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LOCAL SEMIGROUPS

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Connections between topological spaces and locales are well known. This article investigates some similar connections between topological groups and the so called local semigroups. All necessary facts concerning locales can be found in Johnstone [1].

Let us introduce the basic notions. *A frame* is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee\{a \wedge s: s \in S\}$$

holds for all $a \in L$, $S \subseteq L$. A frame homomorphism $K \to L$ is a map preserving finite meets and arbitrary joins. Let *Frm* be the category of frames. If T is a topological space, then the lattice $O(T)$ of open sets of T is a frame. A continuous map $f: S \to T$ determines a frame homomorphism $O(f): O(T) \to O(S)$ sending $V \in O(T)$ to $f^{-1}(V)$. We get a functor $O: Top \to Frm^{op}$, where *Top* is the category of topological spaces and continuous maps. The functor O is full and faithful on the subcategory $Sob \subseteq Top$ of all sober spaces. Then O has a right adjoint $P: Frm^{op} \to Top$ assigning to a frame L the topological space $P(L)$ of prime (i.e. $\wedge$-irreducible and $\neq 1$) elements of L. Open sets of $P(L)$ are $\hat{x} = \{a \in P(L): x \nleq a\}$ where $x \in L$. This fact indicates the importance of the opposite category $Loc = Frm^{op}$. Objects of *Loc* are called *locales*. Locales isomorphic to some $O(T)$ are called *spatial*. Hence *Sob* is isomorphic to the category of spatial locales.

1. BASIC PROPERTIES OF LOCAL SEMIGROUPS

1.1 Definition. *A frame semigroup* is a frame which is a lattice ordered semigroup.

Remark. A frame semigroup $(G, \cdot, \wedge, \bigvee)$ is a nonempty set G with the following properties:

1. $(G, \cdot)$ is a semigroup.
2. $(G, \wedge, \bigvee)$ is a complete lattice.
3. For all $a \in G$, $S \subseteq G$ we have $a \wedge \bigvee S = \bigvee\{a \wedge s: s \in S\}$, $a \,.\, \bigvee S = \bigvee\{a \,.\, s: s \in S\}$ and $\bigvee S \,.\, a = \bigvee\{s \,.\, a: s \in S\}$.

1.2. Definition. A semigroup $(G, \cdot, \wedge)$ which is a $\wedge$-semilattice is called a $\wedge$-*semigroup.*

1.3. Definition. *A homomorphism of frame semigroups* L, M is a map $f: L \to M$ which is a homomorphism of frames and semigroups.

Let *Frs* be the category of frame semigroups and let *Los* denote the opposite category of *Frs*. Objects of *Los* are called *local semigroups.*

Local semigroups isomorphic with $O(T(G))$, where $T(G)$ is the topological space of a topological group G, are called *spatial semigroups on G.*

1.4. Proposition. *Let G, H be topological groups, $f: G \to H$ a continuous homomorphism and $O(f): O(H) \to O(G)$ a map such that $O(f)(U) = f^{-1}(U)$ for $U \in O(H)$. Then:*

1. *If $O(f)$ is a homomorphism of local semigroups, then f is a surjection or $\overline{f(G)} = \overline{H \setminus f(G)} = H$.*

2. *If f is a surjection, then $O(f)$ is a homomorphism of local semigroups.*

Proof. 1. Let $O(f)$ be a homomorphism of local semigroups. If $f(G) \neq H$, then there exists $U \in O(H)$ such that $f(G) \cap U = \emptyset$. Hence $f(G) \cap U^{-1} = \emptyset$, $f^{-1}(U) = f^{-1}(U^{-1}) = \emptyset$ and $\emptyset = O(f)(U) . O(f)(U^{-1}) = O(f)(U . U^{-1}) = f^{-1}(U . U^{-1})$, a contradiction. We have $\overline{f(G)} = H$. Further, if there exists $h \in H \setminus f(G)$, then the existence of $U \in O(H)$, $U \subseteq f(G)$ implies $h . U \cap f(G) \neq \emptyset$. Thus there exist elements $v, g \in G$ and $u \in U$ such that $h . u = f(g)$ and $f(v) = u$. Finally $h = f(g) . [f(v)]^{-1} = f(g . v^{-1})$, a contradiction. We have $U \cap (H \setminus f(G)) \neq \emptyset$ for every $U \in O(H)$, i.e., $\overline{H \setminus f(G)} = H$.

2. $O(f)$ is a homomorphism of locales and for every U, $V \in O(H)$ we have $f^{-1}(U) . f^{-1}(V) = \bigcup\{f^{-1}(u): u \in U\} . \bigcup\{f^{-1}(v): v \in V\} = \bigcup\{f^{-1}(u) . f^{-1}(v): u \in U, \ v \in V\} = \bigcup\{f^{-1}(u . v): u \in U, \ v \in V\} = f^{-1}(U . V)$. Namely, for every elements $u \in U$, $v \in V$ there exist $a, b \in G$ such that $f(a) = u$, $f(b) = v$ and thus $a . b \in f^{-1}(u) . f^{-1}(v) \cap f^{-1}(u . v)$. Finally $f^{-1}(u) . f^{-1}(v) = f^{-1}(u . v)$.

