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TOLERANCES AND CONGRUENCES ON LATTICES

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1. Introduction. The notion of a *tolerance* dates back at least to E. C. Zeeman [22]. A tolerance T on a set X is simply a reflexive symmetric binary relation on X, and a tolerance space is a pair (X, T) where X is a set and T a fixed tolerance on X. These ideas were applied to automata theory by M. A. Arbib ([1], [2]) so that some form of "continuity" could be applied to the state set of a finite automaton. To say that a function $f: X \to X$ on a tolerance space (X, T) is *continuous* is to say ([2], p. 225) that xTy implies $f(x)\,Tf(y)$ for all $x, y \in X$. The set of continuous functions on (X, T) evidently forms a semigroup under composition. With a slight change of perspective, one can fix one's attention on a specified semigroup S of mappings on the set X, and then consider only those tolerances on X that make the mappings continuous. This is the viewpoint that will be adpoted here.

Tolerances on algebraic structures were introduced by B. Zelinka [23]. Though the bibliography is by no means complete, the reader might note that applications to lattice theory appear in [3], [5], [6], [7], [8], [9], [10], and [21]. Results appearing in [3] and [9] on complemented tolerances form the basis on which the present paper rests. Our goal will be to provide a setting general enough so that these results may be extended and related to similar results on congruences that appeared in [14], [15], and [16].

2. Preliminary concepts. Throughout the paper, $\mathscr{L}$ will denote a bounded lattice and $\mathscr{S}$ a semigroup of isotone mappings on $\mathscr{L}$, subject only to the condition that $\mathscr{S}$ contain the identity map. Consider the following conditions that a reflexive symmetric binary relation R on L might enjoy:

(1) $aRb \Rightarrow \phi(a)\,R\,\phi(b)$ for all $\phi \in \mathscr{S}$.

(2) $aRb, cRd \Rightarrow (a \vee c)\,R(b \vee d)$, and $(a \wedge c)\,R(b \wedge d)$.

(3) R is transitive.

We agree to call R a *set- theoretic $\mathscr{S}$-tolerance* if it satisfies (1), an *$\mathscr{S}$-tolerance* if it satisfies (1) and (2), and an *$\mathscr{S}$-congruence* if all three conditions hold. These three classes of relations will be denoted respectively by the symbols $\mathrm{ST}(\mathscr{S}, \mathscr{L})$, $\mathrm{LT}(\mathscr{S}, \mathscr{L})$ and $\mathrm{Con}(\mathscr{S}, \mathscr{L})$, and they will each be ordered by set inclusion. In case $\mathscr{S}$ consists only of the identity map, the simpler symbols $\mathrm{ST}(\mathscr{L})$, $\mathrm{LT}(\mathscr{L})$ and $\mathrm{Con}(\mathscr{L})$

will be employed. It is well known that $\mathrm{ST}(\mathscr{S}, \mathscr{L})$, $\mathrm{LT}(\mathscr{S}, \mathscr{L})$ and $\mathrm{Con}(\mathscr{S}, \mathscr{L})$ each form complete algebraic lattices whose least element is $\Delta = \{(a, a): a \in \mathscr{L}\}$, and whose greatest element is $\mathscr{L} \times \mathscr{L}$. The meet operation is in each case set intersection, while the join operation in $\mathrm{Con}(\mathscr{S}, \mathscr{L})$ coincides with that in $\mathrm{Con}(\mathscr{L})$; indeed, it is easy to see that $\mathrm{Con}(\mathscr{S}, \mathscr{L})$ is necessarily a complete sublattice of $\mathrm{Con}(\mathscr{L})$. Similarly, the join operation $\mathrm{ST}(\mathscr{S}, \mathscr{L})$ is set union, and $\mathrm{ST}(\mathscr{S}, \mathscr{L})$ is a complete sublattice of $\mathrm{ST}(\mathscr{L})$. The join operation in $\mathrm{LT}(\mathscr{S}, \mathscr{L})$ is more difficult to describe. Given a reflexive symmetric relation R on $\mathscr{L}$, let $t(R)$ denote the collection of all ordered pairs of the form $(a \vee c, b \vee d)$ or $(a \wedge c, b \wedge d)$ with aRc and bRd; let $s(R)$ be all ordered pairs of the form $(\phi(a), \phi(b))$ with $\phi \in \mathscr{S}$ and aRb. The $\mathscr{S}$-tolerance *generated* by R is evidently the set of all ordered pairs of the form $(f(a), f(b))$ with aRb and f the composition of finitely many functions of the form s or t. For a family $\{T_\alpha\}$ of elements of $\mathrm{LT}(\mathscr{S}, \mathscr{L})$, it is evident that $\bigvee_\alpha T_\alpha$ in $\mathrm{LT}(\mathscr{S}, \mathscr{L})$ is the $\mathscr{S}$-tolerance generated by $\bigcup_\alpha T_\alpha$. For $a, b \in \mathscr{L}$, it will be convenient to let $T_{\mathscr{S}}(a, b)$ denote the $\mathscr{S}$-tolerance generated by $\{\Delta \cup (a, b) \cup (b, a)\}$. Similarly, we shall let $S_{\mathscr{S}}(a, b)$ denote the set-theoretic $\mathscr{S}$-tolerance and $\theta_{\mathscr{S}}(a, b)$ the $\mathscr{S}$-congruence relation generated by the pair $\{a, b\}$.

