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BIFURCATION POINTS OF REACTION-DIFFUSION SYSTEMS WITH UNILATERAL CONDITIONS

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0. INTRODUCTION

Consider a reaction-diffusion system

(RD)
$$\frac{\partial u}{\partial t} = d_1 \,\Delta u + f(u, v)\,,$$
$$\frac{\partial v}{\partial t} = d_2 \,\Delta v + g(u, v)$$

on Ω with the boundary conditions

(BC)
$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N\,, \quad u = \bar{u}\,, \quad v = \bar{v} \text{ on } \Gamma_D\,.$$

Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega = \Gamma_N + \Gamma_D$, f, g are real functions on $\mathbb{R}^2$, d_1, d_2 are positive parameters (diffusion coefficients) and $\bar{u}$, $\bar{v}$ are constants such that $f(\bar{u}, \bar{v}) = g(\bar{u}, \bar{v}) = 0$, i.e. $\bar{u}$, $\bar{v}$ is a stationary and spatially homogeneous solution of (RD), (BC) (and also of (RD) with the Neumann boundary conditions). In mathematical models arising in biochemistry, morphogenesis, population dynamics etc., the following situation sometimes occurs: For a fixed d_2 (we shall suppose $d_2 = 1$) there is $d_0 > 0$ such that the solution $\bar{U} = [\bar{u}, \bar{v}]$ is stable if $d_1 > d_0$ and unstable if $d_1 < d_0$; moreover, d_0 is the greatest bifurcation point of the corresponding stationary system

(RD_S)
$$0 = d_1 \,\Delta u + f(u, v)\,,$$
$$0 = \Delta v \quad + g(u, v)$$

with (BC), i.e. a branch of spatially nonhomogeneous stationary solutions of (RD), (BC) bifurcates at d_0, $\bar{U}$ from the trivial branch $\{[d, U];\, d \in \mathbb{R},\, U = \bar{U}\}$. Such a situation in the case $n = 1$ (i.e. $\Omega = (0, 1)$, $\Delta u = u_{xx}$, $\Delta v = v_{xx}$) with the Neumann boundary conditions is described in detail for instance in [8], and an analogous information about the stationary solutions can be obtained (by analogous considerations) if the Dirichlet boundary conditions*) or combined boundary conditions

$$u(0) = \bar{u}\,, \quad u_x(1) = 0\,, \quad v(0) = \bar{v}\,, \quad v_x(1) = 0 \tag{0.1}$$

are considered.

In the paper [7], the system (RD) with some unilateral conditions instead of the usual boundary conditions is studied. Problems of such a type are formulated in terms of abstract inequalities on cones in a Hilbert space, the simplest example being the system (RD) on $\Omega = (0, 1)$ $(n = 1)$ subject to the conditions

$$u_x(0) = u_x(1) = 0\,,$$
$$v_x(0) = 0\,, \quad v(1) \geqq \bar{v}\,, \quad v_x(1) \geqq 0\,, \quad (v(1) - \bar{v})\, v_x(1) = 0\,.$$

The sign $\geqq$ can be replaced by $\leqq$, and also some more complicated conditions can be considered. In [7], simple examples are given showing that $\bar{u}$, $\bar{v}$ can be an unstable solution of the corresponding linearized system with unilateral conditions even for $d_1 > d_0$, i.e. that unilateral conditions can have a destabilizing effect in a certain sense. A result about eigenvalues of inequalities was announced and it was shown how an abstract theorem about the destabilizing effect of unilateral conditions follows from it.

The aim of the present paper is to prove (under certain assumptions) the existence of a bifurcation point d_I of $(\mathrm{RD_S})$ with unilateral conditions such that $d_I > d_0$, i.e. the existence of spatially nonconstant stationary solutions bifurcating from the point d_I, $\bar{U}$ lying in the domain of stability of $\bar{U}$ as a solution of (RD), (BC). We do not specify the notion of stability because it is not necessary for our purposes – all the results will be formulated in terms of bifurcation theory. We shall consider mainly the case when the Dirichlet condition for u is prescribed at least on part of the boundary, i.e., the case of purely Neumann conditions is excluded. (For more precise formulation see Remark 1.1.) The simplest case of the corresponding unilateral conditions is

$$\begin{aligned} &u(0) = \bar{u}\,, \quad u_x(1) = 0\,, \\ &v(0) = \bar{v}\,, \quad v(1) \geqq \bar{v}\,, \quad v_x(1) \geqq 0\,, \quad (v(1) - \bar{v})\, v_x(1) = 0\,. \end{aligned} \tag{0.2}$$

In the case of Neumann conditions on the whole boundary we shall obtain either the existence of nonhomogeneous solutions of (RD) with unilateral conditions or the existence of nonhomogeneous solutions to the corresponding shadow system for $d \to +\infty$, introduced in Section 5, i.e. (roughly speaking), the obtained bifurcation point d_I need not be finite.

The precise formulation of the problem in terms of abstract inequalities is given in Section 1. The presentation of main results is the subject of Section 2. An abstract result about the greatest bifurcation point for inequalities is contained in Theorem

*) We want to study only the case when $u = \bar{u}$, $v = \bar{v}$ is a solution; hence, the only possible Dirichlet conditions can be given by the values $\bar{u}$, $\bar{v}$.

2.1, Remark 2.1 deals with the special case of reaction-diffusion problems with unilateral conditions. Theorem 2.2 explains how the greatest bifurcation point of the inequality from Theorem 2.1 can be obtained from the bifurcation point of the equation by a homotopy joining the inequality with the corresponding linearized equation. Further, basic principles of the proof of Theorem 2.2 are explained. The basis for this proof are Lemmas 2.1, 2.2 (which seem to be interesting by themselves and therefore they are also included in Section 2) and a modification of Dancer's global bifurcation result (for the equations) which is given in Section 3. The detailed proof of Theorem 2.2 (i.e. also of Theorem 2.1) is the subject of Section 4. Let us remark that the method of the proof is a modification of that developed in [4, 5, 6]. In Section 5, the case of Neumann conditions on the whole boundary is briefly discussed.

1. NOTATION. ABSTRACT FORMULATION OF THE PROBLEM AND SOME COMMENTS

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and with the corresponding norm $\|\cdot\|$, K a closed convex cone in H with its vertex at the origin. We shall denote by $\tilde{H}$ the Hilbert space $H \times H$ with the inner product $\langle\cdot,\cdot\rangle_\sim$ given by $\langle U, W\rangle_\sim = \langle u, w\rangle + \langle v, z\rangle$ $(U = [u, v],\ W = [w, z])$ and with the corresponding norm $\|\cdot\|_\sim$. The identity mapping in H and $\tilde{H}$ will be denoted by I and $\tilde{I}$, respectively. The interior and the boundary of a set M will be denoted by M^0 and ∂M, respectively. We shall suppose $K \neq H$, $K^0 \neq \emptyset$. The weak convergence and the strong convergence is denoted by $\rightharpoonup$ and $\to$, respectively, $\mathbb{R}$ and $\mathbb{R}^+$ will be the set of all reals and of all positive reals. Throughout the paper we suppose that

(A) A is a linear completely continuous symmetric positive*) operator in H,

(N) $N_1, N_2\colon \tilde{H} \to H$ are nonlinear completely continuous operators,

$$\lim_{\|U\|_\sim \to 0} \frac{N_i(U)}{\|U\|_\sim} = 0 .$$

We shall study the bifurcation problem for the stationary inequality

(SI)
$$u \in H, \quad v \in K,$$
$$du - b_{11}Au - b_{12}Av + N_1(u, v) = 0,$$
$$\langle v - b_{21}Au - b_{22}Av + N_2(u, v), \psi - v\rangle \geqq 0 \quad \text{for all} \quad \psi \in K,$$

where $b_{ij} (i, j = 1, 2)$ are given real numbers satisfying

(SIGN)
$$b_{12}b_{21} < 0, \quad b_{11} \neq 0, \quad b_{22} < 0 .$$

*) I.e. $\langle Au, u\rangle > 0$ for all $u \neq 0$.

Simultaneously the system of equations

$$\text{(SE)} \qquad \begin{aligned} du - b_{11}Au - b_{12}Av + N_1(u, v) &= 0\,, \\ v \quad - b_{21}Au - b_{22}Av + N_2(u, v) &= 0 \end{aligned}$$

will be considered.

Remark 1.1. Consider the reaction-diffusion system (RD) from Introduction with a fixed $d_2 = 1$. Suppose that f, g are twice continuously differentiable, $f(\bar{u}, \bar{v}) = g(\bar{u}, \bar{v}) = 0$ with some $\bar{u} > 0$, $\bar{v} > 0$ and

$$(\text{SIGN}_{fg}) \qquad \frac{\partial f}{\partial v}(\bar{u}, \bar{v}) \frac{\partial g}{\partial u}(\bar{u}, \bar{v}) < 0\,, \quad \frac{\partial f}{\partial u}(\bar{u}, \bar{v}) \neq 0\,, \quad \frac{\partial g}{\partial v}(\bar{u}, \bar{v}) < 0\,.$$

Set

$$b_{11} = \frac{\partial f}{\partial u}(\bar{u}, \bar{v})\,, \quad b_{12} = \frac{\partial f}{\partial v}(\bar{u}, \bar{v})\,,$$

$$b_{21} = \frac{\partial g}{\partial u}(\bar{u}, \bar{v})\,, \quad b_{22} = \frac{\partial g}{\partial v}(\bar{u}, \bar{v})\,.$$