1.5. Corollary. *Let Tog be the category of topological groups and continuous surjective homomorphisms. Then O: Tog $\to$ Los is a contravariant functor.*

Proof follows from 1.4.

1.6. Proposition. *Let G be a topological group. Then $T(G) \in$ Sob if and only if $T(G)$ is a T_0-space.*

Proof follows from the fact that a T_0-space of a topological group is a T_2-space.

Remark. If $M \neq \emptyset$ is a set with a binary operation $\cdot$ and a unary operation $^{-1}$, then we denote $A . B = \{a . b: a \in A, \ b \in B\}$ and $A^{-1} = \{a^{-1}: a \in A\}$ for any $A, B \subseteq M$.

1.7. Definition. Let $(G, \cdot)$ be a group and T a topological space on the set G.

Then T is called *a homogeneous space*, whenever $P . \{g\} \in O(T)$, $\{g\} . P \in O(T)$ for any $P \in O(T)$, $g \in G$.

1.8. Theorem. *Let* $(G, \cdot, ^{-1})$ *be a group and* T *a homogeneous topological space on the set* G. *Then*:

1. *Operation* $\cdot$ *is continuous in* T *if and only if the following implication holds*:

(T1) *If* $\emptyset \neq P \in O(T)$, *then there exist* $R, S \in O(T)$ *such that* $R \neq \emptyset \neq S$ *and* $R . S = P$.

2. *Operation* $^{-1}$ *is continuous in* T *if and only if the following implication holds*:

(T2) *If* $\emptyset \neq P \in O(T)$, *then there exists* $R \in O(T)$ *such that* $R \neq \emptyset$ *and* $R^{-1} = P$.

Proof. 1. $\Rightarrow$: If $a \in P$, then there exist $b, c \in G$ such that $a = b . c$. Further, there exist open sets $P_b, P_c \in O(T)$ such that $b \in P_b$, $c \in P_c$ and $P_b . P_c \subseteq P$. Let us denote $R = \bigcup\{P_b: a \in P\}$, $S = \bigcup\{P_c: a \in P\}$. Then $R, S \in O(T)$, $R \neq \emptyset \neq S$, $R . S = \bigcup\{P_b: a \in P\} . \bigcup\{P_c: a \in P\} \subseteq \bigcup\{P_b . P_c: a \in P\} \subseteq P$. Further $a = b . c \in P_b . P_c \subseteq R . S$, i.e., $P \subseteq R . S$ and consequently $P = R . S$.

$\Leftarrow$: If $\emptyset \neq P \in O(T)$, $a \in P$, $a = b . c$, where $b, c \in G$, then there exist $R, S \in O(T)$ such that $R \neq \emptyset \neq S$ and $P = R . S$. It means that $b . c = r . s$ for suitable elements $r \in R$, $s \in S$ and $R . s . c^{-1} \in O(T)$, $b^{-1} . s . S \in O(T)$, $b \in R . s . c^{-1}$, $c \in b^{-1} . r . S$ and $(R . s . c^{-1}) . (b^{-1} . r . S) \subseteq (R . s) . (b . c)^{-1} . (r . S) = R . s . (s^{-1} . r^{-1}) . . r . S = R . S = P$.

2. $\Rightarrow$: If $a \in P$, then there exists $b \in G$ such that $a = b^{-1}$ and $P_b \in O(T)$ such that $b \in P_b$, $P_b^{-1} \subseteq P$. If we denote $R = \bigcup\{P_b: a \in P\}$, then $R \in O(T)$ and $R^{-1} = [\bigcup\{P_b: a \in P\}]^{-1} = \bigcup\{P_b^{-1}: a \in P\} \subseteq P$. Further $a = b^{-1} \in P_b^{-1} \subseteq R^{-1}$, i.e., $P \subseteq R^{-1}$ and hence $P = R^{-1}$.

$\Leftarrow$: If $\emptyset \neq P \in O(T)$, $a \in P$, $a = b^{-1}$, where $b \in G$, then there exists $R \in O(T)$ such that $R \neq \emptyset$ and $R^{-1} = P$. It means that $b^{-1} = r^{-1}$ for a suitable element $r \in R$, $R . r^{-1} . b \in O(T)$, $b \in R . r^{-1} . b$ and $(R . r^{-1} . b)^{-1} \subseteq b^{-1} . r . R^{-1} = R^{-1} = P$ holds.

1.9. Corollary. *Let* $(G, \cdot, ^{-1})$ *be a group. Then* $T(G)$ *is a topological space of a topological group* G *if and only if* $T(G)$ *is a homogeneous space with properties* (T1) *and* (T2).

Proof immediately follows from 1.8.

