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ON 2-CELL EMBEDDINGS OF GRAPHS WITH MINIMUM NUMBERS OF REGIONS

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By a graph we shall mean a pseudograph in the sense of books [1] and [2]. This means that a graph L is determined if and only if its vertex set $V(L)$, edge set $E(L)$, and its incidence relation between vertices and edges are known. Let L be a graph. We denote by $C(L)$ the set of its components; we denote $c(L) = |C(L)|$. If $U \subseteq V(L)$, then we denote by $\langle U \rangle_L$ the subgraph of L induced by U.

Let G be a connected graph. Denote $\beta(G) = |E(G)| - |V(G)| + 1$. Consider a 2-cell embedding $\mathscr{E}$ of G on an orientable surface of genus g such that $\mathscr{E}$ has r regions. It is well-known (see Theorem 5.1 in [1], for example) that

$$(1) \qquad 2g + r = \beta(G) + 1 .$$

The maximum integer k with the property that there exists a 2-cell embedding of G on an orientable surface of genus k is referred to as the maximum genus $\gamma_M(G)$ of G. It was proved in [8] that $\gamma_M(G) = 0$ if and only if no pair of distinct cycles of G has a vertex in common. As follows from (1), $\gamma_M(G) \leqq [\beta(G)/2]$. (The maximum nonorientable genus of a connected graph has also been studied; as was proved in [9] the maximum nonorientable genus of G equals to $\beta(G)$.) We denote by $\varrho_m(G)$ the minimum integer n with the property that there exists a 2-cell embedding of G on an orientable surface which has n regions. As follows from (1),

$$(2) \qquad \gamma_M(G) = (\beta(G) - \varrho_m(G) + 1)/2 .$$

We say that G is upper embeddable if $\gamma_M(G) = [\beta(G)/2]$. According to (2), G is upper embeddable if and only if $\varrho_m(G) \leqq 2$.

Let G be a connected graph. We denote by x_G the minimum integer k such that there exists a spanning tree T of G with the property that for exactly k components F of $G - E(T)$, $|E(F)|$ is odd.

The following result was proved in [3] and [10]:

Theorem A. *If G is a connected graph, then $\varrho_m(G) = x_G + 1$.*

According to (2), Theorem A can be reformulated as follows:

Theorem A′. *If G is a connected graph, then* $\gamma_M(G) = (\beta(G) - x_G)/2$.

In fact, both in [3] and in [10] Theorem A is formulated for $\gamma_M(G)$ but in [3] in a rather different way.

Corollary A. *A connected graph G is upper embeddable if and only if* $x_G \leqq 1$.

Note that Corollary A was also proved in [4].

If L is a graph, then we denote by $b(L)$ the number of components F of L with the property that $\beta(F)$ is odd. The next theorem was proved in [5]:

Theorem B. *If G is a connected graph, then*

$$x_G = \max_{A \subseteq E(G)} (c(G - A) + b(G - A) - 1 - |A|) .$$

If we combine Theorem B with Corollary A, we get

Corollary B. *A connected graph G is upper embeddable if and only if*

$$c(G - A) + b(G - A) - 2 \leqq |A| \quad \textit{for every} \quad A \subseteq E(G) .$$

In the present paper we shall obtain two generalizations of Theorem B. If a grah G is a spanning subgraph of a graph J, then we denote by $\mathscr{S}(G, J)$ the set of graphs H with the properties that G is a spanning subgraph of H and H is a spanning subgraph of J. Let G, J, and L be graphs; we denote by $b^{\#}_G(L)$ the number of components F' of L with the property that $\beta(F')$ is odd and $E(F') \subseteq E(G)$; moreover, we denote by $b^{\square}_J(L)$ the number of components F'' of L with the property that either $\beta(F'')$ is odd or F'' is not an induced subgraph of J.