First, consider the case $n = 1$, $\Omega = (0, 1)$ and the boundary conditions (0.1). Introduce the space $H = \{u \in W_2^1(0, 1);\ u(0) = 0\}$ with the inner product

$$(1.1) \qquad \langle u, \varphi \rangle = \int_0^1 u_x \varphi_x \,\mathrm{d}x \quad \text{for all} \quad u, \varphi \in H$$

and with the corresponding norm $\|\cdot\|$ which is equivalent on H to the usual norm of the Sobolev space $W_2^1(0, 1)$. Introduce operators $A: H \to H_1$, $N_1, N_2: \tilde{H} \to H$ by

$$\langle Au, \varphi \rangle = \int_0^1 u\varphi \,\mathrm{d}x\,,$$

$$\langle N_1(u, v), \varphi \rangle = -\int_0^1 \left[f(\bar{u} + u, \bar{v} + v) - \frac{\partial f}{\partial u}(\bar{u}, \bar{v})\, u - \frac{\partial f}{\partial v}(\bar{u}, \bar{v})\, v \right] \varphi \,\mathrm{d}x\,,$$

$$\langle N_2(u, v), \psi \rangle = -\int_0^1 \left[g(\bar{u} + u, \bar{v} + v) - \frac{\partial g}{\partial u}(\bar{u}, \bar{v})\, u - \frac{\partial g}{\partial v}(\bar{u}, \bar{v})\, v \right] \psi \,\mathrm{d}x$$

for all $u, v, \varphi, \psi \in H$. It is easy to see that if u, v satisfy (SE) then the couple $u + \bar{u}$, $v + \bar{v}$ is the classical solution of (RD_S), (0.1) (we have $u, v \in C^2(\langle 0, 1 \rangle)$). Let K be a cone in H with its vertex at the origin. It is natural to define the weak solution of the system (RD_S) with unilateral conditions given by $H, K, \bar{u}, \bar{v}$ as a couple $u + \bar{u}$, $v + \bar{v}$, where u, v satisfy (SI) with the operators defined above (see [7], cf. e.g. [2]). Particularly, if we choose $K = \{v \in H;\ v(1) \geqq 0\}$, then it is easy to see that the weak solution of (RD_S) with the conditions given by $H, K, \bar{u}, \bar{v}$ satisfies (RD_S), (0.2) in the classical sense (we have $u, v \in C^2(\langle 0, 1 \rangle)$). (In more complicated examples the weak solution of (RD_S) with conditions given by K need not be a classical solution of the corresponding boundary value problem.)

In the case $n > 1$ we shall consider a bounded domain in $\mathbb{R}^n$ with a lipschitzian boundary $\partial\Omega$. We shall suppose that Γ_D, Γ_N are disjoint sets which are open in $\partial\Omega$ and such that meas $[\partial\Omega \setminus (\Gamma_D \cup \Gamma_N)] = 0$ and meas $\Gamma_D > 0$. We shall define H as the space $\{u \in W_2^1(\Omega);\ u = 0 \text{ on } \Gamma_D \text{ in the sense of traces}\}$ equipped with the inner product

$$\langle u, \varphi \rangle = \int_\Omega \sum_{i=1}^n u_{x_i} \varphi_{x_i} \, dx \quad \text{for all} \quad u, \varphi \in H . \tag{1.2}$$

The corresponding norm is equivalent on H to the usual norm of the Sobolev space $W_2^1(\Omega)$ under the assumption meas $\Gamma_D > 0$. Further, we can define the operators A, N_i as above (but by the integrals over Ω). It is easy to see from the usual regularity argument that if u, v is a solution of (SE) then $u + \bar{u}$, $v + \bar{v}$ is also a (classical) solution of (RD_S), (BC). Analogously as above, we can consider a cone K with its vertex at the origin in H and define the weak solution of (RD_S) with the unilateral conditions given by $H, K, \bar{u}, \bar{v}$ as a couple $u + \bar{u}$, $v + \bar{v}$, where u, v is a solution of (SI). For example, for $K = \{w \in H;\ w \leqq 0 \text{ on } \Gamma_N\}$ the weak solution of (RD_S) with the conditions given by $H, K, \bar{u}, \bar{v}$ is the weak solution of (RD_S) with the boundary conditions

$$u = \bar{u}\,, \quad v = \bar{v} \text{ on } \Gamma_D\,,$$

$$\frac{\partial u}{\partial n} = 0\,, \quad \frac{\partial v}{\partial n} \leqq 0\,, \quad v \leqq \bar{v}\,, \quad \frac{\partial v}{\partial n}(v - \bar{v}) = 0 \text{ on } \Gamma_N\,.$$

Remark 1.2. Speaking about the problem (RD_S) we shall automatically have in mind the weak formulation given by (SE) or (SI). Particularly, in this sense we can speak about critical or bifurcation points of (RD_S) (with (BC) or with the conditions given by $H, K, \bar{u}, \bar{v}$) which will be defined below for (SE), (SI).

Remark 1.3. We shall usually write (SI) and (SE) in the vector form

(SI)~
$$U \in \tilde{K}\,,$$
$$\langle D(d)\,U - B\tilde{A}U + N(U),\ \Phi - U\rangle_\sim \geqq 0 \quad \text{for all} \quad \Phi \in \tilde{K}$$

and

(SE)~
$$D(d)\,U - B\tilde{A}U + N(U) = 0\,,$$

respectively, where

$\tilde{K} = \{U \in \tilde{H};\ U = [u, v],\ v \in K\}$,

$D(d) = \begin{pmatrix} d, 0 \\ 0, 1 \end{pmatrix}$, i.e. $D(d)\,U = [du, v]$ for $U = [u, v]$,

B is the matrix with the elements b_{ij} $(i, j = 1, 2)$,

$\tilde{A}U = [Au, Av]$, i.e. $B\tilde{A}U = [b_{11}Au + b_{12}Av,\ b_{21}Au + b_{22}Av]$,

$N(U) = [N_1(u, v),\ N_2(u, v)]$

for all $U = [u, v] \in \tilde{H}$. Simultaneously, we shall consider the corresponding linearized problems

$(\mathrm{SI_L})^{\sim}$ $$U \in \tilde{K},$$

$$\langle D(d)\, U - B\tilde{A}U,\ \Phi - U\rangle_{\sim} \geqq 0 \quad \text{for all} \quad \Phi \in \tilde{K},$$

$(\mathrm{SE_L})^{\sim}$ $$D(d)\, U - B\tilde{A}U = 0\,.$$

Definition 1.1. A point $d_0 > 0$ is called a *critical point of* $(\mathrm{SI_L})^{\sim}$ or a *critical point of* $(\mathrm{SE_L})^{\sim}$ if there exists a nontrivial solution of $(\mathrm{SI_L})^{\sim}$ or of $(\mathrm{SE_L})^{\sim}$, respectively, with $d = d_0$. We shall denote the set of all solution of $(\mathrm{SI_L})^{\sim}$ and of $(\mathrm{SE_L})^{\sim}$ for $d = d_0$ by $E_I(d_0)$ and $E_B(d_0)$, respectively. A critical point d_0 of $(\mathrm{SE_L})^{\sim}$ is *simple* if the dimension of $\bigcup_{k=1}^{\infty} \operatorname{Ker} (D(d_0)\,\tilde{I} - B\tilde{A})^k$ is one.

Definition 1.2. A point $d_0 > 0$ is called a *bifurcation point of* $(\mathrm{SI})^{\sim}$ or a *bifurcation point of* $(\mathrm{SE})^{\sim}$ if any neighbourhood of $[d_0, 0]$ in $\mathbb{R} \times \tilde{H}$ contains a solution $[d, U]$ of $(\mathrm{SI})^{\sim}$ or $(\mathrm{SE})^{\sim}$, respectively, satisfying $\|U\|_{\sim} \neq 0$.

Remark 1.4. If d_0 is a simple criticial point of $(\mathrm{SE_L})^{\sim}$ then $E_B(d_0)$ is spanned by a single vector W_0. If $E_B(d_0) \cap \tilde{K}^0 \neq \emptyset$ then we can choose W_0 such that $-\ W_0 \in \tilde{K}^0$, $W_0 \notin \tilde{K}$.

Remark 1.5. If d_0 is a bifurcation point of $(\mathrm{SE})^{\sim}$ then d_0 is a critical point of $(\mathrm{SE_L})$; further, if d_n, U_n satisfy $(\mathrm{SE})^{\sim}$, $d_n \to d_0$, $\|U_n\|_{\sim} \to 0$, $W_n = U_n/\|U_n\|_{\sim} \rightharpoonup W$, then $W_n \to W$, $W \in E_B(d_0)$. Analogously, if d_0 is a bifurcation point of $(\mathrm{SI})^{\sim}$ then d_0 is a critical point of $(\mathrm{SI_L})^{\sim}$; if d_n, U_n satisfy $(\mathrm{SI})^{\sim}$, $d_n \to d_0$, $\|U_n\|_{\sim} \to 0$, $W_n = U_n/\|U_n\|_{\sim} \rightharpoonup W$, then $W_n \to W$, $W \in E_I(d_0)$. This will follow from Remark 2.5 by setting $\tau_n = \tau = 1$ (cf. also [6, Remark 1.2]).

Remark 1.6. In the sequel, we shall have in mind the situation mentioned in Introduction when d_0 separates the domain of stability from the domain of instability of the solution $\bar{U}$ of (RD), (BC). In some examples (e.g. in the situation described in [8]) this fact is connected with the following assumption (examples will be studied in a forthcoming paper):

(GC) for any $d > d_0$, all the real eigenvalues of $B\tilde{A} - D(d)\,\tilde{I}$ are negative, for any $d \in (d_0 - \xi, d_0)$ with some $\xi > 0$ there is one positive simple eigenvalue of $B\tilde{A} - D(d)\,\tilde{I}$ and the other real eigenvalues are negative.

Note that (GC) particularly ensures that d_0 is the greatest critical point of $(\mathrm{SE_L})^{\sim}$ (see Definition 1.1).

Remark 1.7. Let us remark that if our space is equipped with the inner product (1.1) or (1.2) then Δ is represented by the identity mapping in H in the weak formulation (Remark 1.1). If meas $\Gamma_D = 0$ then (1.1) does not define an inner product and it is necessary to use the usual inner product

$$\langle u, \varphi\rangle = \int_{\Omega} \Big(\sum_{i=1}^{n} u_{x_i}\varphi_{x_i} + \eta u\varphi\Big)\, \mathrm{d}x \tag{1.3}$$

with a positive η and to replace the expressions in (SE) and (SI) by

$$du - (b_{11} + \eta d) Au - b_{12}Av \ldots,$$
$$v - b_{21}Au - (b_{22} + \eta) Av \ldots.$$

This creates no problem with the second equation or inequality provided η is sufficiently small. Complications arise in the first equation because the coefficient $b_{11} + \eta d$ depends on the variable parameter d. More precisely, see Section 5.

