximate solutions of linear parabolic equations with non-smooth initial data (see [12], etc.).

4. REGULARITY IN (x, t)-VARIABLES

Higher order regularity of the (weak) solution of (1.2)–(1.5) for $u_0 \in L_2(\Omega)$ with respect to the (x, t)-variables in the interior of the domain Q is obtained under stronger regularity assumptions on $a_{ij}(x, t)$, F (from (1.10)) with respect to the (x, t)-variables.

Let $q \geqq 0$ be an integer. We assume

$$\frac{\mathrm{d}^p F}{\mathrm{d}t^p} \in L_2(I, W_2^{\alpha_p}) \quad \text{for all} \quad p = 0, \ldots, q \tag{4.1}$$

where $\alpha_p \geqq 0$ are integers and F is from (1.10). We construct nonnegative integers $\beta_p \geqq 0$ in the following way: Successively for $l = 1, \ldots, q$, where $l = r + s$ (r, s being nonnegative integers), we define

$$\beta_r^{(s)} = \min\left(\beta_{r-1}^{(s-1)}, \bar{\beta}_{r-1}^{(s)}, \bar{\beta}_{r-2}^{(s+1)}, \ldots, \bar{\beta}_0^{(s+r-1)}, \alpha_r\right) + 2k \tag{4.2}$$

where $\bar{\beta}_r^{(s)} = \beta_r^{(s)} - 2k$, $r = 0, \ldots, q$, $1 \leqq s \leqq q - r + 1$. We put $\beta_i^{(0)} = 0$ for $i = 1, \ldots, q + 1$ and

$$\beta_p = \beta_p^{(q+1-p)} \quad \text{for} \quad p = 0, \ldots, q + 1\,. \tag{4.3}$$

We can easily verify $\beta_r^{(s_1)} \geqq \beta_r^{(s_2)}$ for $s_1 > s_2$ and $\beta_j \leqq \beta_{j-1} \leqq \beta_j + 2k$ $(j = 0, \ldots, q)$.

Theorem 4.1. *Let $q \geqq 0$ be an integer. Suppose* (1.7), (4.1), $f \in H^{q+1}(I, V^*)$ (*or* $f \in H^q(I, L_2)$), $u_0 \in L_2(\Omega)$ *and* (1.14) *for* $l_p = \beta_p - 2k$ $(p = 0, \ldots, q)$ *where* β_p *are from* (4.3). *If* u *is a weak solution of* (1.2)–(1.5), *then*

$$t^{q+\gamma} \frac{\mathrm{d}^p u}{\mathrm{d}t^p} \in L_2(I, W_{2,\mathrm{loc}}^{\beta_p}) \quad \textit{for all} \quad p = 0, \ldots, q \quad \textit{and} \quad \gamma > 1/2\,.$$

In particular, when $\alpha_p = 2k(q - p)$ *and* $l_p = 2k(q - p)$ $(p = 0, \ldots, q)$, *then* $(\beta_p = 2k(q - p + 1))$

$$t^{q+\gamma} \frac{\mathrm{d}^p u}{\mathrm{d}t^p} \in L_2(I, W_{2,\mathrm{loc}}^{2k(q-p+1)}) \quad \textit{for all} \quad p = 0, \ldots, q\,.$$

If $F \in C^\infty(Q)$ *and* $a_{ij} \in C^\infty(Q)$, *then* $u \in C^\infty(Q)$.

Proof. We rewrite (3.2) in the form

$$(4.4)\qquad a(t_i;\, \delta_h^p u_i, v) = -(\delta_h^{p+1} u_i, v) - \sum_{j=0}^{p-1} \binom{j}{p} \delta_h^{p-j} a(t_i;\, \delta_h^j u_{i+j-p}, v) + (\delta_h^p F_i, v)$$

for all $v \in \mathscr{D}(\Omega)$ and $p + 1 \leqq i \leqq n$. Here we make use of the a priori estimates (3.1) and the procedure used in the proof of Theorem 3.2. Successively, for $p = 0, \ldots, q$ we apply Theorem 1.1. The structure of the numbers $\beta_s^{(r)}$ corresponds to the following fact. Having a priori estimates for $\delta_h^{j+1} u_i$ in a space $W_{2,\mathrm{loc}}^s$, we come back to (4.4) with $p = j, j - 1, \ldots, 0$ and apply Theorem 1.1 again. At every step we obtain the regularity of $\delta_h^j u_i, \delta_h^{j-1} u_i, \ldots$ which is not smaller than the regularity obtained before, since the right-hand side in (4.4) is possibly more regular in x. Indeed, consider (4.4) for $p = 0$; the second term on the right-hand side vanishes. We have $u_i \in W_{2,\mathrm{loc}}^{2k}$ $(\beta_0^{(1)} = 2k)$. Then (4.4) for $p = 1$ and $u_i \in W_{2,\mathrm{loc}}^{2k}$ yields