1.10. Proposition. *A spatial semigroup on a group* G *has the unit element if and only if every element from* G *has the smallest neighbourhood in the belonging topology.*

Proof. It is sufficient to prove this proposition for $O(T(G))$ and the unit element $1 \in G$. Let E be the unit element in $O(T(G))$ and assume the smallest neighbourhood of 1 does not exist. Then there exists $U \in O(T(G))$ such that $1 \in U \subsetneq E$. If $x \in E \setminus U$, then $x = 1 . x \in E . U = U$, a contradiction.

Let E be the smallest neighbourhood of 1 and $U \in O(T(G))$ an arbitrary neighbourhood. Then $E = \bigcap\{V \in \Sigma: \Sigma$ is a complete system of neighbourhoods of $1\}$ is a normal subgroup in G and $U = \bigcup\{x . E: x \in U\} = U$ holds. It means that E is the unit element of the spatial semigroup on G.

Remark. A local semigroup is a lattice ordered semigroup but no group. Indeed, the fact that a local semigroup G is a group implies that G is a lattice ordered group which contradicts the existence of the greatest and the smallest element in G.

2. LOCAL SEMIGROUPS AND FUNCTOR P

The functor P maps a local semigroup $(L, \cdot)$ onto a topological space $(P(L), \hat{})$. We shall now deal with the natural question to define a group operation $*$ on $P(L)$ such that $(P(L), *, \hat{})$ is a topological group. We want to work with sober spaces and therefore $(P(L), *, \hat{})$ is a T_2-space (see 1.6). In this case $P(L)$ is a totally unordered set and this fact will be our general assumption.

2.1. Examples. 1. If $(G, \cdot)$ is a topological group, then $(O(T(G)), \cdot)$ is a local semigroup. For any prime elements $P, Q \in O(T(G))$ we have $P . Q = G$.

2. Let us introduce an operation $\square$ on $O(T(G))$ such that $X \square Y = G \setminus \overline{P . Q}$, where $X, Y \in O(T(G))$, $X = G \setminus \bar{P}$, $Y = G \setminus \bar{Q}$, $P, Q \subseteq G$. Then $(O(T(G)), \square)$ is a locale but no local semigroup in general.

3. If $(G, \cdot)$ is a topological group, then $(P(T(G)), *)$ is a group, where $(G \setminus \{a\}) * (G \setminus \{b\}) = G \setminus \{a . b\}$ for any $a, b \in G$. We can extend the operation $*$ on to $O(T(G))$ in the following way: $X * Y = \bigcap\{G \setminus \{a . b\}: a \text{ non} \in X, b \text{ non} \in Y\}$ for $X, Y \in O(T(G))$, $X = \bigcap\{G \setminus \{a\}: a \text{ non} \in X\}$ and $Y = \bigcap\{G \setminus \{b\}: b \text{ non} \in Y\}$. Then $(O(T(G)), *)$ is a semigroup. The operation $*$ on $O(T(G))$ is isotone with regard to the set inclusion.

2.2. Lemma. *Let $(L, \cdot)$ be a local semigroup, $(P(L), *)$ a group of all prime elements in L. Let $\tilde{x} = \{c \in P(L): c \varrho x\}$, where $x \in L$ and ϱ is a relation on L, and let $(P(L), \tilde{})$ be a topological space. Then:*

1. *A relation ϱ is compatible with the operation $*$ (i.e., $c \varrho x$, $a \in P(L) \Rightarrow a * c \varrho a * x$) if and only if $\{a\} * \tilde{x} = \tilde{}(a * x)$.*

2. *If ϱ is compatible with the operation $*$, then $(P(L), \tilde{})$ is a homogeneous topological space.*

Proof. 1. $z \in \{a\} * \tilde{x} \Rightarrow z = a * b$ for a suitable element $b \in \tilde{x} \Rightarrow z = a * b$, $b \varrho x \Rightarrow z = a * b$, $a * b \varrho a * x \Rightarrow z \in \tilde{}(a * x)$. On the other hand, $c \varrho x \Rightarrow c \in \tilde{x} \Rightarrow a * c \in \{a\} * \tilde{x} = \tilde{}(a * x) \Rightarrow a * c \varrho a * x$.

2. follows from 1.

2.3. Proposition. *Let $(L, \cdot)$ be a local semigroup, $(P(L), \cdot, \hat{})$ a topological group and $\hat{}: L \to O(P(L))$ a homomorphism of local semigroups $(L, \cdot)$ and $(O(P(L)), \cdot)$. Then $(P(L), \hat{})$ is the discrete topological space.*