Remark 1.8. It would be possible to replace (BC) by the boundary conditions

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{N,1}, \quad u = \bar{u} \text{ on } \Gamma_{D,1}, \quad \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_{N,2}, \quad v = \bar{v} \text{ on } \Gamma_{D,2},$$

where $\Gamma_{N,1} \neq \Gamma_{N,2}$, $\Gamma_{D,1} \neq \Gamma_{D,2}$ in general, and suppose only meas $\Gamma_{D,1} > 0$ (i.e., Neumann conditions can be prescribed for v on the whole boundary, cf. Remark 1.7). But in this case it would be necessary to consider different spaces H_1, H_2 for u, v and the situation would be formally more complicated.

2. MAIN RESULTS

Theorem 2.1. *Let d_0 be a simple critical point of $(\mathrm{SE_L})^\sim$ satisfying (GC) from Remark 1.6, $E_B(d_0) \cap \tilde{K}^0 \neq \emptyset$, and let (A), (N), (SIGN) be fulfilled. Then there exists a bifurcation point d_I of the inequality $(\mathrm{SI})^\sim$ satisfying $d_I > d_0$. In more detail, there is $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ there exist $d(\delta)$, $U(\delta)$ satisfying $(\mathrm{SI})^\sim$, $U(\delta) \in \partial\tilde{K}$, $\|U(\delta)\|^2_\sim = \delta$, $d(\delta) > d_0$ and such that all the limit points d_I*) of $d(\delta)$ for $\delta \to 0_+$ are greater than d_0; $d(\delta)$, $U(\delta)$ do not satisfy $(\mathrm{SE})^\sim_\lambda$.*

Remark 2.1. Consider the system (RD) satisfying the assumptions of Remark 1.1, i.e., particularly, meas $\Gamma_D > 0$ and $(\mathrm{SIGN_{fg}})$ hold. Let H, A, N be from the definition of the weak solution (see Remark 1.1) and let K be a closed convex cone in H with its vertex at the origin. For this special case we can read Theorem 2.1 as follows:

Let d_0 be the greatest critical point of $(\mathrm{RD_S})$, (BC), i.e. the greatest d for which $\lambda = 0$ is an eigenvalue of the problem

$$(\mathrm{RD}_\lambda) \qquad d\,\Delta u + \frac{\partial f}{\partial u}(\bar{u}, \bar{v})\, u + \frac{\partial f}{\partial v}(\bar{u}, \bar{v})\, v = -\lambda\,\Delta u\,,$$
$$\Delta v + \frac{\partial g}{\partial u}(\bar{u}, \bar{v})\, u + \frac{\partial g}{\partial v}(\bar{u}, \bar{v})\, v = -\lambda\,\Delta v$$

with (BC). (This is the classical formulation of $B\tilde{A}U - D(d)\,U = \lambda U$.) Suppose that d_0 is simple (i.e. $\dim \bigcup_{k=1}^{\infty} \operatorname{Ker}(D(d_0)\,\tilde{I} - B\tilde{A})^k = 1$). Let all the real eigenvalues of (RD_λ), (BC) be negative for any $d > d_0$, and for any $d \in (d_0 - \xi, d_0)$ (for some

*) There is at least one.

$\xi > 0$) let there be a unique positive simple eigenvalue of (RD_λ), (BC), the other real eigenvalues being negative. Suppose that there exists a solution u_0, v_0 of (RD_λ), (BC) with $d = d_0$, $\lambda = 0$ satisfying $v_0 - \bar{v} \in K^0$. Then there exists a bifurcation point $d_I > d_0$ of $(\mathrm{RD_S})$ with the conditions given by $H, K, \bar{u}, \bar{v}$. For any $\delta \in (0, \delta_0)$ there are $d_\delta > d_0$ and a nonconstant weak solution u_δ, v_δ of $(\mathrm{RD_S})$ (with $d = d_\delta$) with the conditions given by $H, K, \bar{u}, \bar{v}$, satisfying $\|[u_\delta - \bar{u}, v_\delta - \bar{v}]\|^2_\sim = \delta$, $v_\delta - \bar{v} \in \partial K$; $d_\delta, u_\delta, v_\delta$ do not satisfy $(\mathrm{RD_S})$, (BC). Analogously for $n = 1$, $\Omega = (0, 1)$ and the boundary conditions (0.1). Particularly, for the cone $K = \{v \in H;\ v(1) \geqq 0\}$, d_I is a bifurcation point of $(\mathrm{RD_S})$, (0.2) and $d_\delta, u_\delta, v_\delta$ are solutions of $(\mathrm{RD_S})$, (0.2) satisfying $v_\delta(1) = \bar{v}$ (see Remark 1.1).

For this reformulation of Theorem 2.1 for the case of reaction-diffusion systems it is sufficient to recall Remark 1.1. Let us remark that all our assumptions are satisfied in the situation described in [8]. We shall study the applications more precisely in a forthcoming paper.

Remark 2.2. We shall denote by P and $\tilde{P}$ the projections onto the closed convex cone K in H and $\tilde{K}$ in $\tilde{H}$, respectively, i.e.

$$\|Pu - u\| = \min_{w \in K} \|w - u\|, \quad \|\tilde{P}U - U\|_\sim = \min_{W \in \tilde{K}} \|W - U\|_\sim .$$

Obviously $\tilde{P}U = [u, Pv]$ for all $U = [u, v] \in \tilde{H}$. It is easy to see (precisely see [10]), that P is positive homogeneous ($P(tu) = tPu$ for $t > 0$, $u \in H$), lipschitzian and

$$\text{(P)} \qquad (I - P)\, u = 0 \quad \text{for} \quad u \in K, \quad \|u\|^2 \geqq \langle (I - P)\, u, u \rangle > 0 \quad \text{for} \quad u \notin K,$$

$$(\mathrm{P}, \mathrm{K}^0) \qquad \langle (I - P)\, u, v \rangle < 0 \quad \text{for} \quad u \notin K, \quad v \in K^0 .$$

The same holds for $\tilde{P}, \tilde{I}, \tilde{K}$.

Remark 2.3. It is well-known (see e.g. [10]) and easy to see that for any $V \in \tilde{H}$, $\tilde{P}V$ is the unique point from $\tilde{K}$ satisfying

$$\langle V - \tilde{P}V, W - \tilde{P}V \rangle \leqq 0 \quad \text{for all} \quad W \in \tilde{K} .$$

The inequality in $(\mathrm{SI})^\sim$ can be written as $\langle D^{-1}B\tilde{A}U - D^{-1}N(U) - U, W - U \rangle \leqq 0$ where D^{-1} is the inverse of $D(d)$. It follows (setting $V = D^{-1}B\tilde{A}U - D^{-1}N(U)$) that $(\mathrm{SI})^\sim$ is equivalent to

$$D(d)\, U - \tilde{P}(B\tilde{A}U - N(U)) = 0 . \tag{2.1}$$

Analogously, d is a critical point of $(\mathrm{SI_L})^\sim$ with $U \in E_I(d)$ if and only if

$$D(d) - \tilde{P}B\tilde{A}U = 0 . \tag{2.2}$$

By using the fact that $U \in \tilde{K}$ if and only if $D(d)\, U \in \tilde{K}$ for any $d > 0$, it follows that

$$U \in E_B(d) \cap \tilde{K} \quad \text{if and only if} \quad U \in E_I(d), \quad B\tilde{A}U \in \tilde{K} ; \tag{2.3}$$

(2.4) U satisfies $(\mathrm{SE})^\sim$ and $U \in \tilde{K}$ if and only if U satisfies (SI) and

$$B\tilde{A}U - N(U) \in \tilde{K} .$$

Definition 2.1. For each $\delta > 0$ fixed we shall denote by Z_δ the closure (in $\mathbb{R} \times \tilde{H} \times \mathbb{R}$) of the set of all $[d, U, \tau] \in \mathbb{R}^+ \times \tilde{H} \times (0, 1\rangle$ such that

(a) $$\|U\|^2_\sim = \delta\tau\,,$$

(b) $$D(d)\,U - B\tilde{A}U + \tau(\tilde{I} - \tilde{P})\,B\tilde{A}U + \tau\,R(U) = 0\,,$$

where $R(U) = \tilde{P}B\tilde{A}U - \tilde{P}(B\tilde{A}U - N(U))$,

$$\text{i.e.}\quad R(U) = [N_1(U), R_2(U)] \quad \text{with} \quad R_2(U) =$$
$$= P(b_{21}Au + b_{22}Av) - P(b_{21}Au + b_{22}Av - N_2(U)) \quad \text{for} \quad U = [u, v]\,.$$

Remark 2.4. The assumptions (A), (N) imply that

(R) $$R\colon \tilde{H} \to \tilde{H} \quad \text{is completely continuous}\,,\ \lim_{\|U\|_\sim \to 0} \frac{R(U)}{\|U\|_\sim} = 0$$

because $\tilde{P}$ is lipschitzian (Remark 2.2).

Remark 2.5. If $[d_n, U_n, \tau_n] \in \mathbb{R} \times \tilde{H} \times (0, 1\rangle$ satisfy (b), $\|U_n\|_\sim \neq 0$, $[d_n, U_n, \tau_n] \to [d, 0, \tau]$, $d > 0$, $W_n = U_n/\|U_n\|_\sim \rightharpoonup W$, then $W_n \to W$ and

$$D(d)\,W - B\tilde{A}W - \tau(\tilde{I} - \tilde{P})\,B\tilde{A}W = 0\,.$$

This follows from (b) (with d_n, U_n, τ_n) divided by $\|U_n\|_\sim$ by using the compactness of A, the condition (R) (Remark 2.4) and the fact that $D(d_n)\,W_n \to D(d)\,W$ if and only if $W_n \to W$.