The following theorems are the main results of the present paper:

Theorem 1. *Let G and J be graphs, let G be a spanning subgraph of J, and let J be connected. Then*

$$\min_{H \in \mathscr{S}(G,J)} x_H = \max_{A \subseteq E(J)} (c(J - A) + b^{\#}_G(J - A) - 1 - |A|) .$$

Theorem 2. *Let G and J be graphs, let G be a spanning subgraph of J, and let G be connected. Then*

$$\max_{H \in \mathscr{S}(G,J)} x_H = \max_{A \subseteq E(G)} (c(G - A) + b^{\square}_J(G - A) - 1 - |A|) .$$

If we combine Theorem 1 or Theorem 2 with Theorem A, we can obtain a formula for $\min_{H \in \mathscr{S}(G,J)} \varrho_m(H)$ or for $\max_{H \in \mathscr{S}(G,J)} \varrho_m(H)$, where G and J are the same as in Theorem 1 or in Theorem 2, respectively. Especially, we can obtain two generalizations of Corollary B:

Corollary 1. *If G is a spanning subgraph of a connected graph J, then $\mathscr{S}(G, J)$ contains at least one upper embeddable graph if and only if*

$$c(J - A) + b^{\#}_G(J - A) - 2 \leqq |A| \quad \textit{for every} \quad A \subseteq E(J) .$$

Corollary 2. *If G is a connected spanning subgraph of a graph J, then every graph in $\mathscr{S}(G, J)$ is upper embeddable if and only if*

$$c(G - A) + b_J^{\square}(G - A) - 2 \leqq |A| \quad \textit{for every} \quad A \subseteq E(G)\,.$$

The following notation will be useful for proving Theorems 1 and 2. Let G and J be graphs, and let G be a spanning subgraph of J. If J is connected, then we denote

$$x_{G,J}^{\#} = \min_{H \in \mathscr{S}(G,J)} x_H \quad \text{and} \quad y_{G,J}^{\#} = \max_{A \subseteq E(J)} (c(J - A) + b_G^{\#}(J - A) - 1 - |A|)\,.$$

If G is connected, then we denote

$$x_{G,J}^{\square} = \max_{H \in \mathscr{S}(G,J)} x_H \quad \text{and} \quad y_{G,J}^{\square} = \max_{A \subseteq E(G)} (c(G - A) + b_J^{\square}(G - A) - 1 - |A|)\,.$$

Proof of Theorem 1. We shall prove that $x_{G,J}^{\#} = y_{G,J}^{\#}$ by induction on the number of edges of J. If $E(J) = \emptyset$, then $x_{G,J}^{\#} = 0 = y_{G,J}^{\#}$. Let $E(J) \neq \emptyset$. Assume that for every pair of graphs G' and J' with the properties that G' is a spanning subgraph of J' and J' is connected, if $|E(J')| < |E(J)|$, then $x_{G',J'}^{\#} = y_{G',J'}^{\#}$.

(I) First we wish to prove that $y_{G,J}^{\#} \leqq x_{G,J}^{\#}$. Let $H \in S(G, J)$ and $x_H = x_{G,J}^{\#}$. Then there exists a spanning tree T of H such that exactly x_H components of $H - E(T)$ have odd numbers of edges. There exists $A \subseteq E(J)$ such that $c(J - A) + b_G^{\#}(J - A) - 1 - |A| = y_{G,J}^{\#}$. Denote $A_0 = A \cap E(H)$. Moreover, denote by B_{con} or B_{dis} the set of $F_0 \in C(H - A_0)$ with the following two properties: $\beta(F_0)$ is odd, and $\langle V(F_0)\rangle_T$ is connected or disconnected, respectively. Finally, we denote by $B_{\text{con}}^{\#}$ or $B_{\text{dis}}^{\#}$ the set of $F \in C(H - A)$ with the following three properties: $\beta(F)$ is odd, $E(F) \subseteq E(G)$, and $\langle V(F)\rangle_T$ is connected or disconnected, respectively. The fact that H is spanned by G implies that $B_{\text{con}}^{\#} \subseteq B_{\text{con}}$ and $B_{\text{dis}}^{\#} \subseteq B_{\text{dis}}$.