It will now be assumed that $\mathscr{S}$ contains all meet and join translations of L. In case $\mathscr{S}$ consists only of these translations, we shall write $S_{\mathscr{S}}(a, b)$, $T_{\mathscr{S}}(a, b)$ and $\Theta_{\mathscr{S}}(a, b)$ in the more simple form $S(a, b)$, $T(a, b)$ and $\Theta(a, b)$. Our immediate goal is to provide a slight generalization of [3], Lemma 2.1, p. 374. Before doing so, some terminology is needed. A *quotient* of L is pair a/b of elements of L for which $a \geqq b$. To say that the quotient a/b is $\mathscr{S}$-projective onto c/d will be to say that $c = \phi(a)$ and $d = \phi(b)$ for some $\phi \in \mathscr{S}$. The symbol $a/b \to^{\mathscr{S}} c/d$ will be used to denote this fact. In the case where $\mathscr{S}$ consists only of the identity map and the join and meet translations of $\mathscr{L}$, $\mathscr{S}$-projectivity is the reverse of the notion of weak projectivity that was introduced by Dilworth [11]. We now have

Lemma 1. *Given $T \in \mathrm{ST}(\mathscr{S}, \mathscr{L})$ and $\Theta \in \mathrm{Con}(\mathscr{S}, \mathscr{L})$ with $T \subseteq \Theta$, let Q be the set of all ordered pairs (a, b) such that (1) $a\Theta b$, and (2) $a \vee b/a \wedge b \to^{\mathscr{S}} c/d$ with cTd implies $c = d$. Then $Q \in \mathrm{Con}(\mathscr{S}, \mathscr{L})$ and in $\mathrm{ST}(\mathscr{S}, \mathscr{L})$, is the relative pseudo-complement of T in Θ.*

Proof. If $T' \in \mathrm{ST}(\mathscr{S}, \mathscr{L})$, $T' \subseteq \Theta$ and $T \cap T' = \Delta$, then $aT'b$ clearly implies $a \vee bT'a \vee a$, and $a \wedge aT'a \wedge b$. If $a \vee b/a \wedge b \to^{\mathscr{S}} c/d$ with cTd, and if $\phi \in \mathscr{S}$ is the mapping for which $\phi(a \vee b) = c$ and $\phi(a \wedge b) = d$, then $\phi(a \vee b)\, T'\, \phi(a)$ and $\phi(a)\, T' \phi(a \wedge b)$ together imply that $\phi(a \vee b) = \phi(a) = \phi(a \wedge b)$, so $c = d$. It follows that $T' \subseteq Q$. We would be done now if it could be shown that $Q \in \mathrm{Con}(\mathscr{S}, \mathscr{L})$. In view of [12], Lemma 4, p. 149, we need only establish that

(a) aQb, $\phi \in S$ implies $\phi(a)\, Q\, \phi(b)$, and

(b) $a \geqq b \geqq c$ with aQb and bQc together imply that aQc.