Remark 2.6. Setting $\tau = 0$ and $\tau = 1$ in (b) we obtain $(\mathrm{SE_L})^\sim$ and (2.1) (which is equivalent to $(\mathrm{SI})^\sim$ by Remark 2.3), respectively. For $[d, U, \tau] \in Z_\delta$ we have $\tau = 0$ if and only if $U = 0$ by (a). If $[d, 0, 0] \in Z_\delta$, $d > 0$, then d is a critical point of $(\mathrm{SE_L})^\sim$. Indeed, there exist $[d_n, U_n, \tau_n] \in Z_\delta$ such that $\tau_n > 0$ $(n = 1, 2, \ldots)$, $[d_n, U_n, \tau_n] \to [d, 0, 0]$ and the assertion follows from Remark 2.5 by setting $\tau = 0$.

Remark 2.7. It follows from the compactness of A, N that the set Z_δ is locally compact.

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *be fulfilled. Then for each* $\delta \in (0, \delta_0)$ (*with some* $\delta_0 > 0$ *fixed*) *there exists a closed compact connected subset* $Z^+_{\delta,0}$ *of* Z_δ *containing* $[d_0, 0, 0]$ *and at least one point* $[d(\delta), U(\delta), 1]$. *The following implications are true for all* $[d, U, \tau] \in Z^+_{\delta,0}$:

(c) *if* $[d, U, \tau] \neq [d_0, 0, 0]$ *then* $B\tilde{A}U \notin \tilde{K}$,

(d) *if* $[d, U, \tau] \neq [d_0, 0, 0]$ *then* $d_0 < d \leqq d_m$

with some $d_m > 0$ *independent of* δ. (See Fig. 1.)

Proof of Theorem 2.1. Suppose that Theorem 2.2 is true. Remark 2.6 and Definition 2.1 imply that for any $[d(\delta), U(\delta), 1] \in Z^+_{\delta,0}$ from Theorem 2.2, $d(\delta), U(\delta)$ satisfy $(\mathrm{SI})^\sim$, $\|U(\delta)\|^2_\sim = \delta$ and $D(d(\delta))\,U(\delta) \in \tilde{K}$, i.e. $U(\delta) \in \tilde{K}$. We shall prove $U(\delta) \in \partial\tilde{K}$ for all $\delta \in (0, \delta_0)$ with δ_0 sufficiently small. In the opposite case there are

$d_n = d(\delta_n)$, $U_n = U(\delta_n)$ satisfying $(\mathrm{SI})^{\sim}$ and $\|U_n\|^2_{\sim} = \delta_n \to 0$, $U_n \in \tilde{K}^0$. It follows from (b) from Definition 2.1 (with $d_n, U_n, \tau = 1$) that $\tilde{P}(B\tilde{A}U_n - N(U_n)) \in \tilde{K}^0$, i.e. $B\tilde{A}U_n - N(U_n) \in \tilde{K}^0$. Using (d) from Theorem 2.2 we can suppose $d_n \to d \geqq d_0$, $W_n = U_n/\|U_n\| \to W$. It follows from (b) that $W_n \to W$ and $D(d)\, W - B\tilde{A}W = 0$

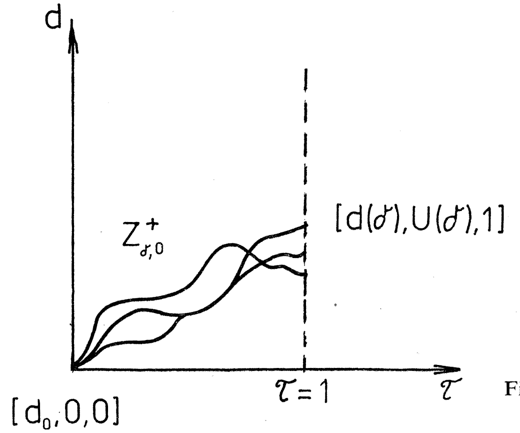

Fig. 1.

(see Remark 2.5). This means d is a critical point of $(\mathrm{SE_L})^{\sim}$, i.e. $d = d_0$ because d_0 is the greatest critical point by the assumption (GC) (see Remark 1.6). We have $B\tilde{A}W_n - (N(U_n)/\|U_n\|_{\sim} \in \tilde{K}^0$ and $B\tilde{A}W_n \notin \tilde{K}$ by (c) from Theorem 2.2. Hence we obtain (using the assumption (N)) that $D(d_0)\, W = B\tilde{A}W \in \partial\tilde{K}$, i.e. also $W \in \partial\tilde{K}$. This contradicts the assumption that d_0 is simple and $E_B(d_0) \cap \tilde{K}^0 \neq \emptyset$ (see Remark 1.4).

The estimate $d(\delta) > d_0$ and the existence of a limit point of $d(\delta)$ for $\delta \to 0+$ follows from the implication (d) of Theorem 2.2. Suppose that there exist δ_n such that $\delta_n \to 0+$, $d(\delta_n) \to d_0$. We can suppose $W_n = U(\delta_n)/\|U(\delta_n)\|_{\sim} \rightharpoonup W$ and (b) from Definition 2.1 implies $W_n \to W$, $D(d_0)\, W - \tilde{P}B\tilde{A}W = 0$ (Remark 2.5). But we know that $U(\delta_n) \in \partial\tilde{K}$, which means $W \in \partial\tilde{K}$ and this contradicts again the assumption that d_0 is simple and $E_B(d_0) \cap \tilde{K}^0 \neq \emptyset$. Hence, all the limit points of $d(\delta)$ for $\delta \to 0+$ are greater than d_0.

It remains to show that $d(\delta)$, $U(\delta)$ do not satisfy $(\mathrm{SE})^{\sim}$ for δ small. If this were not true we should have $d(\delta_n)$, $U(\delta_n)$ satisfying $(\mathrm{SE})^{\sim}$ for some $\delta_n \to 0$, $d(\delta_n) > d_0$, $U(\delta_n) \in \partial\tilde{K}$ and we should obtain a bifurcation point of $(\mathrm{SE})^{\sim}$ greater than d_0 by the previous considerations. This is impossible because d_0 is the greatest critical point by the assumption (GC) (see Remarks 1.5, 1.6).

Main ideas of the proof of Theorem 2.2 (for the details see Section 4; cf. also proof of Theorem 2.2 in [6]): We shall show that the system of equations (a), (b) can be understood as an abstract bifurcation equation in the space $\tilde{H} \times \mathbb{R}$ (more precisely, see Remarks 4.1, 4.2) for which a modification of Dancer's global bifurcation result described in Section 3 (Theorem 3.1) holds. It will follow from it that for each $\delta > 0$ there exist closed connected subsets $Z^+_{\delta,0}$ and $Z^-_{\delta,0}$ of Z_δ starting from $[d_0, 0, 0]$ in the direction $W_0 \notin \tilde{K}$ and $-W_0 \in \tilde{K}^0 \cap E_B(d_0)$, respectively; further, either

(1) $Z^+_{\delta,0}$ contains a point of the type $[d(\delta), U(\delta), 1]$

or

(2) $Z^+_{\delta,0}$ is unbounded in d

or

(3) $Z^+_{\delta,0}, Z^-_{\delta,0}$ meet each other at a point different from $[d_0, 0, 0]$.

Our aim will be to show that the two last cases cannot occur for δ small so that (1) is true. The boundedness of $Z^+_{\delta,0}$ will follow from elementary considerations about the equation (b) (Lemma 4.1, Remark 4.3). The case (3) will be excluded by proving that all the points from $Z^+_{\delta,0}$ (δ small) fulfil (c), (d) and that $B\tilde{A}U \in \tilde{K}^0$ for all $[d, U, \tau] \in Z^-_{\delta,0}$ with $d \geqq d_0$, $[d, U, \tau] \neq [d_0, 0, 0]$. The proof of (c), (d) is based on the following principles:

(i) for an arbitrary $\delta > 0$, the values d are "locally increasing along $Z^+_{\delta,0}$" near $d = d_0$, $\|U\|_\sim = 0$, $\tau = 0$; this is the meaning of Lemma 2.2;

(ii) for $\delta > 0$ small, $B\tilde{A}U$ cannot intersect $\partial\tilde{K}$ with $\tau > 0$ as long as $d \geqq d_0$ for $[d, U, \tau]$ lying on $Z^+_{\delta,0}$ (if this were not true for a sequence $Z^+_{\delta_n,0}$ with $\delta_n \to 0+$ then we should obtain from (b) by the limiting that there exists $W \in \partial\tilde{K} \cap E_B(d_0)$ and this would contradict the simplicity of d_0 and the assumption $E_B(d_0) \cap \tilde{K}^0 \neq \emptyset$); simultaneously, $Z^+_{\delta,0}$ cannot intersect the line $d = d_0$ as long as $B\tilde{A}U \notin \tilde{K}$; this follows from Lemma 2.1; more precisely, see Lemmas 4.2, 4.3.

Lemma 2.1 (cf. [6, Lemma 3.1]). *Let d_0 be a critical point of $(\mathrm{SE_L})^\sim$, $E_B(d_0) \cap \tilde{K}^0 \neq \emptyset$ and let* (A), (N) *be fulfilled. If*

$$D(d_0)\,U - B\tilde{A}U + \tau(\tilde{I} - \tilde{P})\,B\tilde{A}U = 0 \tag{2.5}$$

for some $U \neq 0$, then either $\tau = 0$ or $D(d_0)\,U = B\tilde{A}U \in \tilde{K}$. In any case $U \in E_B(d_0)$. Particularly, $E_I(d_0) = E_B(d_0) \cap \tilde{K}$.