It is not difficult to see that at least $|B_{\text{con}}| - |A_0 - E(T)|$ components of $H - E(T)$ have odd numbers of edges. Thus $x_H \geqq |B_{\text{con}}| - |A_0 - E(T)| \geqq |B_{\text{con}}^{\#}| - |A - E(T)|$. It is clear that

$$c(T - A_0) \geqq c(H - A_0) + |B_{\text{dis}}| \geqq c(J - A) + |B_{\text{dis}}^{\#}|\,.$$

Since T is a tree, $|E(T) \cap A| = c(T - A_0) - 1$. We get that

$$x_{G,J}^{\#} = x_H \geqq |B_{\text{con}}^{\#}| - |A - E(T)| = (b_G(J - A) - |B_{\text{dis}}^{\#}|) - \\ - (|A| - |E(T) \cap A|) \geqq b_G(J - A) + (c(J - A) - c(T - A_0)) - |A| + \\ + (c(T - A_0) - 1) = y_{G,J}^{\#}\,.$$

(II) Now we wish to prove that $x_{G,J}^{\#} \leqq y_{G,J}^{\#}$. Consider an arbitrary $A \subseteq E(J)$ with the properties that

(3) $c(J - A) + b_G^{\#}(J - A) - 1 - |A| = y_{G,J}^{\#}$, and for every $A' \subseteq E(J)$, if A is a proper subset of A', then $c(J - A') + b_G^{\#}(J - A') - 1 - |A'| < y_{G,J}^{\#}$.

We distinguish two cases:

Case 1. Let $A = \emptyset$ and $E(J) - E(G) \neq \emptyset$. Then $y_{G,J}^{\#} = 0$. Let $a \in E(J) - E(G)$.

It follows from (3) that $J - a$ is connected. For every $Z \subseteq E(J - A)$,

$$c((J - a) - Z) + b_G^\#((J - a) - Z) - 1 - |Z| =$$
$$= (c(J - (\{a\} \cup Z)) + b_G^\#(J - (\{a\} \cup Z)) - 1 - |(\{a) \cup Z)|) +$$
$$+ 1 < y_{G,J}^\# + 1 = 1\,.$$

Hence, $y_{G,J-a}^\# \leqq 0$. It follows from the definition of $y_{G,J-a}^\#$ that $y_{G,J-a}^\# \geqq 0$. Therefore, $y_{G,J-a}^\# = 0$. According to the induction hypothesis, $x_{G,J-a}^\# = 0$. Since $a \notin E(G)$, $x_{G,J}^\# = 0 = y_{G,J}^\#$.

Case 2. Let either $A \neq \emptyset$ or $E(G) = E(J)$. We denote by C' the set of all $F \in C(J - A)$ with the property that $\beta(F)$ is odd and $E(F) \subseteq E(G)$. Moreover, we denote $C'' = C(J - A) - C'$.

For every $F \in C'$, let us observe the following fact: Consider an arbitrary $e \in E(F)$. As follows from (3), $F - e$ is connected. For every $Z \subseteq E(F - e)$,

$$y_{G,J}^\# > c(J - (A \cup \{e\} \cup Z)) + b_G^\#(J - (A \cup \{e\} \cup Z)) -$$
$$- 1 - |A \cup \{e\} \cup Z| = (c(J - A) + b_G^\#(J - A) - 1 - |A|) +$$
$$+ (c((F - e) - Z) + b_{F-e}^\#(F - e) - Z) - 1 - |Z|) - 2\,,$$

and thus $y_{F-e,F-e}^\# < 2$. Obviously, $y_{F-e,F-e}^\#$ is identical with y_{F-e} in the sense of [5]. Since $\beta(F - e)$ is even, it follows from Proposition in [5] that $x_{F-e,F-e}^\#$ is even as well, and thus $y_{F-e,F-e}^\# = 0$. Since $|E(F - e)| < |E(J)|$, according to the induction hypothesis, $x_{F-e,F-e}^\# = 0$.