Proof. If $(P(L), \cdot, \hat{})$ is a topological group, then the relation "non $\geqq$" is compatible with the operation $\cdot$ on $P(L)$, because L is a lattice ordered semigroup. With regard to 2.2,1 we have $\hat{x} \,.\, \hat{y} = \{a \,.\, b: a \in \hat{x},\ b \in \hat{y}\} = \bigcup\{\{a\} \,.\, \hat{y}: a \in \hat{x}\} = \bigcup\{\hat{}(a.y): a \in \hat{x}\} = \hat{}(\bigvee\{a.y: a \in \hat{x}\}) = \hat{}(\bigvee\{a: a \in \hat{x}\}) \,.\, \hat{y} = \hat{}(\bigvee\{a: a \in \hat{x}\}) \bigcup\{b: b \in \hat{y}\} = \bigcup\{\hat{}(\bigvee\{a: a \in \hat{x}\}).\ b: b \in \hat{y}\} = \bigcup\{\hat{}(\bigvee\{a \,.\, b: a \in \hat{x}\}): b \in \hat{y}\} = \hat{}(\bigvee\{a \,.\, b: a \in \hat{x},\ b \in \hat{y}\}) = \hat{}(\bigvee\{a: a \in \hat{x}\}) \,.\, \hat{}(\bigvee\{b: b \in \hat{y}\})$, where $\hat{x}$ is denoted by $\hat{}(x)$ in case that x is a long expression. Further, 1.8 (T1) implies that for any $c \in P(L)$, $\hat{c} \neq \emptyset$ there exist $r, s \in P(L)$ such that $\hat{r} \neq \emptyset \neq \hat{s}$ and $\hat{c} = \hat{r} \,.\, \hat{s}$. We have $\hat{c} = \hat{r} \,.\, \hat{s} = \hat{}(\bigvee\{a \,.\, b: a \in \hat{r},\ b \in \hat{s}\})$ and if c non $\geqq \bigvee\{a \,.\, b: a \in \hat{r},\ b \in \hat{s}\}$, then $c \in \hat{c}$, a contradiction. It means that $c \geqq \bigvee\{a \,.\, b: a \in \hat{r},\ b \in \hat{s}\}$ and if there exist $a_1, a_2 \in \hat{r}$, $a_1 \neq a_2$, then $c \gneqq a \,.\, b$, for $b \in \hat{s}$ and either $a = a_1$ or $a = a_2$ holds. But $c \in P(L)$, $a \,.\, b \in P(L)$ which contradicts the fact that $P(L)$ is totally unordered. Finally, $\hat{r}$ is a one point set and thus $(P(L), \cdot, \hat{})$ is a discrete topological group.

The demand to define a group operation $*$ on $P(L)$ such that $(P(L), *, \hat{})$ is a topological group together with Proposition 1.7 implies that $(P(L), \hat{})$ is a regular topological space. We shall restrict ourselves to regular locales.

2.4. Definition. If L is a regular locale and $(L, \cdot)$ is a local semigroup, then $(L, \cdot)$ is called *a regular semigroup.*

We shall use the regularity of a semigroup $(L, \cdot)$ only in the sense that the set $P(L)$ is equal to the set $D(L)$ of all dual atoms in L.

2.5. Proposition. *Let $(L, \cdot)$ be a local semigroup, $(P(L), *)$ a groupoid. Let $x, y \in L$. Then the following assertions are equivalent:*

1. $\hat{x} * \hat{y} \subseteq \hat{}(x \,.\, y)$;
2. $a, b \in L,\ a * b \geqq x \,.\, y \Rightarrow a \geqq x$ *or* $b \geqq y$.

Proof. $1 \Rightarrow 2$: $a * b \geqq x \,.\, y \Rightarrow a * b$ non $\in \hat{}(x \,.\, y) \Rightarrow a * b$ non $\in \hat{x} * \hat{y} \Rightarrow a \geqq x$ or $b \geqq y$.

$2 \Rightarrow 1$: $c \in \hat{x} * \hat{y} \Rightarrow c = a * b,\ a$ non $\geqq x,\ b$ non $\geqq y,\ a, b \in L \Rightarrow a * b$ non $\geqq x \,.\, y \Rightarrow c \in \hat{}(x \,.\, y)$.

2.6. Proposition. *Let $(L, \cdot)$ be a regular semigroup, let $(P(L), *)$ be a groupoid and $x, y \in L$, $c \in P(L)$. Then the following assertions are equivalent:*

1. $\hat{x} * \hat{y} = \hat{}(x \,.\, y)$;
2. $c \vee x \,.\, y = 1$ *if and only if there exist elements $a, b \in P(L)$ such that*

$$\text{(T)} \quad c = a * b\,, \quad a \vee x = 1\,, \quad b \vee y = 1\,.$$

Proof. $1 \Rightarrow 2$: $c \vee x \,.\, y = 1 \Leftrightarrow c$ non $\geqq x \,.\, y \Leftrightarrow c \in \hat{}(x \,.\, y) = \hat{x} * \hat{y} \Leftrightarrow$ there exist $a \in \hat{x}$, $b \in \hat{y}$ such that $c = a * b \Leftrightarrow$ there exist $a, b \in P(L)$ such that $c = a * b$, $a \vee x = 1 = b \vee y$.