$$\delta_h u_i \in W_{2,\mathrm{loc}}^{\beta_1(1)} \quad \text{where} \quad \beta_1^{(1)} = \min(\beta_2^{(0)}, \bar{\beta}_0^{(1)}, \alpha_1) + 2k$$

since $\delta_h a(t_i;\, u_{i-1}, v) = (z_{i-1}, v)$ (Green's formula and (1.10)) where $z_{i-1} \in L_{2,\mathrm{loc}}$. Using the a priori estimate $\delta_h u_i \in W_{2,\mathrm{loc}}^{\beta_1(1)}$, we apply again (4.4) with $p = 0$ since $\delta_h u_i$ on the right-hand side is now more regular. Then we have

$$u_i \in W_{2,\mathrm{loc}}^{\beta_0(2)}, \quad \text{where} \quad \beta_0^{(2)} = \min(\beta_1^{(1)}, \alpha_0) + 2k \geqq \beta_0^{(1)}.$$

In the next step we obtain

$$\delta_h^2 u_i \in W_{2,\mathrm{loc}}^{\beta_2(1)}, \quad \beta_2^{(1)} = \min(\beta_3^{(0)}, \bar{\beta}_1^{(1)}, \bar{\beta}_0^{(2)}, \alpha_2) + 2k,$$
$$\delta_h u_i \in W_{2,\mathrm{loc}}^{\beta_1(2)}, \quad \beta_1^{(2)} = \min(\beta_2^{(1)}, \bar{\beta}_0^{(2)}, \alpha_1) + 2k \quad \text{and}$$
$$u_i \in W_{2,\mathrm{loc}}^{\beta_0(3)}, \quad \beta_0^{(3)} = \min(\beta_1^{(2)}, \alpha_0) + 2k.$$

Thus, successively we obtain $\beta_p = \beta_p^{(q-p+1)}$ for $p = 0, \ldots, q + 1$ and the a priori estimates

$$(4.5)\qquad \|\bar{U}_n^{(p)}(t)\|_{W_{2,\mathrm{loc}}^p}^2 \leqq \frac{C(\Omega')}{\varepsilon^{2q+1}}, \quad \int_\varepsilon^T \left|\frac{\mathrm{d}U_n^{(q)}(t)}{\mathrm{d}t}\right|^2 \mathrm{d}t \leqq \frac{C}{\varepsilon^{2q+1}}$$

for all $n \geqq n_0$, $p = 0, \ldots, q$ and $t \in \langle \varepsilon, T \rangle$ similarly as in the proof of Theorem 3.2. Hence, by the same argument, we deduce the required result. The $C^\infty(Q)$-regularity of the solution is a consequence of the previous results and of the imbedding theorems in the Sobolev spaces $W_2^s(Q)$. Thus, Theorem 4.1 is proved.

Remark 4.1. In the case $a_{ij}(x, t) \equiv a_{ij}(x)$, the formulas for β_p in Theorem 4.1 are simple because of (4.4) where the second member on the right-hand side vanishes. Then we have $\beta_{q+1} = 0$, $\beta_q = 2k$ and

$$\beta_p = \min(\beta_{p+1}, \alpha_p) + 2k \quad \text{for} \quad p = 0, \ldots, q - 1.$$

Remark 4.2. Theorem 4.1 holds true also in the case when $\alpha_p \geqq 0$ $(p = 0, \ldots, q)$ are real numbers, since Theorem 1.1 is valid also for real $l \geqq 0$ (see [7]).

Remark 4.3. In the case $\alpha_p = 2k(q - p)$ $(p = 0, \ldots, q)$, our result in Theorem 4.1

coincides with that of [7] (II, Theorem 5.3 for $k = 2mq$) restricted to the interior of the domain Q.