We have obtained that

(4) if $F \in C'$ and $e \in E(F')$, then $F - e$ is connected and $x_{F-e} = 0\,.$

Let $A = \emptyset$. According to the assumption of Case 2, $G = J$. It follows from (3) that $\beta(J)$ is odd. Then $y_{G,J}^\# = 1$. Statement (4) implies that $x_{G,J}^\# \leqq 1$, and therefore, $x_{G,J}^\# \leqq y_{G,J}^\#$. We shall now assume that $A \neq \emptyset$.

For every $F \in C''$, let us observe the following fact: Denote $G_F = \langle V(F)\rangle_G$. If $Z \subseteq E(F)$, then

$$y_{G,J}^\# \geqq c(J - (A \cup Z)) + b_G^\#(J - (A \cup Z)) - 1 - |A \cup Z| =$$
$$= (c(J - A) + b_G^\#(J - A) - 1 - |A|) +$$
$$+ (c(F - Z) + b_G^\#(F - Z) - 1 - |Z|)\,.$$

This implies that $y_{G_F,F}^\# \leqq 0$, and therefore, $y_{G_F,F}^\# = 0$. Since $A \neq \emptyset$, $|E(F)| < |E(J)|$. According to the induction hypothesis, $x_{G_F,F}^\# = 0$.

We have obtain that

(5) if $F \in C''$, then there exists $H_F \in \mathscr{S}(\langle V(F)\rangle_G, F)$ such that $x_{H_F} = 0$.

We denote by $\tilde{J}$ the graph obtained from the graph $J - (E(J) - A)$ in such a way that for each $F \in C' \cup C''$, the vertices of F are identified into one vertex, say a vertex

v_F. Clearly,

$$V(\tilde{J}) = \{v_F;\ F \in C' \cup C''\} \quad \text{and} \quad E(\tilde{J}) = A\ ;$$

for every $a \in A$ and every $F \in C' \cup C''$, a is incident with v_F in $\tilde{J}$ if and only if a is incident with a vertex of F in J. Denote $D = \{v_F;\ F \in C'\}$.

If L is a connected graph and $W \subseteq V(L)$, then – similarly as in [7] – we denote by $t_W(L)$ the number of isolated vertices w of L such that $w \in W$. It is easy to see that $t_D(\tilde{J} - A_0) \leqq b^{\#}_G(J - A_0)$ for every $A_0 \subseteq A$. This implies that

$$(5) \qquad \max_{A_0 \subseteq A} \left(c(\tilde{J} - A_0) + t_D(\tilde{J} - A_0) - 1 - |A_0|\right) \leqq y^{\#}_{G,J}\,.$$

We denote by $\mathcal{M}$ the set of all subsets C^* of C' with the property that there exists a mapping g of C^* into A such that

(a) v_F and $g(F)$ are incident in $\tilde{J}$ for each $F \in C^*$,

(b) if F_1 and F_2 are distinct elements of C^*, then $g(F_1) \neq g(F_2)$, and

(c) $\tilde{J} - g(C^*)$ is connected.

It immediately follows from Theorem in [7] (or it easily follows also from Lemma 3 in [5]) that

$$(6) \qquad \min_{C_0 \in \mathcal{M}} |C' - C_0| \leqq \max_{A_0 \subseteq A} \left(c(\tilde{J} - A_0) + t_D(\tilde{J} - A_0) - 1 - |A_0|\right).$$

Statements (5) and (6) imply that there exists $C^* \subseteq C'$ such that

$$(7) \qquad |C'| - |C^*| \leqq y^{\#}_{G,J}\,.$$

Consider a mapping g of C^* into A which fulfils (a), (b) and (c). Denote $A^* = g(C^*)$. According to (c), $\tilde{J} - A^*$ is connected, and therefore, $|A| - |A^*| \geqq |C'| + |C''| - 1$. It follows from (3) that

$$(8) \qquad y^{\#}_{G,J} = 2|C'| + |C''| - 1 - |A|\,.$$

Obviously, $|A^*| = |C^*|$. If we combine (7) and (8), we get that $|A| - |A^*| \leqq |C'| + |C''| - 1$. Hence, $|A| - |A^*| = |C'| + |C''| - 1$. This means that $\tilde{J} - A^*$ is a spanning tree of $\tilde{J}$.