Proof. Using the equality $\tilde{P}U = [u, Pv]$, (2.5) can be written as

$$d_0 u - b_{11}Au - b_{12}Av = 0\,, \tag{2.6}$$

$$v - b_{21}Au - b_{22}Av + \tau(I - P)\,(b_{21}Au + b_{22}Av) = 0\,. \tag{2.7}$$

Suppose $U_0 = [u_0, v_0] \in \tilde{K}^0 \cap E_B(d_0)$, i.e.

$$d_0 u_0 - b_{11}Au_0 - b_{12}Av_0 = 0\,, \tag{2.8}$$

$$v_0 - b_{21}Au_0 - b_{22}Av_0 = 0\,. \tag{2.9}$$

Multiplying (2.6), (2.8) by u_0, u and subtracting, we obtain (using the symmetry of A)

$$-b_{12}\langle Av, u_0\rangle + b_{12}\langle Av_0, u\rangle = 0\,. \tag{2.10}$$

Multiplying (2.7), (2.9) by v_0, v and subtracting, we obtain

$$(2.11)\qquad -b_{21}\langle Au, v_0\rangle + b_{21}\langle Au_0, v\rangle + \tau\langle (I-P)(b_{21}Au + b_{22}Av), v_0\rangle = 0\,.$$

It follows from (2.10), (2.11) that $\tau\langle (I-P)(b_{21}Au + b_{22}Av), v_0\rangle = 0$ and therefore either $\tau = 0$ or $b_{21}Au + b_{22}Av \in K$ by (P, K^0) because $v_0 \in K^0$ (see Remark 2.2), i.e. $B\tilde{A}U \in \tilde{K}$. Further, (2.5) and (P) (Remark 2.2) imply $D(d_0)\,U = B\tilde{A}U$. Setting $\tau = 1$ and using (2.3) from Remark 2.3 we obtain $E_I(d_0) = E_B(d_0) \cap \tilde{K}$.

Lemma 2.2 (cf. [6, Lemma 3.2]). *Let* (A), (N), (SIGN) *be fulfilled. Suppose that* $[d_n, U_n, \tau_n] \in \mathbb{R}^+ \times \tilde{H} \times (0, 1\rangle$, $\|U_n\|_\sim \neq 0$

$$(\bar{\mathrm{b}})\qquad D(d_n)\,U_n - B\tilde{A}U_n + \tau_n(\tilde{I} - \tilde{P})\,B\tilde{A}U_n + \tau_n\,R(U_n) = 0$$

$(n = 1, 2, \ldots)$ *and* $[d_n, U_n, \tau_n] \to [d_0, 0, 0]$ *with* $d_0 > 0$, $W_n = U_n/\|U_n\|_\sim \rightharpoonup W_0 = [w_0, z_0]$, $W_0 \notin \tilde{K}$. *Then* $W_n \to W_0$, d_0 *is a critical point of* $(\mathrm{SE_L})^\sim$, $W_0 \in E_B(d_0)$ *and*

$$(2.12)\quad \lim_{n\to\infty} \frac{d_n - d_0}{\tau_n} = -\frac{b_{12}}{b_{21}\|w_0\|^2}\,\langle (I-P)(b_{21}Aw_0 + b_{22}Az_0), z_0\rangle > 0\,.$$

Proof. It follows directly from $(\bar{\mathrm{b}})$ that $W_n \to W_0$, $\|W_0\|_\sim = 1$, $W_0 \in E_B(d_0)$ (see Remark 2.5). Rewriting $(\bar{\mathrm{b}})$ and $(\mathrm{SE_L})^\sim$ with d_0, W_0 in the components, multiplying the individual equations by w_0, w, u_0, u and subtracting similarly as in the proof of Lemma 2.1, we obtain

$$(d_n - d_0)\,\langle w_0, w_0\rangle + \tau_n \frac{\langle N_1(U_n), w_0\rangle}{\|U_n\|_\sim} + \frac{b_{12}}{b_{21}}\,\tau_n\langle (I-P)(b_{21}Aw_n + b_{22}Az_n), z_0\rangle +$$
$$+ \frac{b_{12}}{b_{21}}\,\tau_n \frac{\langle R_2(U_n), z_0\rangle}{\|U_n\|_\sim} = 0\,,$$

where $[w_n, z_n] = W_n$, $[w_0, z_0] = W_0$. Using (N), (R) we obtain

$$(2.13)\qquad \|w_0\|^2 \lim_{n\to\infty} \frac{d_n - d_0}{\tau_n} = -\frac{b_{12}}{b_{21}}\,\langle (I-P)(b_{21}Aw_0 + b_{22}Az_0), z_0\rangle\,.$$

We have $\langle (I-P)(b_{21}Aw_0 + b_{22}Az_0), z_0\rangle = \langle (I-P)\,z_0, z_0\rangle$ because $D(d_0)\,W_0 = B\tilde{A}W_0$. We have $z_0 \notin K$ because $W_0 \notin \tilde{K}$ and (2.12) follows from (2.13), (SIGN) and (P).

3. SOME REMARKS CONCERNING THE KNOWN GLOBAL BIFURCATION RESULTS

Consider a general bifurcation equation of the type

$$(\mathrm{BE})\qquad x - L(\mu)\,x + G(\mu, x) = 0$$

in a real Hilbert space X with the inner product $\langle\cdot,\cdot\rangle_X$ and with the corresponding norm $|||\cdot|||$. Suppose that

(L) for any $\mu \in \mathbb{R}$, $L(\mu)$ is a linear completely continuous operator in X; the mapping $\mu \mapsto L(\mu)$ of $\mathbb{R}$ into the space of linear continuous mappings in X (with the operator norm) is continuous;

(LG) the mapping $M: \mathbb{R} \times X \to X$ defined by $M(\mu, x) = L(\mu)\, x + G(\mu, x)$ is completely continuous;

(G) $\lim_{|||x||| \to 0} (G(\mu, x)/|||x|||) = 0$ uniformly on bounded subsets of $\mathbb{R}$.

Well-known global bifurcation results of P. H. Rabinowitz [9] and E.N. Dancer [1] deal with the special case $L(\mu) = \mu L$ but can be modified for the case of the equation (BE). We shall formulate precisely such a modification of Dancer's result (see Theorem 3.1) which will be essential for the proof of Theorem 2.2.

Remark 3.1. Denote by C the closure (in $\mathbb{R} \times X$) of the set of all nontrivial solutions of (BE), i.e.

$$C = \overline{\{[\mu, x] \in \mathbb{R} \times X;\ |||x||| \neq 0,\ (\text{BE}) \text{ is fulfilled}\}}\,.$$

Suppose that μ_0 is a simple critical point of

$$(\text{BE}_\text{L}) \qquad x - L(\mu)\, x = 0\,,$$

$x_0 \in B_\text{L}(\mu_0)$, $|||x_0||| = 1$, i.e. x_0 is a nontrivial solution of $(\text{BE}_\text{L})^\sim$ with $\mu = \mu_0$ and $\dim \bigcup_{k=1}^{\infty} \text{Ker}\,(I - L(\mu_0))^k = 1$ (cf. the notation from Section 1 for (SE)). If $[\mu_n, x_n] \in C$, $\mu_n \to \mu_0$, $|||x_n||| \neq 0$, $|||x_n||| \to 0$ and $x_n/|||x_n||| \rightharpoonup \bar{x}$, then $x_n/|||x_n||| \to \bar{x}$ and either $\bar{x} = x_0$ or $\bar{x} = -x_0$. This is easy to see if we divide (BE) (with μ_n, x_n) by $|||x_n|||$ and use (L), (G).

Remark 3.2. In what follows, we shall consider the situation from Remark 3.1. Let C_0 be the component of C containing $[\mu_0, 0]$. It is well-known that $C_0 \neq \emptyset$ under our assumptions (see [9]).

Choose $\eta \in (0, 1)$ and set

$$K_\eta = \{[\mu, x] \in \mathbb{R} \times X;\ |\langle x, x_0\rangle_X| > \eta |||x|||\}\,,$$
$$K_\eta^+ = \{[\mu, x] \in K_\eta;\ \langle x, x_0\rangle_X > 0\}\,, \quad K_\eta^- = K_\eta \setminus K_\eta^+\,.$$

(see Fig. 3.1a). It follows from Remark 3.1 that there exists $R > 0$ such that

$$(C \setminus \{[\mu_0, 0]\}) \cap B_R(\mu_0, 0) \subset K_\eta\,,$$

where $B_R(\mu_0, 0) = \{[\mu, x] \in \mathbb{R} \times X;\ |\mu - \mu_0| + |||x||| \leqq R\}$.*) For each $r \in (0, R\rangle$ denote by D_r^+ and D_r^- the components containing $[\mu_0, 0]$ of the set $\{[\mu_0, 0]\} \cup \cup (C \cap B_r(\mu_0, 0) \cap K_\eta^+)$ and $\{[\mu_0, 0]\} \cup (C \cap B_r(\mu_0, 0) \cap K_\eta^-)$, respectively. Denote by $C_{0,r}^+$ and $C_{0,r}^-$ the components of $\overline{C_0 \setminus D_r^-}$ and $\overline{C_0 \setminus D_r^+}$, respectively, containing $[\mu_0, 0]$.

*) Cf. the considerations in [9, pp. 495, 496] which remain valid if we consider the general case $L(\mu)$ instead of μL, we replace the assumption that μ_0 is a simple characteristic value by the assumption that μ_0 is a simple critical point.

Define

$$C_0^+ = \overline{\bigcup_{0<r\leq R} C_{0,r}^+}, \quad C_0^- = \overline{\bigcup_{0<r\leq R} C_{0,r}^-}.$$

According to Remark 3.1 the sets C_0^+, C_0^- are independent of the choice of $\eta \in (0, 1)$, they are connected and $C_0 \supset C_0^+ \cup C_0^-$ *). See Figs. 2a–c: Fig. 2a shows how the projection of C_0 into X can look like, Figs. 2b) and c) show the corresponding projections of C_0^+ and C_0^-, respectively, into X.

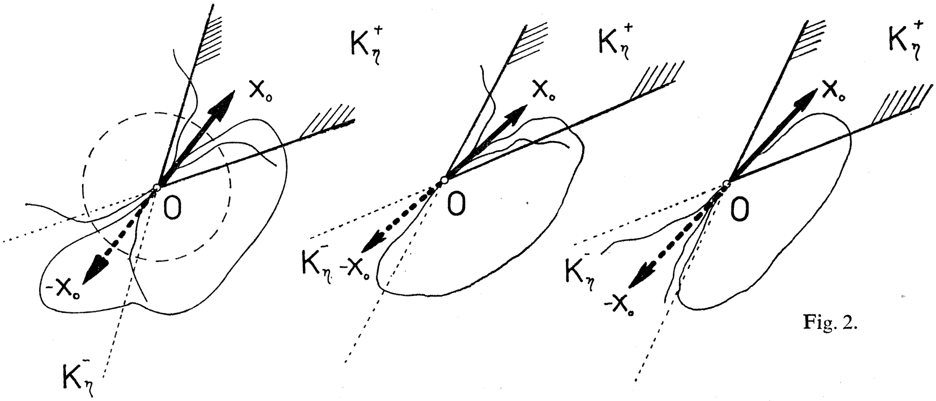

Fig. 2.