Remark 4.4. By the same argument as in Theorem 3.3 we can specify the structure of the constants $C(\Omega')$, C in the a priori estimates (4.5). In fact, we can obtain

$$C(\Omega') \leqq C_1(\Omega') \left(|u_0|^2 + \sum_{j=0}^{p} \left\| \frac{\mathrm{d}^j F}{\mathrm{d}t^j} \right\|^2_{L_2(I, W_2^{\alpha_j})} \right).$$

Then the assertion of Theorem 4.1 can be rewritten in the form

$$\left\| t^{q+\gamma} \frac{\mathrm{d}^p u}{\mathrm{d}t^p} \right\|^2_{L_2(I, W^{\beta_p}_{2,\mathrm{loc}})} \leqq C_1(\Omega') \left(|u_0|^2 + \sum_{j=0}^{p} \left\| \frac{\mathrm{d}^j F}{\mathrm{d}t^j} \right\|^2_{L_2(I, W_2^{\alpha_j})} \right).$$

5. APPLICATION OF REGULARITY RESULTS TO THE CONVERGENCE OF ROTHE'S METHOD

The regularity results obtained in Sections 3 and 4 yield stronger results on the convergence of Rothe's method.

Theorem 5.1. *Let the assumptions of Theorem* 3.1 *be satisfied. If* $V \subset H$ *is compact, then* $U_n^{(p)} \to \mathrm{d}^p u/\mathrm{d}t^p$ *in* $C(\langle \varepsilon, T \rangle, H)$ *for* $p = 0, \ldots, q$ *and*

$$\varrho_n(p) \equiv \left| U_n^{(p)}(t) - \frac{\mathrm{d}^p u(t)}{\mathrm{d}t^p} \right|^2 + \int_\varepsilon^T \left\| \bar{U}_n^{(p)}(s) - \frac{\mathrm{d}^p u(s)}{\mathrm{d}s^p} \right\|^2 \mathrm{d}s \leqq$$

$$\leqq C \left(\frac{1}{n} + \sum_{j=0}^{p} \left| U_n^{(j)}(\varepsilon) - \frac{\mathrm{d}^j u(\varepsilon)}{\mathrm{d}t^j} \right|^2 + \sum_{j=0}^{p} \left\| \bar{f}_n^{(j)} - \frac{\mathrm{d}^j f}{\mathrm{d}t^j} \right\|^2_{L_2(\langle \varepsilon, T \rangle, V^*)} \right) \quad \textit{for} \quad p = 0, \ldots, q\,.$$

In particular, $\varrho_n(p) \to 0$ *for* $n \to \infty$.

Proof. Let us denote $a^{(p)}(t; u, v) = (\mathrm{d}^p/\mathrm{d}t^p)\, a(t; u, v)$ and $a_n(t; u, v) = a(t_i; u, v)$ for $t_{i-1} < t < t_i$, $i = 1, \ldots, n$, $a_n(0; u, v) = a(0; u, v)$ for all $u, v \in V$ and $p = 0, \ldots, q$. Due to the assumptions of Theorem 3.1, we have

$$\text{(5.1)} \qquad |\delta_h^p\, a(t; u, v) - a^{(p)}(t; u, v)| \leqq Ch \|u\| \, \|v\| \quad \left(h = h_n = \frac{T}{n} \right)$$

for all $p = 0, \ldots, q - 1$, $u, v \in V$ and $n \geqq n_0$. We rewrite (4.4) in the form

(5.2)

$$\left(\frac{\mathrm{d}U_n^{(p)}(t)}{\mathrm{d}t}, v \right) + a_n(t; \bar{U}_n^{(p)}(t), v) = (\bar{f}_n^{(p)}(t), v) - \sum_{j=0}^{p-1} \binom{j}{p} \delta_h^{p-j} a_n(t; \bar{U}_n^{(p)}(t - (p-j)\, h_n), v)$$

for all $v \in V$. Hence we conclude (the variable t is omitted

$$\frac{\mathrm{d}}{\mathrm{d}t} |U_n^{(p)} - U_m^{(p)}|^2 + a_n(t; \bar{U}_n^{(p)} - \bar{U}_m^{(p)}, \bar{U}_n^{(p)} - \bar{U}_m^{(p)}) \leqq (U_n^{(p+1)} - U_m^{(p+1)}, U_n^{(p)} -$$

$$- \bar{U}_n^{(p)} + (U_m^{(p)} - \bar{U}_m^{(p)})) - a_n(t; \bar{U}_m^{(p)}, \bar{U}_n^{(p)} - \bar{U}_m^{(p)}) +$$

$$+ a_m(t; \bar{U}_m^{(p)}, \bar{U}_n^{(p)} - \bar{U}_m^{(p)}) - \sum_{j=1}^{p-1} \binom{j}{p} \delta_{h_n}^{p-j} a_n(t; \bar{U}_n^{(j)} - \bar{U}_m^{(j)}, \bar{U}_n^{(p)} - \bar{U}_m^{(p)}) +$$