To establish (a), we must show that if aQb

$$\phi(a) \vee \phi(b)/\phi(a) \wedge \phi(b) \to^{\mathscr{S}} c/d \quad \text{with} \quad cTd$$

implies $c = d$. But

$$\phi(a \vee b) \geqq \phi(a) \vee \phi(b) \geqq \phi(a) \wedge \phi(b) \geqq \phi(a \wedge b)$$

says that $\phi(a \vee b)/\phi(a \wedge b) \to^{\mathscr{S}} c/d$, and from this $c = d$ follows. To establish (b), note that if $a \geqq b \geqq c$ with aQb and bQc, then $a\theta c$ is clear from $\theta \in \mathrm{Con}(\mathscr{S}, \mathscr{L})$, and $a/c \to^{\mathscr{S}} x/z$ with xTz forces the existence of an element y such that $a/b \to^{\mathscr{S}} x/y$ and $b/c \to^{\mathscr{S}} y/z$. But now $x = y = z$, thus completing the proof.

This now leads immediately to

Theorem 2. *Each of* $\mathrm{ST}(\mathscr{S}, \mathscr{L})$, $\mathrm{LT}(\mathscr{S}, \mathscr{L})$ *and* $\mathrm{Con}(\mathscr{S}, \mathscr{L})$ *is a pseudocomplemented lattice. For a given* $T \in \mathrm{ST}(\mathscr{S}, \mathscr{L})$, T^* *as computed in* $\mathrm{ST}(\mathscr{S}, \mathscr{L})$ *is necessarily an* $\mathscr{S}$-congruence on L.

3. Tolerance kernels. Let $T \in \mathrm{ST}(\mathscr{S}, \mathscr{L})$. The *kernel* of T is defined by the rule $\{x \in \mathscr{L}\colon xT0\}$. This is evidently an order ideal of $\mathscr{L}$; if $T \in \mathrm{LT}(\mathscr{S}, \mathscr{L})$, then $\ker(T)$ is even an ideal of $\mathscr{L}$. In any event, it is always 0-*projective* (see [14]) in the sense that if $K = \ker(T)$, then

$$a \in K\,, \quad b \leqq a \vee x \quad \text{with} \quad b \wedge x = 0 \quad \text{together imply} \quad b \in K\,.$$

The idea of a distributive ideal appears to be due to O. Ore [19]. To say that the ideal D is distributive amounts to saying that the binary relation $\theta(D)$ defined by

$$x\,\theta(D)\,y \quad \text{if and only if} \quad x \vee d = y \vee d$$

for some $d \in D$ is a congruence on L. We shall also need the notion of a standard ideal [13]. This is an ideal S for which the binary relation

$$x\,\theta(S)\,y \quad \text{if and only if} \quad x \vee y = (x \wedge y) \vee s$$

for some $s \in S$ is a congruence on $\mathscr{L}$. H. J. Bandelt ([3], p. 377) mentions the fact that if $\mathscr{L}$ is section complemented, then $\mathrm{LT}(\mathscr{L}) = \mathrm{Con}(\mathscr{L})$. We shall now extend this by showing that in such a lattice if the kernel of a set-theoretic tolerance T is an ideal, then $T \in \mathrm{Con}(\mathscr{L})$. To this end, we note the following generalization of [9], Theorem 1, p. 55:

Theorem 3. *Let D be a distributive ideal of L for which* $\theta(D) \in \mathrm{Con}(\mathscr{S}, \mathscr{L})$. *If T is the smallest set-theoretic $\mathscr{S}$-tolerance having kernel D, then* $T = \theta(D)$.

Proof. Evidently, $T \subseteq \theta(D)$, so assume that $a\,\theta(D)\,b$. Then $(a \vee b)\,\theta(D)\,(a \wedge b)$ and $(a \vee b) \vee d = (a \wedge b) \vee d$ for some $d \in D$ so that $dT0$ implies $[(a \wedge b) \vee d]\,T[(a \wedge b) \vee 0]$, and $(a \vee b) \wedge [(a \wedge b) \vee d]\,T(a \vee b) \wedge [(a \wedge b) \vee 0]$. But $(a \vee b) \wedge [(a \wedge b) \vee d] = (a \vee b) \wedge [(a \vee b) \vee d] = a \vee b$, and $(a \vee b) \wedge [(a \wedge b) \vee 0] = a \wedge b$, from which $(a \vee b)\,T(a \wedge b)$, and consequently aTb follows. This shows that $\theta(D) \subseteq T$.