$2 \Rightarrow 1$: $c \in \hat{}(x \,.\, y) \Leftrightarrow c$ non $\geqq x \,.\, y \Leftrightarrow c \vee x \,.\, y = 1 \Leftrightarrow$ there exist $a, b \in P(L)$ such that $c = a * b$, $a \vee x = 1 = b \vee y \Leftrightarrow$ there exist $a, b \in P(L)$ such that $c = a * b$, a non $\geqq x$, b non $\geqq y \Leftrightarrow c \in \hat{x} * \hat{y}$.

2.7. Proposition. *Let* $(L, \cdot)$ *be a local semigroup, let* $(P(L), *)$ *be a group and* $x, y \in L$. $a, b \in P(L)$. *Then:*

1. $a * \hat{x} = \hat{}(a . x)$ *if and only if the following condition is fulfilled:* $b \geqq x \Leftrightarrow \Leftrightarrow a * b \geqq a . x$.

2. $a \geqq y,\ b \geqq x,\ a * \hat{x} = \hat{}(a . x) \Rightarrow a * b \geqq y . x$.

Proof. 1a. If $a * b \geqq a . x$, then $a * b$ non $\in \hat{}(a . x)$ and thus $a * b$ non $\in a * \hat{x}$. It means that b non $\in \hat{x}$ and $b \geqq x$. If $\geqq x$, $a * b$ non $\geqq a . x$, then $a * b \in \in \hat{}(a . x) = a * \hat{x}$ and this implies $a * b = a * k$ for a suitable $k \in P(L)$, k non $\geqq x$. Finally, we have $b = k$ non $\geqq x$, a contradiction.

1b. If $p \in a * \hat{x}$, then $p = a * b$ for a suitable $b \in P(L)$, b non $\geqq x$ and thus $a * b$ non $\geqq a . x$. This implies $p = a * b \in \hat{}(a . x)$.

If $p \in \hat{}(a . x)$, then p non $\geqq a . x$ and if p non $\in a * \hat{x}$, then $a^{-1} * p$ non $\in \hat{x}$. Finally, we have $a^{-1} * p \geqq x$ and $p \geqq a . x$, a contradiction.

2. If $a \geqq y$, then $a . x \geqq y . x$ and if $b \geqq x$, $a * \hat{x} = \hat{}(a . x)$, then 1. yields $a * b \geqq a . x \geqq y . x$.

If $(L, \cdot)$ is a local semigroup and $(P(L), *)$ is a group, then we can extend the operation $*$ on to L in the following way:

We put $a * b = \bigwedge\{p * q: p, q \in P(L),\ p \geqq a,\ q \geqq b\}$ for any $a, b \in L$. Then $(L, *)$ is a semigroup and $*$ is an isotone operation on L.

2.8. Proposition. *Let* $(L, \cdot)$ *be a local semigroup, let* $(L, *)$ *be a semigroup with an isotone operation* $*$ *such that* $(P(L), *)$ *is a group. Then* $a * \hat{x} = \hat{z}$ *for any* $a \in P(L)$, $x \in L$ *and* $z = \bigwedge\{a * b: b \in P(L),\ b \geqq x\}$.

Proof. Let $c \in a * \hat{x}$. Then $c = a * d$ for a suitable element $d \in P(L)$, d non $\geqq x$. If c non $\in \hat{z}$, then $a * d \geqq \bigwedge\{a * b: b \in P(L),\ b \geqq x\}$. We have $a * b \geqq a * x$ for any $b \geqq x$ and it means that $a * d \geqq a * x$ and $d \geqq x$, a contradiction. Thus $a * \hat{x} \subseteq \hat{z}$.

If $c \in \hat{z}$, then c non $\geqq \bigwedge\{a * b: b \geqq x\}$. If c non $\in a * \hat{x}$, then $c = a * d$, $d \in P(L)$ implies $d \geqq x$. Further, $c = a * (a^{-1} * c)$ and $a^{-1} * c \geqq x$. Finally, $c \in \{a * b: b \geqq x\}$ and $c \geqq \bigwedge\{a * b: b \geqq x\}$, a contradiction. This implies $\hat{z} \subseteq a * \hat{x}$.

2.9. Proposition. *Let* $(G, \cdot)$ *be a topological group and let* $\cdot, *$ *be operations on* $O(T(G))$ *such that* $A . B = \{a . b: a \in A,\ b \in B\}$, $A * B = \bigcap\{G \setminus \{a . b\}: a$ non $\in A,\ b$ non $\in B\}$ *for any* $A, B \subseteq O(T(G))$. *Then we have for any* $X, Y, A \in O(T(G))$, $A = G \setminus \{a\}$, $a \in G$:

1. $\hat{X} * \hat{Y} = \hat{}(X . Y)$;

2. $A * X = \{a\} . X = \bigwedge\{C \in PO(T(G)): C \supseteq \{a\} . X$ *and* $A * \hat{X} = \hat{}(\{a\} . X)$.