$$+ \sum_{j=0}^{p-1} \binom{j}{p} \left|\delta_{h_m}^{p-j} a_m(t; \bar{U}_m^{(j)}, \bar{U}_n^{(p)} - \bar{U}_m^{(p)}) - a^{(p-j)}(t; \bar{U}_m^{(j)}, \bar{U}_n^{(p)} - \bar{U}_m^{(p)})\right| +$$

$$+ \sum_{j=0}^{p-1} \binom{j}{p} \left|a^{(p-j)}(t; \bar{U}_m^{(j)}, \bar{U}_n^{(p)} - \bar{U}_m^{(p)}) - \delta_{h_n}^{p-j} a_n(t; \bar{U}_m^{(j)}, \bar{U}_n^{(p)} - \bar{U}_m^{(p)})\right| +$$

$$+ (\bar{f}_n^{(p)}(t) - \bar{f}_m^{(p)}(t), \bar{U}_n^{(p)} - \bar{U}_m^{(p)}) .$$

We integrate it over (ε, t). Using the estimates

$$|U_n^{(p)}(t) - \bar{U}_n^{(p)}(t)| \leqq 2|U_n^{(p+1)}(t)| \frac{1}{n}, \quad \int_\varepsilon^T |U_n^{(p+1)}(s)|^2 \, ds \leqq C(\varepsilon),$$

$$|a_n(t; v, w) - a_m(t; v, w)| \leqq C\left(\frac{1}{n} + \frac{1}{m}\right) \|v\| \, \|w\| , \quad \|\bar{U}_n^{(j)}(t)\| \leqq C(\varepsilon)$$

for all $j = 0, \ldots, q$, $n \geqq n_0$, $t \in (\varepsilon, T)$ (see (3.1)), and (5.1), we obtain

$$|U_n^{(p)}(t) - U_m^{(p)}(t)|^2 + \int_\varepsilon^T \|\bar{U}_n^{(p)}(s) - \bar{U}_m^{(p)}(s)\|^2 \, ds \leqq \tag{5.3}$$

$$\leqq C(\varepsilon) \Bigg(|U_n^{(p)}(\varepsilon) - U_m^{(p)}(\varepsilon)|^2 + \frac{1}{n} + \frac{1}{m} + \sum_{j=0}^{p-1} \int_\varepsilon^T \|\bar{U}_n^{(j)}(s) - \bar{U}_m^{(j)}(s)\|^2 \, ds +$$

$$+ \int_\varepsilon^T \left\|\bar{f}_n^{(p)}(s) - \frac{d^p f(s)}{ds^p}\right\|_*^2 ds + \int_\varepsilon^T \left\|\bar{f}_m^{(p)}(s) - \frac{d^p f(s)}{ds^p}\right\|_*^2 ds +$$

$$+ C_1 \int_\varepsilon^t |U_n^{(p)}(s) - U_m^{(p)}(s)|^2 \, ds .$$

In virtue of Gronwall's Lemma, we may put $C_1 = 0$. The a priori estimates (2.12) and the compactness of the imbedding $V \hookrightarrow H$ imply

$$U_n^{(p)} \to \frac{d^p u}{dt^p} \quad \text{in} \quad C(\langle \varepsilon, T \rangle, H) \quad \text{for all} \quad p = 0, \ldots, q$$

(see the proof of Theorem 3.1). Thus, taking the limit as $m \to \infty$ in (5.3), successively for $p = 0, \ldots, q$, we obtain the required result. In the case $p = 0$ the term with the summation in (5.3) vanishes. The convergence

$$\sum_{j=0}^{q} \left\|\bar{f}_n^{(j)} - \frac{d^j f}{dt^j}\right\|_{L_2((\varepsilon, T), V^*)} \to 0$$

is a consequence of Lemma 1.1.

In the following theorem we prove the convergence of Rothe's method (in the interior of Q) in more regular functional spaces. For simplicity, we shall assume $a(t; u, v) = a(u, v)$ for all $u, v \in V$.

Theorem 5.2. *Let the assumptions of Theorem* 4.1 *be satisfied and let* $a_{ij}(x, t) \equiv$