Corollary 4. *If L is section complemented, then* $\mathrm{LT}(\mathscr{S}, \mathscr{L}) = \mathrm{Con}(\mathscr{S}, \mathscr{L})$;

furthermore, if the kernel of the set-theoretic tolerance T is an ideal of $\mathscr{L}$, then $T \in \mathrm{Con}(\mathscr{S}, \mathscr{L})$.

Proof. It suffices to establish the second assertion, so let $T \in \mathrm{ST}(\mathscr{S}, \mathscr{L})$ with $K = \ker(T)$ an ideal of $\mathscr{L}$. A slight modification of [14], Theorem 4.2, p. 293 shows K to be a standard (hence distributive) ideal of $\mathscr{L}$. By the Theorem, $\theta(K) \subseteq T$. On the other hand, if aTb, and if t is a complement of $a \wedge b$ in $[0, a \vee b]$, then $t \in K$ shows that $a\ \theta(K)\ b$, and from this we obtain $T = \theta(K)$.

Remark 5. The assertion that $K = \ker(T)$ be an ideal of $\mathscr{L}$ is really needed. To see this, let $\mathscr{L} = \{0, a, b, 1\}$ be a four element Boolean algebra with $0 < a,\ b < 1$. If T is the set-theoretic tolerance obtained by collapsing the quotients $a/0$, $b/0$, $1/a$ and $1/b$, then $T \notin \mathrm{Con}(\mathscr{L})$, and $\ker(T) = \{0, a, b\}$ is not an ideal of $\mathscr{L}$.

4. Complementary tolerances. The question of the meaning of an $\mathscr{S}$-tolerance having a complement in $\mathrm{LT}(\mathscr{S}, \mathscr{L})$ or $\mathrm{Con}(\mathscr{S}, \mathscr{L})$ will now be addressed. In view of Theorem 2, we may restrict ourselves to the consideration of $\mathscr{S}$-congruences. The basic groundwork was laid by H. J. Bandelt ([3], Lemma 2.3, p. 375) and we begin by extending that result to the present context.

Lemma 6. *Let $T_1, T_2 \in \mathrm{LT}(\mathscr{S}, \mathscr{L})$ with $T_1 \cap T_2 = \Delta$. Then $T = T_1 \vee T_2$ is given by the rule aTb if and only if there exists elements c_i of $L\ (i = 1, 2)$ such that $a \vee b = c_1 \vee c_2$, $a \wedge b = c_1 \wedge c_2$, and $c_i T_i a \wedge b$.*

Proof. By [3], Lemma 2.3, p. 375, $T = T_1 \vee T_2$ in $\mathrm{LT}(\mathscr{L})$. Thus we need only establish that $T \in \mathrm{LT}(\mathscr{S}, \mathscr{L})$. To do this, assume aTb, and let $\phi \in S$. Then

$$\phi(a \vee b) = \phi(c_1 \vee c_2) \geqq \phi(c_1) \vee \phi(c_2) \geqq \phi(c_1) \wedge \phi(c_2) \geqq$$
$$\geqq \phi(c_1 \wedge c_2) = \phi(a \wedge b)\,.$$

Using the fact that $\phi(c_1) \wedge \phi(c_2)\,(T_1 \cap T_2)\,\phi(a \wedge b)$, we have $\phi(a \wedge b) = \phi(c_1) \wedge \phi(c_2)$. A similar argument produces $\phi(a \vee b) = \phi(c_1) \vee \phi(c_2)$, and from this $\phi(a)\,T\,\phi(b)$ follows.

We may now generalize [9], Theorem 3, p. 57 and [16], Thoerem 2, p. 88 by noting

Theorem 7. *An $\mathscr{S}$-congruence T has a complement in $\mathrm{LT}(\mathscr{S}, \mathscr{L})$ if and only if there is a central element z of L such that $T = \theta(0, z)$ and $\theta(z, 1) \in \mathrm{Con}(\mathscr{S}, \mathscr{L})$.*