Proof. 1. $(O(T(G)), \cdot)$ is a regular semigroup, $(O(T(G)), *)$ is a semigroup (see Example 2.1) and according to 2.6 it is sufficient to prove that for any $X, Y \in O(T(G))$ and any $C = G \setminus \{c\}$, $c \in G$ we have: $C \cup X . Y = G \Leftrightarrow$ there exist $a, b \in G$ such that $C = (G \setminus \{a\}) * (G \setminus \{b\}) = G \setminus \{a . b\}$ and $a \in X$, $b \in Y$. This immediately follows from the fact that $c \in X . Y$.

2a. $A * X = \bigcap\{G \setminus \{a . b\}: b \notin X\} = \bigcap\{G \setminus \{c\}: a^{-1} . c \notin X\} = \bigcap\{G \setminus \{c\}: c \notin$ $\notin \{a\} . X\} = \bigwedge\{C \in PO(T(G)): C \supseteq \{a\} . X\}$.

2b. $C \in A * \hat{X} \Leftrightarrow C = A * B$ for a suitable $B \in \hat{X} \Leftrightarrow C = G \setminus \{a . b\}$ for a suitable $b \in X \Leftrightarrow C$ non $\supseteq \{a\} . X \Leftrightarrow C \in \hat{}(\{a\} . X)$.

2.10. Definition. Let $(L, \cdot)$ be a regular semigroup, $(L, *)$ a semigroup with an isotone operation $*$ and $(P(L), *)$ a group. Then $(L, \cdot)$ is called *a homogeneous semigroup*.

Remark. If $(L, \cdot)$ is a homogeneous semigroup, then the topological space $(P(L), \hat{})$ is homogeneous with regard to the operation $*$ (see 2.8). $P(L)$ is a totally unordered set with regard to the order on L.

2.11. Definition. If a homogeneous semigroup $(L, \cdot)$ fulfils the condition (T) from 2.6, then we say that $(L, \cdot)$ *has the property* (T).

2.12. Theorem. *Let $(L, \cdot)$ be a homogeneous semigroup. Then:*

1. *If $(L, \cdot)$ has the property* (T), *then the operation $*$ is continuous in the space $(P(L), \hat{})$ if and if the following condition is fulfilled:*

(C1) *For any $0 \neq x \in L$ there exist $y, z \in L$ such that $y \neq 0 \neq z$ and $\hat{x} = \hat{}(y . z)$.*

2. *The operation $^{-1}$ inverse to $*$ on $P(L)$ is continuous in the space $(P(L), \hat{})$ if and only if the following condition is fulfilled:*

(C2) *There exists an isotone map $i: L \to L$ such that the restriction i on $P(L)$ is the operation $^{-1}$ inverse to $*$.*

Proof. 1. $\Leftarrow$: If $x \in L$, $\hat{x} \neq \emptyset$, then $x \neq 0$ and the existence of elements $y, z \in L$ such that $y \neq 0 \neq z$, $\hat{x} = \hat{}(y . z)$ follows from (C1). This fact and 2.6 imply $\hat{x} = \hat{}(y . z) = \hat{x} * \hat{z}$ and thus $\hat{y} \neq \emptyset \neq \hat{z}$ holds. With regard to Theorem 1.8,1 we have that $*$ is a continuous operation on $(P(L), \hat{})$.

$\Rightarrow$: If $*$ is a continuous operation on $(P(L), \hat{})$ and $0 \neq x \in L$, then in the case $\hat{x} \neq \emptyset$ there exist $y, z \in L$ such that $\hat{y} \neq \emptyset \neq \hat{z}$ and $\hat{x} = \hat{y} * \hat{z}$. Further, $\hat{x} = \hat{y} * \hat{z} = \hat{}(y . z)$ and $y \neq 0 \neq z$ follows from 2.6. The condition (C1) is fulfilled. In the case $\hat{x} = \emptyset$ we have $\hat{x} = \emptyset = \emptyset * \emptyset = \hat{x} * \hat{x} = \hat{}(x . x)$ (see 2.6) and (C1) is fulfilled again.

2. $\Leftarrow$: If $x \in L$, $\hat{x} \neq \emptyset$, then $x \neq 0$ and (C2) implies $\hat{}(i(x)) = \{p \in P(L): p \text{ non} \geqq i(x)\} = \{i(q) \in P(L): i(q) \text{ non} \geqq i(x)\} = \{q^{-1}: q \text{ non} \geqq x\} = (\hat{x})^{-1}$, because if $q \in P(L)$ is such that $q^{-1} = p$, then $i(q) = i(p^{-1}) = (p^{-1})^{-1} = p$. We have $i(x) \neq \emptyset$ and according to Theorem 1.8,2 the operation $^{-1}$ is continuous on $(P(L), \hat{})$.