$\equiv a_{ij}(x)$ *for all* $|i|, |j| \leqq k$. *Then the estimates*

$$\varrho_n \equiv \sum_{p=0}^{q} \left\| \bar{U}_n^{(p)} - \frac{\mathrm{d}^p u}{\mathrm{d}t^p} \right\|_{Y_p}^2 \leqq C(\Omega') \left(\frac{1}{n} + \sum_{p=0}^{q} \left(\left| U_n^{(p)}(\varepsilon) - \frac{\mathrm{d}^p u(\varepsilon)}{\mathrm{d}t^p} \right|^2 + \right. \right.$$

$$\left. \left. + \left\| \bar{F}_n^{(p)} - \frac{\mathrm{d}^p F}{\mathrm{d}t^p} \right\|_{Z_p}^2 \right) + \sum_{p=0}^{q} \left\| \bar{f}_n^{(p)} - \frac{\mathrm{d}^p f}{\mathrm{d}t^p} \right\|_{L_2(\langle \varepsilon, T \rangle, V^*)} \right)$$

are true where $Y_p \equiv L_2(\langle \varepsilon, T \rangle, W_{2,\mathrm{loc}}^{\gamma_p})$, $Z_p \equiv L_2(\langle \varepsilon, T \rangle, W_2^{\alpha_p})$, $\gamma_p = \min \{\gamma_{p+1}, \alpha_p\} + 2k$ *for* $p = 0, \ldots, q - 1$ *and* $\gamma_q = k$. *In particular,* $\varrho_n \to 0$ *with* $n \to \infty$.

Proof. Taking the limit as $n \to \infty$ in (5.2) (where the last term vanishes), we deduce

$$\left(\frac{\mathrm{d}^{p+1} u(t)}{\mathrm{d}t^{p+1}}, v \right) + a \left(\frac{\mathrm{d}^p u(t)}{\mathrm{d}t^p}, v \right) = \left(\frac{\mathrm{d}^p F(t)}{\mathrm{d}t^p}, v \right)$$

for all $v \in \mathscr{D}(\Omega)$, $t \in I$ and $p = 0, \ldots, q - 1$. In the case $p = q$, this identity holds for a.e. $t \in I$. Then, subtracting it from (5.2), we conclude

$$a \left(\bar{U}_n^{(p)}(t) - \frac{\mathrm{d}^p u(t)}{\mathrm{d}t^p}, v \right) = - \left(\bar{U}_n^{(p+1)}(t) - \frac{\mathrm{d}^{p+1} u(t)}{\mathrm{d}t^{p+1}}, v \right) + \left(\bar{F}_n^{(p)}(t) - \frac{\mathrm{d}^p F(t)}{\mathrm{d}t^p}, v \right)$$

for all $v \in \mathscr{D}(\Omega)$, $p = 0, \ldots, q - 1$. Thus $\bar{U}_n^{(p)}(t) - \mathrm{d}^p u(t)/\mathrm{d}t^p$ (t is fixed) is a weak solution of the linear elliptic equation. In virtue of Theorem 1.1 and the a priori estimates (4.4) ($\beta_p - k \leqq \gamma_p \leqq \beta_p$ – see Remark 4.1) we successively obtain

$$\left\| \bar{U}_n^{(p)}(t) - \frac{\mathrm{d}^p u(t)}{\mathrm{d}t^p} \right\|_{W_{2,\mathrm{loc}}^{\gamma_p}}^2 \leqq C(\Omega') \left(\left\| \bar{U}_n^{(p)}(t) - \frac{\mathrm{d}^p u(t)}{\mathrm{d}t^p} \right\|^2 + \right. \tag{5.4}$$

$$\left. + \left\| \bar{U}_n^{(p+1)}(t) - \frac{\mathrm{d}^{p+1} u(t)}{\mathrm{d}t^{p+1}} \right\|_{W_{2,\mathrm{loc}}^{\bar{\gamma}_p}}^2 + \left\| \bar{F}_n^{(p)}(t) - \frac{\mathrm{d}^p F(t)}{\mathrm{d}t^p} \right\|_{W_2^{\alpha_p}}^2 \right)$$

where $\bar{\gamma}_p = \gamma_p - 2k$. Since $\bar{\gamma}_p \leqq \gamma_{p+1}$ we have

$$\left\| \bar{U}_n^{(p+1)}(t) - \frac{\mathrm{d}^{p+1} u(t)}{\mathrm{d}t^{p+1}} \right\|_{W_{2,\mathrm{loc}}^{\bar{\gamma}_p}}^2 \leqq \left\| \bar{U}_n^{(p+1)}(t) - \frac{\mathrm{d}^{p+1} u(t)}{\mathrm{d}t^{p+1}} \right\|_{W_{2,\mathrm{loc}}^{\gamma_{p+1}}}^2$$

($p = 0, \ldots, q - 1$). Thus, the right-hand side in (5.4) is an element of $L_1(\langle \varepsilon, T \rangle)$ because of (4.5) and Theorem 4.1. Let us integrate (5.4) over (ε, T). In virtue of Theorem 5.1, successively for $p = q - 1, \ldots, 0$, from (5.4) we obtain the required result.

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