Proof. If T, T' are complements in $\mathrm{LT}(\mathscr{S}, \mathscr{L})$, then by Lemma 6, there exist elements z, z' such that $zT0$, $z'T'0$, and $1 = z \vee z'$. Since $0(T \cap T')\,z \wedge z'$, we must have $z \wedge z' = 0$. For a given $x \in \mathscr{L}$, $zT0$ implies $(x \vee z) \wedge (x \vee z')\,T(x \vee 0) \wedge (x \vee z') = x$. Similarly, $(x \vee z) \wedge (x \vee z')\,T'\,x$, whence $(x \vee z) \wedge (x \vee z') = x$. Now $zT0$ implies $1 = z \vee z'T0 \vee z' = z'$. Since also $1T'z$, a dual argument produces $x = (x \wedge z) \vee (x \wedge z')$. By [14], Theorem 7.2, p. 299, z is central. We must still argue that $T = \theta(0, z)$. From the proof of Theorem 3, $\theta(0, z) \subseteq T$. To establish the reverse inclusion, note that aTb implies $a \vee zTb \vee z$. Now $zT'1$ with $z \leqq (a \vee z) \wedge (b \vee z) \leqq (a \vee z) \vee (b \vee z) \leqq 1$ forces $a \vee zT'b \vee z$, and from this $a \vee z = b \vee z$ follows, thereby showing that $T = \theta(0, z)$. Dually, $T' = \theta(z, 1)$. The converse is clear.

Note that [16], Theorem 2, p. 88 is valid for $\mathcal{S}$-congruences, as well as congruences; furthermore, [9], Theorem 3, p. 57 may be generalized by removing the restriction that the lattice be modular as follows:

Theorem 8. *Let z, z' be complementary elements of $\mathcal{L}$, with $T = T_{\mathcal{S}}(0, z)$ and $T' = T_{\mathcal{S}}(0, z')$. The following conditions are then equivalent*:

(a) *z is central, $\theta(0, z) = \theta_{\mathcal{S}}(0, z)$ and $\theta(z, 1) = \theta_{\mathcal{S}}(z, 1)$.*

(b) $\theta_{\mathcal{S}}(0, z) \cap \theta_{\mathcal{S}}(0, z') = \Delta$.

(c) $T \cap T' = \Delta$.

(d) *T and T' are complements in* $\mathrm{LT}(\mathcal{S}, \mathcal{L})$.

Proof. (b) $\Rightarrow$ (c) is clear, and so is (c) $\Rightarrow$ (d). To establish (d) $\Rightarrow$ (a), note that by Theorem 7, there is a central element z_1 such that $T = \theta(0, z_1)$ with $T' = \theta(z_1, 1)$. Evidently, $z \leqq z_1$, and the fact that $z_1 \wedge z' \, (T \cap T') \, 0$ implies that $z = z_1$. This leaves us to establish (a) $\Rightarrow$ (b). In that $[0, z'] = \ker(\theta(z, 1))$, Theorem 3 now shows that $T = \theta_{\mathcal{S}}(0, t)$ and $T' = \theta_{\mathcal{S}}(0, z')$, so (b) is clear.

In view of Theorems 7 and 8, it is appropriate at this point to introduce the notion of an $\mathcal{S}$-central element. This is a central element z for which $\theta(0, z) = \theta_{\mathcal{S}}(0,z)$ and $\theta(0, z) = \theta_{\mathcal{S}}(0, z)$. It follows from Theorem 8 that the $\mathcal{S}$-central elements of L form a Boolean sublattice of L, which will be denoted as $Z_{\mathcal{S}}(L)$. If $\mathcal{S}$ consists of only the join and meet translations, then z $\mathcal{S}$-central is equivalent to z being central, and $Z_{\mathcal{S}}(L)$ becomes the center of L.

If $T \in \mathrm{LT}(\mathcal{S}, \mathcal{L})$ has a complement in $\mathrm{LT}(\mathcal{S}, \mathcal{L})$, then it also has a complement in $\mathrm{Con}(\mathcal{S}, \mathcal{L})$. The converse is not true, and this may be seen by letting $T = T(0,z)$, where z is a neutral element of $\mathcal{L}$ that has no complement in $\mathcal{L}$. If $\mathcal{L}$ is complemented, the converse does, however, follow from [16], Theorem 2, p. 88.