Further, we shall suppose that

(Ind) $\quad \operatorname{ind}(I - L(\mu_0 + \varepsilon)) \neq \operatorname{ind}(I - L(\mu_0 - \varepsilon))$ for all $\varepsilon \in (0, \varepsilon_0)$

with some $\varepsilon_0 > 0$ **), where I is the identity in X and $\operatorname{ind}(I - L(\mu))$ is the Leray-Schauder index of $I - L(\mu)$ with respect to the origin (see eg. [11]).

Theorem 3.1 (cf. [1, Theorem 2]). *Let μ_0 be a simple critical point of* (BE$_L$) *and let* (L), (LG), (G), (Ind) *be fulfilled. Then either both C_0^+ and C_0^- are unbounded or $C_0^+ \cap C_0^-$ contains a point different from $[\mu_0, 0]$, where C_0^+, C_0^- are introduced in Remark* 3.2.

Proof can proceed in the same way as that of Theorem 2 in [1]. The only difference is that in [1] it is supposed that $L(\mu) = \mu L$ and μ_0 is a simple characteristic value of L (which is the same as the simple critical point in this case). In this special situation (Ind) is fulfilled automatically and therefore it is not explicitly supposed (see [1]). But in the proof only the properties (L), (LG), (G) together with (Ind) are used.

Remark 3.3. It follows from the definition of C_0^+, C_0^- and from Theorem 3.1 that for any $r > 0$ there exist $[\mu, x] \in B_r(\mu_0, 0) \cap K_\eta^+ \cap C_0^+$ and $[\mu, x] \in B_r(\mu_0, 0) \cap \cap K_\eta^- \cap C_0^-$, respectively. Particularly, it follows from here and Remark 3.1 that

$$\text{there exist } [\mu_n, x_n] \in C_0^+ \text{ such that } \mu_n \to \mu_0, \quad |||x_n||| \to 0, \quad \frac{x_n}{|||x_n|||} \to x_0;$$

**) It is supposed that the indices in (Ind) are well-defined for $\varepsilon \in (0, \varepsilon_0)$.

there exist $[\mu_n, x_n] \in C_0^-$ such that $\mu_n \to \mu_0$, $|||x_n||| \to 0$, $\frac{x_n}{|||x_n|||} \to -x_0$.

The same assertion holds also for any nonempty subset of the type $C_0^+ \setminus C_1^+$, where C_1^+ is any closed connected subset of C_0^+ such that $\overline{C_0^+ \setminus C_1^+} \cap C_1^+ = [\mu_0, 0]$; analogously for subsets of C_0^-.

4. PROOF OF THEOREM 2.2

In this section the assumptions of Theorem 2.1 will be automatically considered to be fulfilled and notation from Sections 1, 2 will be used.

Remark 4.1. We shall use the results of Section 3 for the special case of the space $X = \tilde{H} \times \mathbb{R}$ (with the points $[U, \varepsilon]$) and the operators $L(\mu): X \to X$, $G_\delta: \mathbb{R} \times \times X \to X$ (for any $\mu \in \mathbb{R}$, $\delta > 0$) defined by

$$(*) \qquad L(\mu)\, x = L(\mu)(U, \varepsilon) = [D(\mu)\, B\tilde{A}U, 0]\,,$$

$$G_\delta(\mu, x) = G_\delta(\mu, U, \varepsilon) =$$

$$= \left[\frac{\varepsilon}{1+\varepsilon}(\tilde{I} - \tilde{P})\, D(\mu)\, B\tilde{A}U + \frac{\varepsilon}{1+\varepsilon} D(\mu)\, R(U), -\frac{1+\varepsilon}{\delta}\|U\|_\sim^2\right]$$

$$\text{for all} \quad \mu \in \mathbb{R}\,, \quad x = [U, \varepsilon] \in X\,.$$

This means

$$(4.1) \qquad L(\mu)(U, \varepsilon) = [\mu b_{11}Au + \mu b_{12}Av,\ b_{21}Au + b_{22}Av,\ 0]\,,$$

$$(4.2) \qquad G_\delta(\mu, U, \varepsilon) = \left[\frac{\mu\varepsilon}{1+\varepsilon} N_1(U), \frac{\varepsilon}{1+\varepsilon}(I - P)(b_{21}Au + b_{22}Av) + \right.$$

$$\left. + \frac{\varepsilon}{1+\varepsilon} R_2(U), -\frac{1+\varepsilon}{\delta}\|U\|_\sim^2\right].$$

We shall consider the point $\mu_0 = 1/d_0$ and $x_0 = [W_0, 0]$, where $W_0 \in E_B(d_0) \cap \cap (-\tilde{K}^0)$ (see the assumptions of Theorem 2.1 and Remark 1.4). It is easy to see that the operators $(*)$ satisfy (L), (LG), (G) (for any $\delta > 0$) and μ_0 is a simple critical point of $(\mathrm{BE_L})$, $x_0 \in B_L(\mu_0)$. In the described case, we shall write C_δ, $C_{\delta,0}$, $C_{\delta,0}^+$, $C_{\delta,0}^-$ instead of C, C_0, C_0^+, C_0^- for the sets from Remarks 3.1, 3.2.

Remark 4.2. If we set $d = 1/\mu$, $\tau = \varepsilon/(1 + \varepsilon)$, then the equation (BE) (with G_δ instead of G) is equivalent to (a), (b) from Definition 2.1 for $d \neq 0 \neq \mu$, $\varepsilon \geqq 0$. We have $\varepsilon \geqq 0$ for all $[\mu, U, \varepsilon] \in C_{\delta,0}$ because of $[\mu_0, 0, 0] \in C_{\delta,0}$, (a) cannot be fulfilled with $\varepsilon \in (-1, 0)$ and $C_{\delta,0}$ is connected.

Lemma 4.1. *There exist $\delta_0 > 0$, $\xi_0 > 0$ such that $\mu \geqq \xi_0$ for all $[\mu, U, \varepsilon] \in C_{\delta,0}$ with $\delta \in (0, \delta_0)$.*

Proof. In the opposite case there exist $[\mu_n, U_n, \varepsilon_n] \in C_{\delta_n,0}$ satisfying $\mu_n > 0$,

$\|U_n\|_\sim > 0$, $\varepsilon_n > 0$ $(n = 1, 2, \ldots)$, $\delta_n \to 0$, $\mu_n \to 0$, $\varepsilon_n/(1 + \varepsilon_n) \to \tau$, $W_n = U_n/\|U_n\|_\sim \rightharpoonup W$. We have $\|U\|_\sim \to 0$ by (a) (see Remark 4.2). Setting $W_n = [w_n, z_n]$, rewriting (BE) into the components (see (4.1), (4.2)) and dividing by $\|U_n\|_\sim$ we obtain

$$w_n - \mu_n b_{11} A w_n - \mu_n b_{12} A z_n + \mu_n \frac{\varepsilon_n}{1 + \varepsilon_n} \frac{N_1(U_n)}{\|U_n\|_\sim} = 0\,, \tag{4.3}$$

$$z_n - b_{21} A w_n - b_{22} A z_n + \frac{\varepsilon_n}{1 + \varepsilon_n} (I - P)(b_{21} A w_n + b_{22} A z_n) + \frac{\varepsilon_n}{1 + \varepsilon_n} \frac{R_2(U_n)}{\|U_n\|_\sim} = 0\,. \tag{4.4}$$

Using (N), (R) (Remark (2.4)) and the compactness of A we obtain $w_n \to w$, $z_n \to z$, i.e. $W_n \to W$, $\|W\|_\sim = 1$. It follows from (4.3) that $w_n \to 0$ and therefore $\|z_n\| \to 1$. Multiplying (4.4) by Az_n we obtain a contradiction because the obtained left-hand side should be positive for n large in virtue of the positivity of A, (SIGN), (R) and (P) (Remark 2.2).

Remark 4.3. For any $\delta \in (0, \delta_0)$ (with δ_0 from Lemma 4.1) we can define

$$Z^+_{\delta,0} = \overline{\left\{[d, U, \tau];\ d = \frac{1}{\mu},\ \tau = \frac{\varepsilon}{1 + \varepsilon},\ [\mu, U, \varepsilon] \in C^+_{\delta,0}\right\}}\,,$$

$$Z^-_{\delta,0} = \overline{\left\{[d, U, \tau];\ d = \frac{1}{\mu},\ \tau = \frac{\varepsilon}{1 + \varepsilon},\ [\mu, U, \varepsilon] \in C^-_{\delta,0}\right\}}\,,$$

$$Z_{\delta,0} = Z^+_{\delta,0} \cup Z^-_{\delta,0}\,.$$

It follows from Lemma 4.1 and Remarks 2.7, 3.2, 4.2 that $Z^+_{\delta,0}$, $Z^-_{\delta,0}$ are connected closed compact subsets of Z_δ from Definition 2.1, $[d_0, 0, 0] \in Z^+_{\delta,0} \cap Z^-_{\delta,0}$.

Remark 4.4. It follows from Remarks 3.1, 4.1 and from the definition of $Z_{\delta,0}$ (Remark 4.3) that

if $[d_n, U_n, \tau_n] \in Z_{\delta,0}$, $d_n \to d_0$, $\|U_n\|_\sim \to 0$ (i.e. also $\tau_n/\|U_n\|_\sim \to 0$ by (a)), $U_n/\|U_n\|_\sim \rightharpoonup \overline{W}$, then $W_n \to \overline{W}$ and either $\overline{W} = W_0$ or $\overline{W} = -W_0$

(i.e. either $B\tilde{A}W_n \to D(d_0)\, W_0$ or $B\tilde{A}W_n \to -D(d_0)\, W_0$ because $D(d_0)\, W_0 = B\tilde{A}W_0$).