For every $F \in C^*$, we can choose an edge $e(F)$ of F with the property that $e(F)$ and $g(F)$ are adjacent in J. Denote

$$\tilde{A} = A^* \cup \{e(F);\ F \in C^*\}\,.$$

Since the edges $A - A^*$ form a spanning tree of $\tilde{J}$, it follows from (3) and (4) that there exists $\tilde{H} \in \mathcal{S}(G - \tilde{A}, J - \tilde{A})$ such that $x_{\tilde{H}} \leqq |C'| - |C^*| \leqq y^{\#}_{G,J}$. It follows from the definition of $\tilde{A}$ that there exists $H \in \mathcal{S}(G, J)$ such that $x_H \leqq x_{\tilde{H}}$. Hence, $x^{\#}_{G,J} \leqq x_H \leqq y^{\#}_{G,J}$, which completes the proof of Theorem 1.

Remark. Many ideas in the proof of Theorem 1 have been derived from those in the second proof of Theorem 1 of [5], which is Theorem B of the present paper.

Proof of Theorem 2. We shall prove that $x^{\square}_{G,J} = y^{\square}_{G,J}$ by using Theorem B.

(I) First we wish to prove that $x^{\square}_{G,J} \geqq y^{\square}_{G,J}$. Consider $A \subseteq E(G)$ such that

$$c(G - A) + b^{\square}_J(G - A) - 1 - |A| = y^{\square}_{G,J},$$

and for every proper subset A_0 of A,

$$c(G - A_0) + b^{\square}_J(G - A_0) - 1 - |A_0| < y^{\square}_{G,J}.$$

This implies that every component of $G - A$ is an induced subgraph of G. We denote by C^* the set of all $F \in C(G - A)$ such that $\beta(F)$ is even and F is not an induced subgraph of J. This means that for every $F \in C^*$ we can choose an edge $e_F \in E(J)$ such that if e_F is incident with a vertex u in J, then $u \in V(F)$. We denote by H the graph in $\mathscr{S}(G, J)$ which is obtained from G by adding all the edges e_F, $F \in C^*$. Clearly, $b(H - A) = b^{\square}_J(G - A)$, and thus

$$c(H - A) + b(H - A) - 1 - |A| =$$
$$= c(G - A) + b^{\square}_J(G - A) - 1 - |A| \leqq y^{\square}_{G,J}.$$

According to Theorem B,

$$x_H \geqq c(H - A) + b(H - A) - 1 - |A|,$$

and therefore, $x^{\square}_{G,J} \geqq x_H \geqq y^{\square}_{G,J}$.

(II) Now we wish to prove that $x^{\square}_{G,J} \leqq y^{\square}_{G,J}$. There exists $H \in \mathscr{S}(G, J)$ such that $x^{\square}_{G,J} = x_H$. As follows from Theorem B, there exists $A \subseteq E(H)$ such that

$$x_H = c(H - A) + b(H - A) - 1 - |A|.$$

Put $A^* = A \cap E(G)$.

Consider an arbitrary $F \in C(H - A)$. Denote $F^* = \langle V(F) \rangle_G$. If F^* is connected, then $b(F) \leqq b^{\square}_J(F^*)$, and therefore, $c(F) + b(F) \leqq c(F^*) + b^{\square}_J(F^*)$. If F^* is disconnected, then $c(F) + b(F) \leqq 2 \leqq c(F^*) \leqq c(F^*) + b^{\square}_J(F^*)$.

Therefore,

$$c(H - A) + b(H - A) - 1 - |A| \leqq$$
$$\leqq c(G - A) + b^{\square}_J(G - A) - 1 - |A| \leqq y^{\square}_{G,J}.$$

We have that $x^{\square}_{G,J} = x_H \leqq y^{\square}_{G,J}$, which completes the proof of Theorem 2.

Remark. Theorem 2 was proved in [6] under the condition that J is a complete graph (with no loop or multiple edge) and $x^{\square}_{G,J} \leqq 1$. The proof of Theorem 2 is based on the ideas of the proof of Theorem 2 of [6].

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