5. Stone lattices. The term Stone lattice will be used here to denote a bounded pseudocomplemented lattice in which the pseudocomplement of each element has a complement. In [15] and [16], an investigation was launched into lattices $\mathcal{L}$ in which $\mathrm{Con}(\mathcal{L})$ is a Stone lattice. This investigation was carried out in terms of axioms involving weak projectivity, and was thus related to properties of the lattice $\mathcal{L}$. It is equally possible to think of the problem in terms of $\mathcal{S}$-congruences, and investigate the situation in terms of properties of the semigroup $\mathcal{S}$ of isotone mappings of $\mathcal{L}$. It turns out that the results and proofs carry over with only minor modifications, so that some of our discussion here will consist only of an indication of the possibilities. The analogue of Axiom (A) of [15] is

$$(\mathcal{S}, A)\ a/0 \rightarrow^{\mathcal{S}} c/d \quad \textit{with} \quad c > s \quad \textit{implies} \quad c/d \rightarrow^{\mathcal{S}} a_1/a_2 \quad \textit{for}$$

$$\textit{suitable elements} \quad a_1, a_2 \quad \textit{such that} \quad a \geqq a_1 > a_2 .$$

[15], Lemma 1, p. 178 now translates as

Lemma 7. *Axiom $(\mathcal{S}, A)$ is equivalent to the assertion that for every $\mathcal{S}$-congruence relation θ on L, $a\theta^*0$ iff the interval $[0, a]$ contains only trivial classes of θ.*

Axiom (B) becomes

$(\mathcal{S}, B)$ *$a > b$ implies the existence of an element t such that $t\theta 1$ but $t \ngeq a$, where θ is the $\mathcal{S}$-congruence relation generated by a/b.*

The proof of [15], Theorem 3, p. 179 now carries over to produce

Theorem 9. *The following conditions are equivalent:*

(1) *For each $\mathcal{S}$-congruence relation θ on L, $a\theta^*0 \Leftrightarrow a$ is a lower bound for $\{t \in L: t\theta 1\}$.*

(2) *$\mathcal{L}$ satisfies $(\mathcal{S}, A)$ and $(\mathcal{S}, B)$.*

The analog of [16], Theorem 4, p. 89 may now be stated as

Theorem 10. (1) *Let $\mathcal{L}$ satisfy the duals of axioms $(\mathcal{S}, A)$ and $(\mathcal{S}, B)$. If* $\mathrm{LT}(\mathcal{S}, \mathcal{L})$ *is a Stone lattice, then the kernel of every $\mathcal{S}$-tolerance on $\mathcal{L}$ has a supremum in $\mathcal{L}$. Furthermore, if $\mathcal{L}$ is complemented, then* $\mathrm{LT}(\mathcal{S}, \mathcal{L})$ *a Stone lattice is equivalent to* $\mathrm{Con}(\mathcal{S}, \mathcal{L})$ *being a Stone lattice.*

(2) *Let $\mathcal{L}$ and its dual satisfy $(\mathcal{S}, A)$ and $(\mathcal{S}, B)$. If the kernel of each $\mathcal{S}$-congruence of $\mathcal{L}$ has a supremum in $\mathcal{L}$, then* $\mathrm{LT}(\mathcal{S}, \mathcal{L})$ *is a Stone lattice.*

Proof. (1) follows from Theorem 2, Theorem 7, and the proof of [16], Theorem 4(1), p. 89. To establish (2), let $T \in \mathrm{LT}(\mathcal{S}, \mathcal{L})$. By Theorem 2, T^*, $T^{**} \in \mathrm{Con}(\mathcal{S}, \mathcal{L})$. By hypothesis, $z = \bigvee\{\ker(T^{**})\}$ and $z' = \bigvee\{\ker(T^*)\}$ both exist in L. By Theorem 9, we must have $zT^{**}0$, $z'T^*0$, zT^*1, and $z'T^{**}1$. It is now apparent that z and z' are complementary elements of $\mathcal{L}$. By Theorem 8, z is central and T^*, T^{**} are complements in $\mathrm{LT}(\mathcal{S}, \mathcal{L})$.

A rather curious connection exists between results in [17] and $\mathrm{LT}(\mathcal{S}, \mathcal{L})$ being a Stone lattice. In what follows, when we speak of $Z_{\mathcal{S}}(\mathcal{L})$ being a *complete* sublattice of $\mathcal{L}$, we shall mean that for any family $\{z_\alpha\}$ of elements of $Z_{\mathcal{S}}(\mathcal{L})$, both $z_0 = \bigwedge_\alpha z_\alpha$ and $z_1 = \bigvee_\alpha z_\alpha$ exist in $\mathcal{L}$, and are themselves $\mathcal{S}$-central.