$\Rightarrow$: If $^{-1}$ is a continuous operation in $(P(L), \hat{})$, then we define a map $i: L \to L$ in the following way: If $x \in L$, $\hat{x} = \emptyset$, then we put $i(x) = 0$ and if $\hat{x} \neq \emptyset$, then we put $i(x) = \bigvee\{y \in L: \hat{y} = \hat{x}^{-1}\}$. For $\hat{x} \neq \emptyset$ there exists $y \in L$ such that $\hat{y} \neq \emptyset$, $\hat{y} = \hat{x}^{-1}$ (see 1.8,2) and thus $i(x) > 0$. Now, we prove that i is an isotone map. If $a, b \in L$, $a \leqq b$, then a) if $\hat{a} = \emptyset$, then $i(a) = 0 \leqq i(b)$; b) if $\hat{a} \neq \emptyset$, then $\hat{b} \neq \emptyset$ and according to 1.8,2 there exist $y_a, y_b \in L$ such that $\hat{y}_a = \hat{a}^{-1}$, $\hat{y}_b = \hat{b}^{-1}$. For $a_0 = \bigvee\{l \in L: \hat{l} = \hat{y}_a\}$, $b_0 = \bigvee\{l \in L: \hat{l} = \hat{y}_b\}$ we have $\hat{a}_0 = \hat{y}_a$, $\hat{b}_0 = \hat{y}_b$ and $a_0 = \bigwedge A_0$, $b_0 = \bigwedge B_0$, where $A_0 = \{p \in P(L): p \geqq a_0\}$, $B_0 = \{p \in P(L): p \geqq b_0\}$. Further, $A_0 \supseteq B_0$ and $i(a) = a_0 = \bigwedge A_0 \leqq \bigwedge B_0 = b_0 = i(b)$ holds.

Finally, for any $p \in P(L)$ we have $i(p) = \bigvee\{y \in L: \hat{y} = \hat{p}^{-1}\} = \bigwedge\{y \in L: \hat{y} = P(L) \setminus \{p^{-1}\}\} = p^{-1}$ because for $y \in L$ such that $\hat{y} = P(L) \setminus \{p^{-1}\}$ we have $p^{-1} \geqq y$ and $\hat{p}^{-1} = P(L) \setminus \{p^{-1}\}$. An isotone map $i: L \to L$ has the properties $i/P(L) = {}^{-1}$ and (C2).

2.13. Corollary. *If* $(L, \cdot)$ *is a homogeneous semigroup with the properties* (T), (C1) *and* (C2), *then* $(P(L), *, \hat{})$ *is a topological group.*

Proof follows from 2.12.

2.14. Definition. If $(L, \cdot)$ is a homogeneous semigroup with the properties (T), (C1) and (C2), then $(L, \cdot)$ is called *a topologizable group.*

The category of topologizable groups with homomorphisms of local semigroups such that the restrictions on to groups of prime elements are group homomorphisms will be denoted by $\mathscr{P}$. If *Tog* is the category of topological groups, then the functor O maps any $(G, \cdot) \in Tog$ onto the topologizable semigroup $(O(G), \cdot) \in \mathscr{P}$ (see 2.1,3 and 2.9). The functor P maps any $(L, \cdot) \in \mathscr{P}$ onto the topological group $(P(L), *, \hat{}) \in Tog$. Now we can investigate the functors $O: Tog \to \mathscr{P}^{op}$ and $P: \mathscr{P}^{op} \to Tog$.

2.15. Theorem. *The functor* $P: \mathscr{P}^{op} \to Tog$ *is a right adjoint to the functor* $O: Tog \to \mathscr{P}^{op}$.

Proof. Let $L \in \mathscr{P}$ and let us construct the map $e: L \to OP(L)$ such that $e(l) = \hat{l}$ for any $l \in L$. Then e is a homomorphism of locales and according to 2.6 for any $x, y \in L$ we have $e(x \cdot y) = \hat{}(x \cdot y) = \hat{x} * \hat{y} = e(x) * e(y)$. It means that e is a homomorphism of local semigroups $(L, \cdot)$ and $(O\,P(L), *)$. Further, the restriction e_P of the map e on to $P(L)$ is the map $e_P: (P(L), *) \to (POP(L), \otimes)$, where $e_P(p) = \hat{p} = P(L) \setminus \{p\}$ and $e_P(p) \otimes e_P(q) = (P(L) \setminus \{p\}) \otimes (P(L) \setminus \{q\}) = P(L) \setminus \{p * q\}$ holds for any $p, q \in P(L)$ (see 2.1,2). This implies $e_P(p) \otimes e_P(q) = e_P(p * q)$ and similarly $e_P(p^{-1}) = P(L) \setminus \{p^{-1}\} = (P(L) \setminus \{p\})^{-1} = [e_P(p)]^{-1}$. Hence e is a morphism of $\mathscr{P}$.

Now, if $(G, +) \in Tog$ and $f: L \to O(G)$ is a morphism of $\mathscr{P}$, then we define a map $\tilde{f}: G \to L$ such that $\tilde{f}(g) = \bigvee\{a \in L: g \text{ non} \in f(a)\}$. We can prove in the category of locales that $\tilde{f}: G \to P(L)$, $\tilde{f}$ is continuous and $\tilde{f}^{-1}(\hat{x}) = f(x)$ for any $x \in L$. If we introduce a map $O(\tilde{f}): OP(L) \to O(G)$, then $O(\tilde{f})(\hat{x}) = \tilde{f}^{-1}(\hat{x})$ holds for any $x \in L$. We can prove again in the category of locales that $O(\tilde{f})$ is a homomorphism of locales with the property $O(\tilde{f}) \,.\, e = f$ and $\tilde{f}$ is the unique map with this property.