We have $W_0 \notin \tilde{K}$, $-W_0 \in \tilde{K}^0$ and therefore $\overline{W} = W_0$ and $\overline{W} = -W_0$ means $U_n \notin \tilde{K}$ and $U_n \in \tilde{K}^0$, respectively, for n sufficiently large. Particularly, it follows from here and Remarks 3.3, 4.3 that

(4.5) there exist $[d_n, U_n, \tau_n] \in Z^+_{\delta,0}$ such that $U_n \notin \tilde{K}$, $d_n \to d_0$, $\|U_n\|_\sim \to 0$,

$$\frac{\tau_n}{\|U_n\|_\sim} \to 0\,, \quad \frac{U_n}{\|U_n\|_\sim} \to W_0 \quad \left(\text{i.e. also } \frac{B\tilde{A}U_n}{\|U_n\|_\sim} \to D(d_0)\, W_0\right);$$

(4.6) there exist $[d_n, U_n, \tau] \in Z^-_{\delta,0}$ such that $U_n \in \tilde{K}^0$, $d_n \to d_0$, $\|U_n\|_\sim \to 0$,

$$\frac{\tau_n}{\|U_n\|} \to 0, \frac{U_n}{\|U_n\|_\sim} \to -W_0 \quad \left(\text{i.e. also } \frac{B\tilde{A}U_n}{\|U_n\|_\sim} \to -D(d_0)\, W_0\right);$$

the same assertion holds for any nonempty subset of the type $Z^+_{\delta,0} \setminus Z^+_{\delta,1}$, where $Z^+_{\delta,1}$ is any closed connected subset of $Z^+_{\delta,0}$ such that $\overline{(Z^+_{\delta,0} \setminus Z^+_{\delta,1})} \cap Z^+_{\delta,1} = [d_0, 0, 0]$; analogously for subsets $Z^-_{\delta,0} \setminus Z^-_{\delta,1}$ of $Z^-_{\delta,0}$.

Remark 4.5. It is known (see e.g. [11]) that $\operatorname{ind}(I - T) = (-1)^{\gamma(T)}$ for any compact linear operator T in a Banach space, where $\gamma(T)$ is the sum of the algebraic multiplicities of all the real eigenvalues of T which are greater than 1, i.e. the sum of all positive eigenvalues of the operator $T - I$. It is easy to see that $\lambda \neq 0$ is a simple eigenvalue of $L(\mu)$ (for some $\mu > 0$) and $[U, \varepsilon]$ is the corresponding eigenvector if and only if λ is a simple eigenvalue of the operator $D(\mu)\, B\tilde{A}$ and U is the corresponding eigenvector, $\varepsilon = 0$. Note that λ is an eigenvalue of $D(\mu)\, B\tilde{A}$ if and only if $\lambda - 1$ is an eigenvalue of $B\tilde{A} - D(d)\,\tilde{I}$ with $d = 1/\mu$ (for $\mu \neq 0$). It follows from here and the assumption (GC) (Remark 1.6) that the assumption (Ind) from Section 3 for $\mu_0 = 1/d_0$ is fulfilled in our situation.

Lemma 4.2 (cf. [6, Lemma 4.1]. *There exists $\delta_0 > 0$ such that for all $[d, U, \tau] \in Z^+_{\delta,0}$, $\delta \in (0, \delta_0)$ the implications* (c), (d) *from Theorem* 2.2 *hold, where $Z^+_{\delta,0}$ is the set from Remark* 4.3.

Proof. Denote by $Z^+_{\delta,1}$ the component of the set

$$\{[d, U, \tau] \in Z^+_{\delta,0};\ d \geqq d_0\}$$

containing $[d_0, 0, 0]$. There exist $[d_n, U_n, \tau_n] \in Z^+_{\delta,0}$ such that $[d_n, U_n, \tau_n] \to [d_0, 0, 0]$, $\|U_n\|_\sim > 0$, $U_n \notin \tilde{K}$, $\tau_n > 0$, $B\tilde{A}U_n/\|U_n\|_\sim \to D(d_0)\, W_0 \notin \tilde{K}$ (see Remark 4.4). Lemma 2.2 implies $d_n > d_0$ for n sufficiently large.

Particularly,

(4.7) $\quad Z^+_{\delta,1}$ contains points $[d, U, \tau]$ with $B\tilde{A}U \notin \tilde{K}$ for any $\delta > 0$.

Let us prove that

(C) $\quad$ there exists $\delta_0 > 0$ such that (c) is valid for all

$$[d, U, \tau] \in Z^+_{\delta,1}, \quad \delta \in (0, \delta_0).$$

Suppose the contrary. Then it follows from (4.7) and from the connectedness of $Z^+_{\delta,1}$ that there exist δ_n and $[d_n, U_n, \tau_n] \in Z^+_{\delta,1}$ such that

(4.8) $$\delta_n > 0, \quad \delta_n \to 0, \quad d_n \geqq d_0, \quad [d_n, U_n, \tau_n] \neq [d_0, 0, 0], \quad B\tilde{A}U_n \in \partial\tilde{K}.$$

We have $\|U_n\|_\sim > 0$ because in the opposite case d_n would be critical points of (SE_L) by Remarks 2.6, 4.2 and there is no critical point greater than d_0 by the assumption (GC) (see Remark 1.6). Further, $\|U_n\|_\sim \to 0$ by (a) and we can suppose $W_n =$

$= U_n/\|U_n\|_\sim \rightharpoonup W$, $\tau_n \to \tau$, $d_n \to \bar{d}$ ($\{d_n\}$ is bounded by Lemma 4.1 and Remark 4.3). But $[d_n, U_n, \tau_n]$ satisfy (b) and it follows by Remark 2.5 that $W_n \to W$,

$$D(\bar{d})\, W - B\tilde{A}W + \tau(\tilde{I} - \tilde{P})\, B\tilde{A}W = 0\,.$$

We obtain $B\tilde{A}W \in \partial\tilde{K}$ from (4.8), and (P) (Remark 2.2) implies that $\bar{d}$ is a critical point of (SE_L), $W \in E_B(\bar{d})$. Hence, $\bar{d} = d_0$. But $D(d_0)\, W = B\tilde{A}W \in \partial\tilde{K}$, i.e. $W \in \partial\tilde{K}$ and this contradicts the assumption that d_0 is simple, $E_B(d_0) \cap \tilde{K}^0 \neq \emptyset$ (see Remark 1.4).

Further, we shall prove that

(D) *there exists* $\delta_0 > 0$ *such that* (d) *is valid for all* $[d, U, \tau] \in Z^+_{\delta,1}$, $\delta \in (0, \delta_0)$.

It follows from Lemma 4.1 and the definition of $Z^+_{\delta,0}$ that d's are bounded by $d_m = \xi^{-1}$ for $[d, U, \tau] \in Z^+_{\delta,0}$ with δ small. Suppose that (D) is not true. Then there exist $\delta_n > 0$, $[d_n, U_n, \tau_n] \in Z^+_{\delta_n,1}$ such that

$$\delta_n > 0\,, \quad \delta_n \to 0\,, \quad \|U_n\|_\sim > 0\,, \quad d_n = d_0\,. \tag{4.9}$$

We can suppose that (c) holds on $Z^+_{\delta_n,1}$ and $W_n = U_n/\|U_n\|_\sim \rightharpoonup W$, $\tau_n \to \tau$. It follows from (b) for $[d_n, U_n, \tau_n]$ that $W_n \to W$,

$$D(d_0)\, W - B\tilde{A}W + \tau(\tilde{I} - \tilde{P})\, B\tilde{A}W = 0 \tag{4.10}$$

(see Remark 2.5). We have $B\tilde{A}U_n \notin \tilde{K}$ since (c) holds on $Z^+_{\delta_n,1}$, i.e. $B\tilde{A}W \notin \tilde{K}^0$. If $B\tilde{A}W \in \partial\tilde{K}$ then (4.10) together with (P) imply $D(d_0)\, W = B\tilde{A}W$, i.e. $W \in E_B(d_0) \cap \partial\tilde{K}$ and this contradicts the simplicity of d_0 and $E_B(d_0) \cap \tilde{K}^0 \neq \emptyset$ (see Remark 1.4). Hence $B\tilde{A}W \notin \tilde{K}$. Now, in the case $\tau > 0$, (4.10) contradicts Lemma 2.1, but the case $\tau = 0$ is impossible by Lemma 2.2 because $(d_n - d_0)/\tau_n = 0$ and $W \notin \tilde{K}$.

Now, it is sufficient to show that

(4.11) there exists $\delta_0 > 0$ such that $Z^+_{\delta,1} = Z^+_{\delta,0}$ for each $\delta \in (0, \delta_0)$.

It is sufficient to take δ_0 such that (c), (d) hold for all $[d, U, \tau] \in Z^+_{\delta,1}$, $\delta \in (0, \delta_0)$ (see (C), (D)). If $\delta \in (0, \delta_0)$, $Z^+_{\delta,1} \neq Z^+_{\delta,0}$ then $Z^+_{\delta,1} \cap \overline{(Z^+_{\delta,0} \setminus Z^+_{\delta,1})} = \{[d_0, 0, 0]\}$ by (d). It follows from here and (c) that there exists $[d_n, U_n, \tau_n] \in Z^+_{\delta,0} \setminus Z^+_{\delta,1}$ with $d_n < d_0$, $\tau_n > 0$, $U_n \notin \tilde{K}$, $[d_n, U_n, \tau_n] \to [d_0, 0, 0]$, $U_n/\|U_n\| \to W_0 \notin \tilde{K}$ (see Remark 4.4). This contradicts Lemma 2.2 and the proof of Lemma 4.2 is complete.

Lemma 4.3 (cf. [6, Lemma 4.2]). *There exists* $\delta_0 > 0$ *such that for all* $[d, U, \tau] \in Z^-_{\delta,0}$, $\delta \in (0, \delta_0)$, *the following implication holds*:

(c^-) *if* $[d, U, \tau] \neq [d_0, 0, 0]$, $d \geqq d_0$, *then* $B\tilde{A}U \in \tilde{K}^0$.