Theorem 11. *If* $\mathrm{LT}(\mathcal{S}, \mathcal{L})$ *is a Stone lattice, or if $\mathcal{L}$ is complemented and* $\mathrm{Con}(\mathcal{S}, \mathcal{L})$ *is a Stone lattice, then $Z_{\mathcal{S}}(\mathcal{L})$ is a complete sublattice of $\mathcal{L}$.*

Proof. Let $\{z_\alpha\}$ be a family of $\mathcal{S}$-central elements, $T_\alpha = T(0, z_\alpha)$, and $T = \bigcap_\alpha T_\alpha$. Then $T \cap T_\alpha^* = \Delta$ implies $T^{**} \cap T_\alpha^* = \Delta$, so that $T^{**} \subseteq \bigcap_\alpha T_\alpha = T$. It follows that T has a complement in $\mathrm{LT}(\mathcal{S}, \mathcal{L})$, and by Theorem 7, there is an $\mathcal{S}$-central element z such that $T = T(0, z)$. Evidently, $z = \bigwedge_\alpha z_\alpha$. A dual argument applies to suprema.

If the conditions of Theorem 10 (2) hold, then it is clear that for each element a of L, the smallest $\mathcal{S}$-central element dominating a is the supremum of the kernel of $\theta_{\mathcal{S}}(0, a)$. We also mention that [15], Theorems 6 and 7, pp. 180–181 can be put in the present framework, as well as some earlier work by C. S. Johnson [18] and B. J. Thorne [20].

6. Complete tolerances. We generalize a definition of R. Wille [21] by saying that a tolerance T on L is *complete* if for each $x \in L$, there correspond elements x_T, x^T such that

$$\{y \in L\colon xTy\} = [x_T, x^T]\,.$$

If we now define $\phi_T(x) = x_T$, and $\phi_T^+(x) = x^T$, we may note that

$$\phi_T(x) \leqq y \Leftrightarrow x \wedge yTx \Leftrightarrow yTx \vee y \Leftrightarrow x \leqq \phi_T^+(y)\,.$$

This shows that ϕ_T is *residuated* in the sense of [4], p. 11. Note that T is completely determined by ϕ_T in that xTy is equivalent to the assertion that $x \geqq \phi_T(y)$ and $y \geqq \phi_T(x)$. Suppose now that ϕ is a decreasing residuated mapping on L with ϕ^+ its associated residual mapping. By [4], Theorem 2.9, p. 13, ϕ^+ is an increasing mapping. Thus we have

$$\phi(x) \leqq x \leqq \phi^+(x)$$

for every $x \in L$. Let us define a binary relation T on L by the rule

$$xTy \quad \text{if and only if} \quad x \geqq \phi(y) \quad \text{and} \quad y \geqq \phi(x)\,.$$

Note that xTy is also equivalent to $x \leqq \phi^+(y)$ and $y \leqq \phi^+(x)$. Because of this, it is easy to see that T is a complete tolerance on L for which $\phi = \phi_T$ and $\phi^+ = \phi_T^+$. It is even true that if x_iTy_i for all i, and if $x = \bigvee_i x_i$, $y = \bigvee_i y_i$ both exist, then xTy, with a similar assertion valid for arbitrary existing infima. [4], Theorem 15.1, p. 144 establishes a bijection between complete congruences and idempotent decreasing residuated maps. The corresponding result for complete tolerances may now be stated as

Theorem 12. *Given the complete tolerance relation T, if ϕ_T is defined by the rule $\phi_T(x) = \bigwedge\{y \in \mathscr{L}\colon yTx\}$, then ϕ_T is a decreasing residuated mapping. The correspondence $T \to \phi_T$ sets up a bijection between complete tolerances and decreasing residuated mappings. Under this bijection, complete congruences correspond to idempotent decreasing residuated mappings.*

[4], Exercise 15.2., p. 149 may now be used to deduce

Corollary 13. *Any complete tolerance on a bounded section semicomplemented lattice is in fact a congruence.*

The result of Corollary 13 remains valid if $\mathscr{L}$ satisfies the condition that:

$a > b$ implies the existence of an element t such that $a/b \to t/0$, $t \leqq a$, but $t \nleqq b$. A further investigation of complete tolerances is most certainly called for, but this will be reserved for a later paper.

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