Now, we prove that $O(\tilde{f})$ is a morphism in $\mathscr{P}$: For any $x, y \in L$ we have $O(\tilde{f})(\hat{x} * \hat{y}) = \tilde{f}^{-1}(\hat{x} * \hat{y}) = \tilde{f}^{-1}(\hat{}(x \,.\, y)) = f(x \,.\, y) = f(x) + f(y) = \tilde{f}^{-1}(\hat{x}) + \tilde{f}^{-1}(\hat{y}) = O(\tilde{f})(\hat{x}) + O(\tilde{f})(\hat{y})$. Further, for the restriction $O(\tilde{f})_P: (POP(L), \otimes) \to (PO(G), \oplus)$ of the map $O(\tilde{f})$ on to $POP(L)$ (the operations $\otimes$ and $\oplus$ are described in 2.1,3) and for any $p, q \in P(L)$ we have:

a) $O(\tilde{f})_P\,[(P(L) \setminus \{p\}) \otimes (P(L) \setminus \{q\})] = O(\tilde{f})_P\,[P(L) \setminus \{p * q\}] = O(\tilde{f})_P\,(\hat{}(p * q)) = \tilde{f}_P^{-1}(\hat{}(p * q)) = f_P(p * q) = f_P(p) \oplus f_P(q) =$

$= \tilde{f}_P^{-1}(\hat{p}) \oplus \tilde{f}_P^{-1}(\hat{q}) = O(\tilde{f})(\hat{p}) \oplus O(\tilde{f})(\hat{q}) = [O(\tilde{f})(P(L) \setminus \{p\})] \oplus$
$\oplus [O(\tilde{f})(P(L) \setminus \{q\})]$ where $\tilde{f}_P^{-1}(f_P)$ is the restriction of $\tilde{f}^{-1}(f)$ onto $POP(L)$ ($P(L)$, respectively).

b) $[O(\tilde{f})_P (P(L) \setminus \{q\})]^{-1} = [O(\tilde{f})_P (\hat{q})]^{-1} = [\tilde{f}_P^{-1}(\hat{q})]^{-1} = [f_P(q)]^{-1} =$
$= f_P(q^{-1}) = \tilde{f}_P^{-1}(\hat{q}^{-1}) = O(\tilde{f})_P (\hat{q}^{-1}) = O(\tilde{f})_P (P(L) \setminus \{q^{-1}\}) =$
$= O(\tilde{f})_P [P(L) \setminus \{q\})^{-1}]$.

Finally, we prove that $\tilde{f}: G \to P(L)$ is a group homomorphism: Let $p, q \in P(L)$, $p \neq q$ and let $f(p) = f(q) = G \setminus \{g\}$, where $g \in G$. Then $p \vee q = 1$ and $G = f(1) = f(p \vee q) = f_P(p) \vee f_P(q) = G \setminus \{g\}$, a contradiction. We conclude that f_P is an injection. Now, for $g, h \in G$ we have

a) $f_P . \tilde{f}(g + h) = f(\bigvee\{a \in L: g + h \text{ non} \in f(a)\}) = \bigvee\{f(a): f(a) \subseteq$
$\subseteq G \setminus \{g + h\}\} = G \setminus \{g + h\} = (G \setminus \{g\}) * (G \setminus \{h\}) = \bigvee\{f(a): f(a) \subseteq$
$\subseteq G \setminus \{g\}\} * \bigvee\{f(a): f(a) \subseteq G \setminus \{h\}\} = f(\bigvee\{a: g \text{ non} \in f(a)\}) *$
$* f(\bigvee\{a: h \text{ non} \in f(a)\}) = f(\bigvee a: g \text{ non} \in f(a)\} * \bigvee\{a: h \text{ non} \in f(a)\}) =$
$= f_P(\tilde{f}(g) * \tilde{f}(h))$ and this implies that $\tilde{f}(g + h) = \tilde{f}(g) + \tilde{f}(h)$.

b) $f_P . \tilde{f}(g^{-1}) = f(\bigvee\{a \in L: g^{-1} \text{ non} \in f(a)\}) = \bigvee\{f(a): f(a) \subseteq G\{g^{-1}\}\} =$
$= G \setminus \{g^{-1}\} = (G \setminus \{g\})^{-1} = [\bigvee\{f(a): g \text{ non} \in f(a)\}]^{-1} =$
$= [f(\bigvee\{a \in L: g \text{ non} \in f(a)\})]^{-1} = f_P[(\bigvee\{a \in L: g \text{ non} \in f(a)\})]^{-1} = f_P[\tilde{f}(g)]^{-1}$
and this implies that $\tilde{f}(g^{-1}) = [f(g)]^{-1}$.

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