Proof. $Z^-_{\delta,0}$ contains the points $[d, U, \tau]$ with $B\tilde{A}U \in \tilde{K}^0$ (see Remark 4.4). If the assertion of Lemma 4.3 were not true then it would follow from the connectedness of $Z^-_{\delta,0}$ that there exist δ_n and $[d_n, U_n, \tau_n] \in Z^-_{\delta_n,0}$ satisfying (4.8) or (4.9) with $B\tilde{A}U_n \notin \tilde{K}$. This would lead to a contradiction as in the proof of Lemma 4.2.

Proof of Theorem 2.2 (cf. [6, proof of Theorem 2.2]): Let δ_0 be such that (c),

(d) and (c^-) hold on $Z^+_{\delta,0}$ and $Z^-_{\delta,0}$, respectively, for any $\delta \in (0, \delta_0)$ (see Lemmas 4.2, 4.3). But all these conditions can be fulfilled simultaneously only in the case $Z^+_{\delta,0} \cap \cap Z^-_{\delta,0} = \{[d_0, 0, 0]\}$, i.e. $C^+_{\delta,0} \cap C^-_{\delta,0} = \{[\mu_0, 0]\}$ (see Remark 4.3). Hence, $C^+_{\delta,0}$ is unbounded by Theorem 3.1 and Remark 4.5. It follows from here, from the definition of $Z^+_{\delta,0}$ and (d) that $Z^+_{\delta,0}$ contains at least one point of the type $[d(\delta), U(\delta), 1]$.

5. NEUMANN CONDITIONS

Let us mention briefly the case of Neumann conditions, i.e.

$$\text{(NC)} \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,,$$

instead of (BC). We shall consider only the model case $n = 1$, $\Omega = (0, 1)$, K is a closed convex cone in $W^1_2(0, 1)$. In this case we must work in $W^1_2(0, 1)$ and hence (1.1) is not the inner product. Take $\eta > 0$ and define the inner product

$$\langle u, \varphi \rangle_\eta = \int_0^1 (u_x \varphi_x + \eta u \varphi)\, dx$$

which is equivalent to the usual inner product

$$\langle u, \varphi \rangle = \int_0^1 (u_x \varphi_x + u \varphi)\, dx\,.$$

Let us denote the corresponding norms by $\|\cdot\|_\eta$ and $\|\cdot\|$, respectively, and let H_η, H be the space $W^1_2(0, 1)$ equipped with the norm $\|\cdot\|_\eta$, $\|\cdot\|$, respectively. Then analogously as in Definition 2.1 for each $\delta > 0$ fixed we may denote by Z^η_δ the closure (in $\mathbb{R} \times H_\eta \times H_\eta \times \mathbb{R}$) of the set of all $[d, u, v, \tau] \in \mathbb{R}^+ \times H_\eta \times H_\eta \times (0, 1\rangle$ such that

$$\text{(a')} \qquad \|u\|^2 + \|v\|^2 = \delta\tau\,,$$

$$\text{(b')} \qquad du - (b_{11} + d_0\eta)\, Au - b_{12} Av + \tau N_1(u, v) = 0\,,$$

$$v - b_{21} Au - (b_{22} + \eta)\, Av +$$

$$+ \tau(I - P)\,(b_{21} Au + (b_{22} + \eta)\, Av) + \tau R_2(u, v) = 0\,,$$

where

$$R_2(u, v) = P(b_{21} Au + (b_{22} + \eta)\, Av) -$$

$$- P(b_{21} Au + (b_{22} + \eta)\, Av - N_2(u, v))\,.$$

Remark 5.1. Setting $\tau = 0$, $d = d_0$ in (b') we obtain (SE_L) (with $d = d_0$). For $\eta > 0$ sufficiently small, the coefficients $\bar{b}_{11} = b_{11} + \eta d_0$, $\bar{b}_{12} = b_{12}$, $\bar{b}_{21} = b_{21}$, $\bar{b}_{22} = b_{22} + \eta$ satisfy again the assumption (SIGN) and hence analogously as in Section 4 we may apply the results of Section 3. For any $\delta > 0$ sufficiently small we obtain $d_\eta > d_0$, $u^\eta \in H$, $v^\eta \in \partial K$ such that $\|u^\eta\|^2 + \|v^\eta\|^2 = \delta$ and

$$(5.1) \qquad d_\eta u^\eta - (b_{11} + d_\eta \eta) Au^\eta + \eta(d_\eta - d_0) Au^\eta - b_{12}Av^\eta + N_1(u^\eta, v^\eta) = 0,$$

$$(5.2) \qquad v^\eta - P(b_{21}Au^\eta + (b_{22} + \eta) Av^\eta - N_2(u^\eta, v^\eta)) = 0 \quad \text{in} \quad H_\eta,$$

i.e.

$$(5.1') \quad \int_0^1 d_\eta u_x^\eta \varphi_x + [\eta(d_\eta - d_0) u^\eta - b_{11}u^\eta - b_{12}v^\eta + n_1(u^\eta, v^\eta)] \varphi \, dx = 0 \quad \text{for all} \quad \varphi \in H,$$

$$(5.2') \quad \int_0^1 \{v_x^\eta(\psi_x - v_x^\eta) - [b_{21}u^\eta + b_{22}v^\eta - n_2(u^\eta, v^\eta)] (\psi - v^\eta)\} \, dx \geqq 0 \quad \text{for all} \quad \psi \in K,$$

where $-n_1(u, v) = f(u, v) - b_{11}u - b_{12}v$, $n_2(u, v) = g(u, v) - b_{21}u - b_{22}v$ (see Remark 1.1).

Now, let $\eta \to 0+$ (for $\delta > 0$ fixed). Then either

(i) there exist η_n ($n = 1, 2, \ldots$) such that $\eta_n \to 0$ and $d_n = d_{\eta_n} \to d(\delta)$ for some finite $d(\delta)$,

or

(ii) $$d_\eta \to +\infty \quad \text{for} \quad \eta \to 0+ .$$

In the case (i) we can suppose $u^n = u^{\eta_n} \rightharpoonup u(\delta)$, $v^n = v^{\eta_n} \rightharpoonup v(\delta)$ and we obtain from (5.1), (5.2) also $u^n \to u(\delta)$, $v^n \to v(\delta)$. The limiting process in (5.1′), (5.2′) gives

$$(5.3) \quad \int_0^1 \{d(\delta)\, u_x(\delta)\, \varphi_x - [b_{11}\, u(\delta) + b_{12}\, v(\delta) - n_1(u(\delta), v(\delta))]\, \varphi\} \, dx = 0 \quad \text{for all} \quad \varphi \in H,$$

$$(5.4) \quad \int_0^1 \{v_x(\delta)\, (\psi_x - v_x(\delta)) - [b_{21}\, u(\delta) + b_{22}\, v(\delta) - n_2(u(\delta), v(\delta))]\, (\psi - v(\delta))\} \, dx \geqq 0 \quad \text{for all} \quad \psi \in K.$$

Hence, in the case (i) we obtain a nonhomogeneous solution $u(\delta)$, $v(\delta)$ of (RD) with the conditions given by $H = W_2^1(0, 1)$, K, $\bar{u}$, $\bar{v}$ satisfying $v(\delta) \in \partial K$, $\|u(\delta)\|^2 + \|v(\delta)\|^2 = \delta$ as in Theorem 2.1. In the case (ii) the situation is different. It follows from (5.1′) by setting $\varphi = u^\eta$ that

$$(5.5) \quad \int_0^1 [d_\eta(u_x^\eta)^2 - b_{11}(u^\eta)^2 - \eta(d - d_0)(u^\eta)^2 - b_{12}v^\eta u^\eta + n_1(u^\eta, v^\eta)\, u^\eta] \, dx = 0 .$$

It follows from here (dividing (5.5) by d_η and letting $\eta \to 0+$) that $\int_1^0 (u_x^\eta)^2 \, dx \to 0$ and therefore $u^\eta \to \xi(\delta)$ in $W_2^1(0, 1)$, where $\xi(\delta)$ is a constant depending on δ only. Further, (5.2) yields $v^\eta \to v(\delta)$ in $W_2^1(0, 1)$. We have $\xi(\delta)^2 + \|v(\delta)\|^2 = \delta$. Writing (5.1) as the classical differential equation (cf. Remark 1.1), integrating it over (0,1) and taking the limit $\eta \to 0$ we obtain

$$(5.6) \qquad u = \xi(\delta) \quad (= \text{constant function on } (0, 1))\,,$$

$$\int_0^1 [b_{11}\, \xi(\delta) + b_{12}\, v(\delta) - n_1(\xi(\delta), v(\delta))]\, \mathrm{d}x = 0\,.$$

The limiting process $\eta \to 0+$ in (5.2') gives

$$(5.7) \qquad \int_0^1 \{v_x(\delta)\,(\psi_x - v_x(\delta)) - [b_{21}\, \xi(\delta) + b_{22}\, v(\delta) -$$

$$- n_2(\xi(\delta), v(\delta))]\,.\,(\psi - v(\delta))\}\, \mathrm{d}x \geqq 0 \quad \text{for all} \quad \psi \in K\,.$$

The system

$$(5.8) \qquad u = \xi \quad (= \text{const.})\,,$$

$$\int_0^1 [b_{11}\xi + b_{12}v - n_1(\xi, v)]\, \mathrm{d}x = 0\,,$$

$$(5.9) \qquad v \in K\,,$$

$$\int_0^1 v_x(\psi_x - v_x) - [b_{21}\xi + b_{22}v - n_2(\xi, v)]\,(\psi - v)\, \mathrm{d}x \geqq 0 \quad \text{for all} \quad \psi \in K$$

can be called the shadow system to (RD) with the conditions given by $W_2^1(0, 1)$, $K, \bar{u}, \bar{v}$ for $d \to +\infty$ (cf. [8], where the shadow system to the classical reaction-diffusion system for $d_2 \to +\infty$ is studied). Hence, in the case (ii) we obtain a non-homogeneous solution $\xi(\delta)$, $v(\delta)$ of the shadow system for $d \to +\infty$ to (RD) with our unilateral conditions, satisfying $v(\delta) \in \partial K$, $\xi(\delta)^2 + \|v(\delta)\|^2 = \delta$.

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